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ANALOGA OF MENGER'S THEOREM FOR POLAR AND POLARIZED GRAPHS

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In this paper we shall study polar and polarized graphs. These concepts were introduced by F. ZÍTEK [6]. Their importance for the applications of the graph theory was pointed out by J. ČERNÝ [1] and K. ČULÍK [2]. The definitions of these concepts arre to be found in [5].

In a polar graph we can define two special cases of paths, namely homopolar paths and heteropolar paths.

A homopolar path joining vertices a and b in a polar graph G is a sequence $a = u_1, e_1, u_2, e_2, \ldots, e_{n-1}, u_n = b$, where $u_1, \ldots, u_n$ are vertices, $e_1, \ldots, e_{n-1}$ are edges, the neighbouring elements of the sequence are incident and for $i = 1, \ldots, n - 2$ the edges e_i, e_{i+1} are incident with the same pole of u_{i+1}.

A heteropolar path joining vertices a and b in a polar graph G is defined analogously as a homopolar path, only the edges e_i, e_{i+1} are incident with distinct poles of u_{i+1} for $i = 1, \ldots, n - 2$.

In polarized graphs we shall distinguish SS-edges, NN-edges and SN-edges. An SS-edge joins southern poles of two vertices, an NN-edge joins northern poles of two vertices, an SN-edge joins the southern pole of one vertex with the northern pole of another one.

A homopolar path consisting only of SS-edges (or NN-edges) is called an SS-path (or NN-path, respectively). A heteropolar path consisting of SN-edges joining a and b and incident with the southern (or northern) pole of a and the northern (or southern, respectively) pole of b is called a heteropolar SN-path (or NS-path, respectively) from a to b.

Two vertices a and b of a polar or polarized graph G are called X-connected, if and only if either $a = b$, or they are joined by an X-path, where X stands for "homopolar", "heteropolar", "SS-", "NN-", "heteropolