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ISOMORPHISMS OF POLAR AND POLARIZED GRAPHS

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At the Conference on Graph Theory held at Štiřín in May 1972, F. ZÍTEK read a lecture [4] in which he suggested to study two new types of graphs – polar and polarized graphs. The importance of these graphs for the application of the graph theory in the railway traffic was pointed out by J. ČERNÝ in his lecture [1] at the same conference. K. ČULÍK [2] mentioned the graphs of logic networks and remarked that also in this direction polar and polarized graphs are applicable.

In this paper we shall study some properties of these graphs which are related to the concept of isomorphism. All graphs that we investigate here are undirected.

Definition. A *polar graph* (undirected, without loops and multiple edges) is an ordered quintuple $\langle V, E, P, \varkappa, \lambda \rangle$, where V, E, P are sets, $\varkappa$ is a mapping of the set V into the set of unordered pairs of distinct elements of P and λ is a mapping of the set E into the set of unordered pairs of distinct elements of P such that the following conditions are satisfied:

(1) For any $u \in V$, $v \in V$, $u \neq v$, we have $\varkappa(u) \cap \varkappa(v) = \emptyset$.
(2) For any $e \in E$, $f \in E$, $e \neq f$ we have $\lambda(e) \neq \lambda(f)$.
(3) For any $p \in P$ there exist $v \in V$ such that $p \in \varkappa(v)$.

The elements of the set V are called *vertices*, the elements of the set E are called *edges*, the elements of the set P are called *poles*. If $p \in P$, $v \in V$, $p \in \varkappa(v)$, we say that *the pole P belongs to the vertex v*. If $p \in P$, $e \in E$ and $p \in \lambda(e)$, we say that the edge e is incident with the pole p.

This was the formal definition of a polar graph. We shall explain it intuitively. We obtain a polar graph from an undirected graph by decomposing at each vertex v the set of edges incident with it into two disjoint subsets. To each of these sets we assign an abstract element – a pole of the vertex v. Therefore any vertex of a polar graph has two poles, no two distinct vertices have a pole in common (condition (1)) and any edge is incident with two poles. We shall say also that an edge joins two poles instead of saying that it is incident with them. Condition (2) expresses the absence of multiple edges. Condition (3) means that each pole belongs to some vertex.

Now we shall introduce the formal definition of a polarized graph.

Definition. A *polarized graph* (undirected, without loops and multiple edges) is the ordered quintuple $\langle V, E, P, \vec{\varkappa}, \lambda \rangle$, where V, E, P are sets, $\vec{\varkappa}$ is a mapping of the set V into the set of ordered pairs of distinct elements of P and λ is a mapping of the set E into the set of unordered pairs of distinct elements of P for which the same conditions (1), (2), (3) as in the definition of a polar graph are satisfied. The elements of V are called *vertices*, the elements of E are called *edges*, the elements of P are called *poles*. If $\vec{\varkappa}(v) = [p_1, p_2]$ for some $v \in V$, $p_1 \in P$, $p_2 \in P$, then p_1 is called the *southern pole* of v and p_2 is called *the northern pole* of v.

Thus we obtain a polarized graph from a polar graph by declaring at each vertex v one of its poles to be the southern and the other to be the northern pole.

When drawing polar or polarized graphs it is useful to draw a vertex as a magnetic needle (which has also two poles). A polar graph has both poles of this needle white, a polarized graph has the southern poles white, the northern poles black.

We shall give an example of an application of polar graphs; this example will be taken from the railway traffic, as mentioned in [1] and [4]. The vertices of a polar graph are railway stations on a certain territory, an edge joining two vertices is the segment of the railway line joining these stations and not going through any other station. Then the poles of a vertex are two sides from which the trains come into the station. If two edges are incident with different poles of the same vertex, this means that the corresponding railway line segments come into the station from different sides and that a train coming by one of them can continue its travel by the other following always the same direction. If two edges are incident with the same pole of a vertex, this means that the corresponding railway line segments come into the station from the same side and that a train coming by one of them can continue its travel by the other only by changing its direction (i.e. by changing the engine).

Now we shall define some further concepts.

A *quasi-loop* is an edge joining two different poles of the same vertex. (If we wished to extend the definitions of polar and polarized graphs onto graphs with loops, then a loop would be an edge joining a pole with itself.)

A *complete polar graph* is a polar graph in which any two distinct poles are joined by an edge.

Therefore between any two distinct vertices of such a graph there are four edges, because an edge must go from each pole of one vertex to each pole of the other. At each vertex there is a quasi-loop. If such a graph has n vertices, it has $2n^2 - n$ edges.

A *complete polarized graph* is obtained from a complete polar graph by an evident way.

Given a non-polar graph G, then to make G polar means to add two poles to each vertex v of G, to decompose the set of edges incident with v into two disjoint subsets and to assign the poles to these subsets.

Given a polar graph G, then to polarize G means at each vertex of G to declare one pole to be the southern and the other to be the northern pole.

Given a non-polar graph G, then to polarize G means first to make G polar and then to polarize the polar graph thus obtained.

If $G_1 = \langle V_1, E_1, P_1, \varkappa_1, \lambda_1 \rangle$, $G_2 = \langle V_2, E_2, P_2, \varkappa_2, \lambda_2 \rangle$ are polar graphs, then *an isomorphism of* G_1 *onto* G_2 is a one-to-one mapping φ of $V_1 \cup E_1 \cup P_1$ onto $V_2 \cup \cup E_2 \cup P_2$ such that

(a) V_1 is mapped by φ onto V_2, E_1 is mapped onto E_2, P_1 is mapped onto P_2;
(b) if $\varkappa_1(v) = \{p_1, p_2\}$ for some $p_1 \in P_1$, $p_2 \in P_1$, $v \in V_1$, then $\varkappa_2(\varphi(v)) = \{\varphi(p_1), \varphi(p_2)\}$;
(c) if $\lambda_1(e) = \{p_1, p_2\}$ for $p_1 \in P_1$, $p_2 \in P_2$, $e \in E_1$, then $\lambda_2(\varphi(e)) = \{\varphi(p_1), \varphi(p_2)\}$.

An isomorphism of G_1 onto G_2 is defined analogously if G_1 and G_2 are polarized graphs; only in (b) we consider the directed pair $[p_1, p_2]$ instead of the undirected pair $\{p_1, p_2\}$ (and naturally also $[\varphi(p_1), \varphi(p_2)]$ and $\vec{\varkappa}$ instead of $\varkappa$).

We see that for polarized graphs, each isomorphism maps a southern pole always onto a southern pole and a northern pole always onto a northern pole. An isomorphism of polar graphs is uniquely determined by giving the images of poles; the images of vertices and edges may be derived from them. An isomorphism of polarized graphs is uniquely determined by giving the images of vertices; the images of poles (because each vertex has exactly one northern pole and exactly one southern pole) are derivable from it and so are the images of edges.

An isomorphism of a graph (polar or polarized) onto itself is called *an automorphism* of this graph. It is easy to prove that the automorphisms of a graph form a group under their superposition.

Theorem 1. *Let K_n be the complete polar graph with n vertices $v_1, \ldots, v_n$, let $\mathfrak{A}(K_n)$ be the group of all automorphisms of K_n. Then $\mathfrak{A}(K_n)$ is isomorphic to the direct product of the groups $\mathfrak{S}_n$, $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$, where $\mathfrak{S}_n$ is the symmetric group of order n and $\mathfrak{P}_i$ for $i = 1, \ldots, n$ is a cyclic group of order 2.*

Remark. The order of $\mathfrak{A}(K_n)$ is then equal to $2^n \,.\, n!$.

Proof. Let each vertex v_i for $i = 1, \ldots, n$ have the poles $p_1(v_i)$, $p_2(v_i)$. Let α_i for $i = 1, \ldots, n$ be the automorphism of K_n such that $\alpha_i(p_1(v_i)) = p_2(v_i)$, $\alpha_i(p_2(v_i) = = p_1(v_i)$ while the other poles of K_n (therefore also all vertices) are fixed. Further, to any permutation π of the number set $\{1, \ldots, n\}$ let β_π be the automorphism of K_n such that $\beta_\pi(p_1(v_i)) = p_1(v_{\pi(i)})$, $\beta_\pi(p_2(v_i)) = p_2(v_{\pi(i)})$. Any α_i generates a cyclic group $\mathfrak{P}_i$ of order 2. The mappings β_π form a group $\mathfrak{B}$ isomorphic to the group of all permutations of $\{1, \ldots, n\}$, i.e. to the symmetric group $\mathfrak{S}_n$ of order n. The groups $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$, $\mathfrak{B}$ have pairwise no elements in common except for the identical automorphism of K_n. Evidently each automorphism of K_n is a superposition of mappings from these groups and $\mathfrak{A}(K_n)$ is the direct product of them, q.e.d.

Theorem 2. *Let K'_n be the complete polarized graph with n vertices. Then the group of all automorphisms of K'_n is isomorphic to the symmetric group $\mathfrak{S}_n$ or order n.*

Proof. As each automorphism of a polarized graph maps the northern pole of a vertex again onto the northern pole of some vertex and a southern pole onto a southern pole, it is uniquely determined by giving the images of vertices, i.e. by the permutation of the vertex set induced by this automorphism. Therefore, to any permutation of the vertex set of K'_n there corresponds exactly one automorphism of K'_n (as K'_n is complete) and the group of automorphisms of K'_n is isomorphic to the group of all permutations of a set with n elements, which is the symmetric group of order n.

Theorem 3. *Let G be a non-polar graph without isolated vertices. Then G can be made polar in such a way that the group of automorphisms of the resulting polar graph is isomorphic to the group of automorphisms of G.*

Proof. Let us make G polar so that at any vertex v one pole of v is incident with all edges which are incident with v in G while the other pole is incident with no edge (is isolated). We obtain a polar graph G_P. As no vertex of G is isolated, each vertex of G_P has one pole isolated, another non-isolated. Any automorphism of G_P is uniquely determined by images of vertices, because if an image of a vertex v is given, we know that the isolated or non-isolated pole of v is mapped respectively onto the isolated or non-isolated pole of the image of v. And, if a permutation of the vertex set of G induces an automorphism of G, it induces obviously also an automorphism of G_P. Hence the assertion holds. If G has an isolated vertex u, the assertion does not hold; a counterexample is a graph G consisting only of one isolated vertex u. The group of automorphisms of G consists only of the identical mapping, but the group of automorphisms of G_P has order 2; in addition to the identical automorphism it contains the automorphism interchanging the poles of u.

Theorem 4. *Let G be a non-polar graph. Then G can be polarized in such a way that the group of automorphisms of the resulting polarized graph is isomorphic to the group of automorphisms of G.*

Proof. Let us make G polar in the same way as in the proof of Theorem 3. Then let us declare all non-isolated poles to be northern, the isolated poles of non-isolated vertices to be southern. If a vertex is isolated, its northern pole is chosen arbitrarily. As already mentioned above, any automorphism of a polarized graph is uniquely determined by the images of vertices, therefore the assertion holds. (Even the isolated vertices do not matter, because no automorphism interchanging poles of a vertex can exist.)

Theorem 5. *Let $\mathfrak{G}$ be a finite or countable group. Then there exists a polar graph G and a polarized graph G' so that the groups of automorphisms of both of these graphs are isomorphic to $\mathfrak{G}$.*

Proof. According to a theorem of FRUCHT [3], to G there exists a non-polar graph without isolated vertices whose group of automorphisms is isomorphic to G. Hence Theorem 5 is a consequence of Theorems 3, 4.

Theorem 6. *Let G be a non-polar graph with a Hamiltonian path. Then G can be polarized so that the resulting graph has no automorphisms except the identical automorphism.*

Proof. Let H be a Hamiltonian path in G. Let the vertices of H (and also of G) be $v_1, \ldots, v_n$, let the edges of H be $v_i v_{i+1}$ for $i = 1, \ldots, n - 1$. Let us polarize G so that for $1 \leqq i < j \leqq n$ the edge $v_i v_j$ is incident with the northern pole of v_i and with the southern pole of v_j; let the resulting polarized graph be G_P. Thus if we have a path P joining v_k and v_l for some k, l which goes from the northern pole of v_k to the southern pole of v_l, any edge of P joins a northern pole of one vertex with a southern pole of another one and any two edges of P incident in G with a certain vertex are incident in G_P with different poles of this vertex, then $k < l$ and the subscripts of vertices on P form an increasing sequence if we go from v_k to v_l along P. If $\varphi(v_k) = v_r$, $\varphi(v_l) = v_s$ in an automorphism φ of G_P, then there exists a path P' from v_r to v_s with the same properties as P (k substituted by r, l substituted by s) namely, the image of P in φ and therefore $r < s$. Thus φ preserves the ordering of subscripts and as φ is one-to-one, it must be the identical mapping.

An example is in Fig. 1 where G is the complete graph with five vertices. A counterexample showing that for graphs without Hamiltonian paths the assertion does not hold in general is a star with six or more vertices. Let c be the center of such a star. For a vertex $v \neq c$ there are four possible cases: (a) the southern pole of v is joined with the southern pole of c, (b) the southern pole of v is joined with the northern pole of c, (c) the northern pole of v is joined with the southern pole of c, (d) the northern pole of v is joined with the northern pole of c. If there are at least five vertices different from the center, there must be at least two vertices of them for which the same case occurs and they can be interchanged by a non-identical automorphism of this star.

Theorem 7. *Let G be a non-polar graph having a Hamiltonian path which is neither a circuit with less than six vertices, nor a snake with less than four vertices. Then G can be made polar so that the resulting polar graph has no automorphisms except the identical automorphism.*

Proof. First, let us polarize G as in the proof of Theorem 6. Then we consider the resulting graph as polar (we do not distinguish the southern and northern poles). If this polar graph G_P has no automorphisms except the identical automorphism, the proof is complete. If not, we shall distinguish three cases: (a) G is a circuit (at least with seven vertices); (b) G is a snake (at least with four vertices); (c) the other possibilities. If G is a circuit, then two cases are possible at any vertex v of G: either the edges incident with v in G are incident in the polar graph G_P with the same pole of v, or with different poles. Let the vertices of G be $v_1, \ldots, v_n$ and the edges $v_i v_{i+1}$ for $i = 1, \ldots, n-1$, $v_n v_1$. Then G can be made polar so that the first case occurs at v_1, v_3, v_4, the second case at all other vertices (Fig. 2 for $n = 6$). It is easy to prove that for the graph thus obtained the assertion holds. If G is a snake with the vertices $v_1, \ldots, v_n$ and edges $v_i v_{i+1}$ for $i = 1, \ldots, n-1$, we can make it polar so that both the edges at v_2 are incident with the same pole of v_2 while at $v_3, \ldots, v_{n-1}$ the edges are

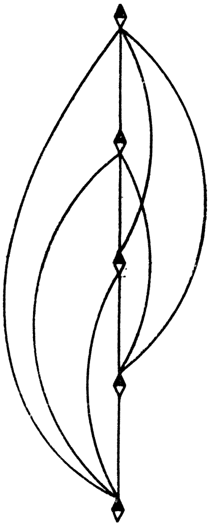

Fig. 1.

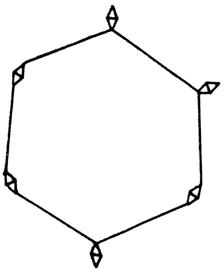

Fig. 2.

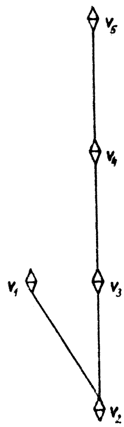

Fig. 3.

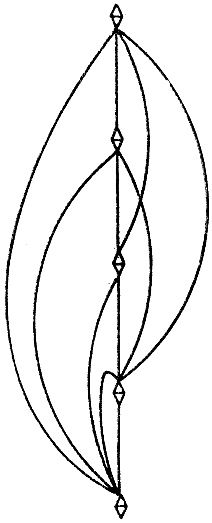

Fig. 4.

incident with different poles (Fig. 3). For such a graph the assertion holds. Now take the last case. Let H be a Hamiltonian path in G, let its vertices be $v_1, \ldots, v_n$ and edges $v_i v_{i+1}$ for $i = 1, \ldots, n-1$. Consider the polarization of G mentioned at the beginning of the proof. If an edge goes from v_1 into a vertex v_i, $i \neq 2$, $i \neq n$, then take the least i with this property and change the graph G_P so that the edge $v_1 v_i$ is incident with the other pole of v_i than the edge $v_{i-1} v_i$ (in G_P they were incident with the same pole, because both 1 and $i - 1$ are less than i). We have a circuit C with the edges $v_j v_{j+1}$ for $j = 1, \ldots, i-1$ and $v_1 v_j$. This circuit has the length i and at the vertex v_1 the edges of C are incident with the same pole, at the other vertices with different poles. Such a circuit is evidently exactly one in this graph, therefore any automorphism must map C onto itself, v_1 onto itself. It also maps v_n onto itself as the only vertex with an isolated pole and not belonging to C. If such an automorphism φ is non-identical, we must have $\varphi(v_2) = v_i$, $\varphi(v_i) = v_2$. The vertex v_2 is connected with v_n by a simple path P of the length $n = 2$ and at each vertex of P the edges of P are incident with different poles (this path P is obtained from H by deleting v_1).

However, any path connecting v_i with v_n with this property can have a length at most $n - i$, therefore such a non-identical automorphism φ cannot exist. If v_1 is joined in G only with v_2 and v_n, then there exists a Hamiltonian circuit consisting of the Hamiltonian path H and the edge $v_1 v_n$, therefore there exist at least n

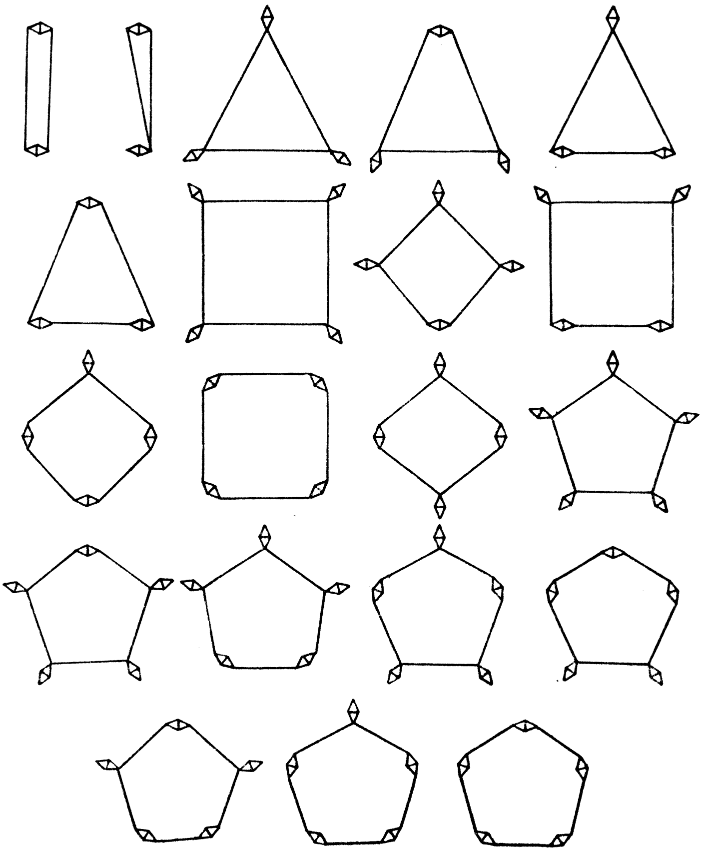

Fig. 5.

Hamiltonian paths obtained from it by deleting an edge. We can choose another Hamiltonian path which has the required property (as G is not a circuit). If v_1 is joined in G with no other vertex than v_2, we take the least j such that v_j is joined with a vertex v_i, $i > j$, $i \neq j + 1$, $i \neq n$ and use the same argument for v_j instead of v_1. If for any j the vertex v_j is joined only with v_{j-1}, v_{j+1} and possibly with v_n, we change the subscripts so that v_i becomes v'_{n-i+1} for $i = 1, \ldots, n$; as G is not a snake, some of the above mentioned cases must occur.

Fig. 4 shows the case of the complete graph with five vertices. In Fig. 5, there are all possible polar graphs obtained by making circuits with less than six vertices polar; from this figure it is easy to see that each of them has a non-identical automorphism. In Fig. 6, the same is done for snakes with less than four vertices.

In the end of this paper we shall prove some analoga of Whitney's theorem on reconstructing a graph from the incidence relation on its edge set.

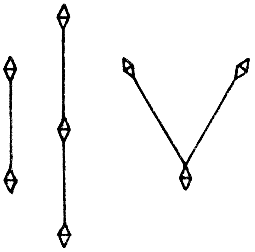

Fig. 6.

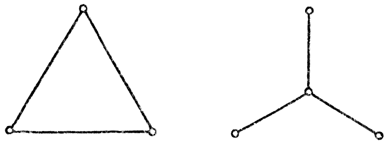

Fig. 7.

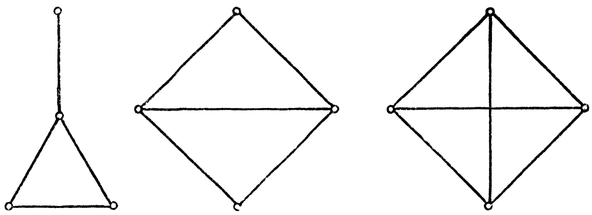

Fig. 8.

H. WHITNEY (quoted in [3]) has proved that every edge isomorphism of two undirected graphs is induced by a vertex isomorphism, with the exception of the graphs in Figs. 7 and 8. A vertex isomorphism of graphs is an isomorphism in the usual sense. An edge isomorphism is a one-to-one mapping of the edge set of one graph onto the edge set of another graph preserving the incidence relation (two edges are in this relation, if and only if they have a common end vertex). The graphs in Fig. 7 are edge-isomorphic, but not vertex-isomorphic. Any one of the graphs in Fig. 8 has an edge automorphism which is not induced by a vertex automorphism, but it cannot be edge-isomorphic with a graph with which it is not vertex-isomorphic. From Whitney's theorem it follows that any graph which does not contain a connected component isomorphic with a graph from Fig. 9 can reconstructed from its edge set with the relation of incidence given on it.

For a polar graph we shall consider three relations on the edge set. Two edges e_1, e_2 of G are in the relation ϱ_1, if and only if there exists a pole in G with which they both are incident. They are in the relation ϱ_2, if and only if there exists a vertex v in G

such that e_1 is incident with one pole of v while e_2 is incident with the other. Finally, they are in the relation $\varrho_2^{(2)}$, if and only if there exist two vertices u, v in G such that e_1 joins one pole of u with one pole of v and e_2 joins the other pole of u with the other pole of v; obviously $\varrho_2^{(2)} \subset \varrho_2$. We shall investigate how much information about G can be obtained from these relations.

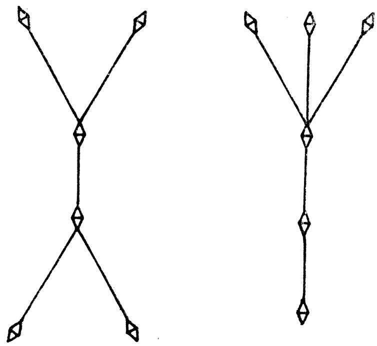

Fig. 9.

Given a polar graph G, then the pole graph of G is a non-polar graph G^* whose vertices are the poles of G and in which two vertices are joined by an edge, if and only if the corresponding poles are joined by an edge in G.

Theorem 8. *Let the edge set E of a polar graph G and the relation ϱ_1 on it be given. Then we can reconstruct the pole graph G^* of G if and only if no connected component of G is isomorphic to any one of the graphs in Fig. 7.*

Proof. The relation ϱ_1 on the edge set E of G corresponds to the usual incidence relation on the vertex set of G. Therefore the theorem follows from Whitney's theorem.

It is easy to see that from ϱ_1 no more information about G can be obtained. We cannot recognize which pairs of poles belong to the same vertex and which do not.

Theorem 9. *Let G be a polar graph such that at each vertex of G one pole is incident at least with three edges, the other at least with two edges. Let the edge set E of G and the relation ϱ_2 on it be given. Then we can uniquely reconstruct the graph G.*

Proof. Let four edges e_1, e_2, e_3, f_1 from E satisfy $(e_i, f_1) \in \varrho_2$ for $i = 1, 2, 3$. There are four possible cases which are in Fig. 9. But if there is also an edge $f_2 \neq f_1$ such that $(e_i, f_2) \in \varrho_2$ for $i = 1, 2, 3$, then only the first case in this figure is possible: otherwise f_1 and f_2 would form a double edge. Therefore to any pair e, f of elements of E we find a pair $\{R(e, f), S(e, f)\}$ of subsets of E such that $e \in R(e, f)$, $f \in S(e, f)$,

$(e',f') \in \varrho_2$ for each $e' \in R(e,f)$ and $f' \in S(e,f)$, the sets $R(e,f)$, $S(e,f)$ being the maximal subsets of E with these properties. The condition of the theorem implies that always one of these sets has at least three elements while the other has at least two elements. Now it follows that to each pair with these properties there corresponds a vertex of G; different pairs are associated with different vertices. The sets which form such a pair correspond to the poles of this vertex. Therefore, for any edge we know the vertices and the poles with which it is incident and hence we can reconstruct G.

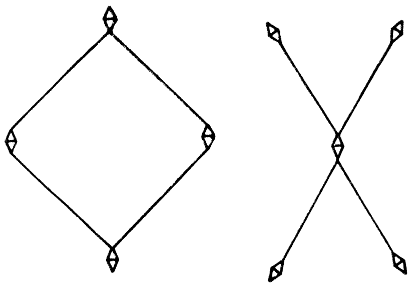

Fig. 10.

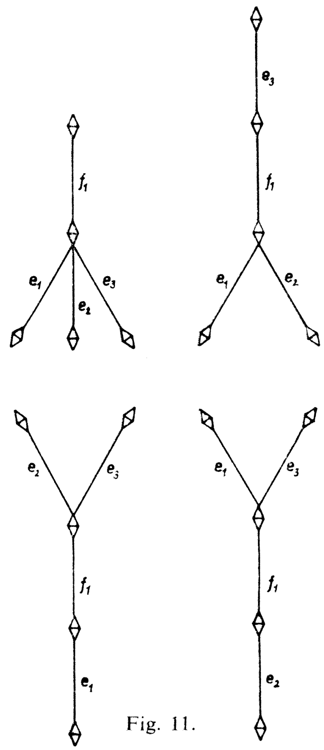

Fig. 11.

In Figs. 10 and 11 we see pairs of graphs with the same set E of edges and with the same relation ϱ_2 on it which are not isomorphic. These graphs are counterexamples showing that if the condition of Theorem 9 is not satisfied, this theorem does not hold in general.

As we have mentioned above, in a polar graph two vertices can be joined with more than one (at most four) edges. If G is a polar graph, then let $\mathfrak{D}(G)$ be the class of graphs obtained from G by deleting, for any pair of vertices joined by more than one edge, all edges except one. Also quasi-loops are deleted.

Theorem 10. *Let G be a polar graph. Let the class $\mathfrak{D}(G)$ contain a graph G', none of whose connected components is isomorphic to any of the graphs in Fig.* 12. *Let the edge set E of G and the relations ϱ_1, ϱ_2, $\varrho_2^{(2)}$ on it be given. Then G can be uniquely reconstructed.*

Proof. First we shall construct a graph G' from $\mathfrak{D}(G)$; if possible, we construct a graph satisfying the condition. If two edges e_1, e_2 join the same pair of vertices, they are either in the relation $\varrho_1 \cap \varrho_2$ (Fig. 14) or in the relation $\varrho_2^{(2)}$ (Fig. 15); there-

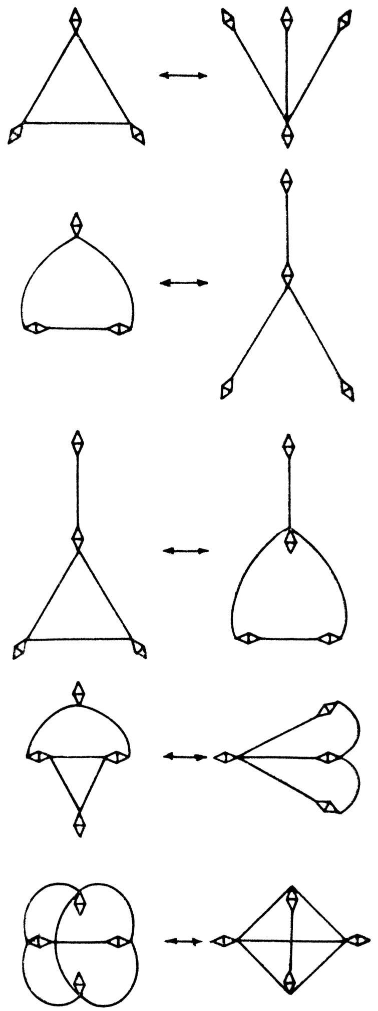

Fig. 12.

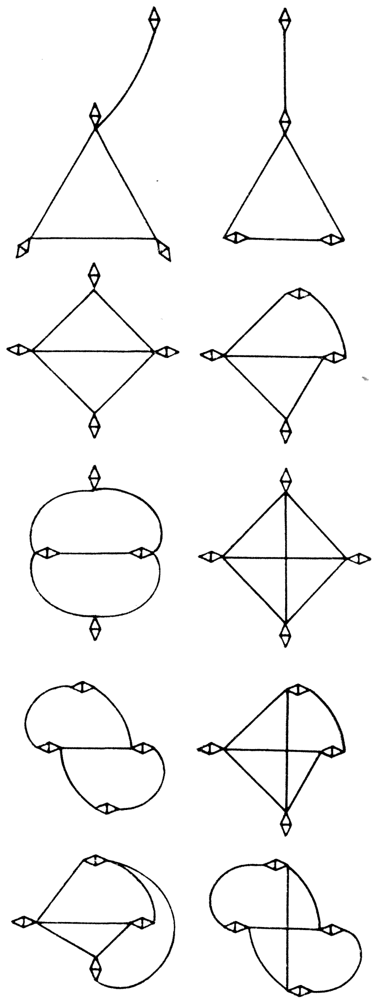

Fig. 13.

fore we can recognize, which pairs of edges join the same pair of vertices. Also we can easily recognize quasi-loops; they are such edges e that $(e, e) \in \varrho_2$. Thus we can construct $G' \in \mathfrak{D}(G)$. Then we treat the graph G' as non-polar; the relation $\varrho_1 \cup \varrho_2$ is the usual incidence relation on it. If none of the connected components of G' is isomorphic (as a non-polar graph) to any one of the graphs in Figs. 7 and 8, the graph G' can be uniquely reconstructed according to Whitney's theorem as non-polar. Then we use the relations ϱ_1, ϱ_2 to make it polar. In the end we add the edges which were deleted when constructing G'; this is also easy since we know ϱ_1, ϱ_2 and $\varrho_2^{(2)}$. Thus it remains to study singular cases, when some of the connected components of G' is isomorphic to some of the graphs on Figs. 7 and 8. If it is isomorphic to some of the graphs on Fig. 8, it can be reconstructed as a non-polar graph, but the location of edges cannot be uniquely reconstructed (see the last chapter of [3]). In these cases, given a triple of edges any two of which are incident, the crucial step is to decide whether they form a triangle or a star. But if any two of such edges are in ϱ_2, or one pair is in ϱ_2 and two others in ϱ_1, they must form a triangle. And if we reconstruct a triangle from one of the graphs in Figs. 7 and 8 with the exact location of edges, we can easily reconstruct also the rest of the graph, again with the exact location of edges. Therefore we have difficulties only with polar graphs, some connected component of which is isomorphic (as a non-polar graph) with some of the graphs on Figs. 7 and 8 and is such that any three edges of it, any two of which are in $\varrho_1 \cup \varrho_2$, have the property that either any two of them are in ϱ_1, or one pair of them is in ϱ_1 and two others in ϱ_2. In Figs. 12 and 13 we see all of the possible components. In Fig. 12 we see the pairs of non-isomorphic graphs with the same edge set and the same relations ϱ_1 and ϱ_2. A graph G', a connected component of which is isomorphic to

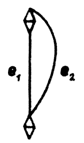

Fig. 14.

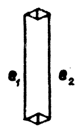

Fig. 15.

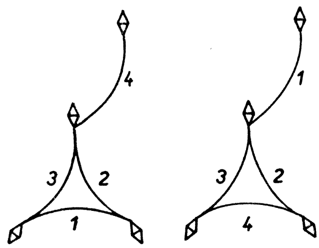

Fig. 16.

some of them, cannot be uniquely reconstructed. If we define an edge-isomorphism of polar graphs as a one-to-one mapping of the edge set of one graph onto the edge set of another preserving ϱ_1 and ϱ_2, we may say that these graphs are edge-isomorphic, but not vertex-isomorphic. In Fig. 14 we see graphs with the property that any graph edge-isomorphic with such a graph is vertex-isomorphic with it, but there exist edge-automorphisms of such graphs which are not induced by vertex-automorphisms. These graphs can be reconstructed uniquely, but not with the exact location of edges. In Fig. 16 we see the two possible reconstructions of one of these graphs (edges are numbered); in both the cases we obtain isomorphic graphs, but the location of edges is different. Thus the theorem is proved.

We have used the graphs from $\mathfrak{D}(G)$, because Whitney's theorem holds only for graphs without loops and multiple edges. A dilligent reader can construct the list of all connected graphs with the property that any graph from $\mathfrak{D}(G)$ is isomorphic to a certain graph in Fig. 12 (by adding edges).

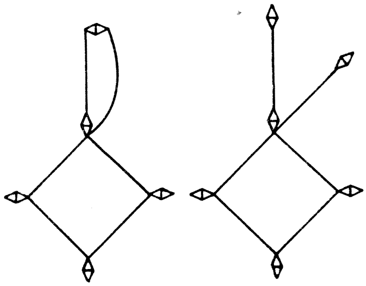

Fig. 17.

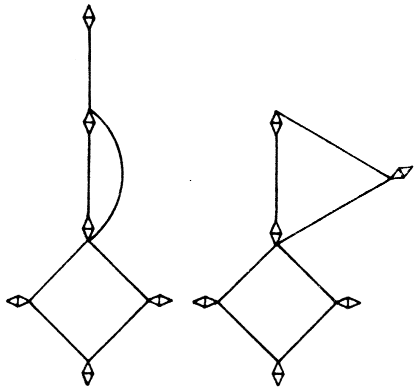

Fig. 18.

In Figs. 17 and 18 we see two conterexamples showing that without $\varrho_2^{(2)}$ a graph cannot be uniquely reconstructed. These pairs of graphs have the same E, ϱ_1 and ϱ_2, but different $\varrho_2^{(2)}$.

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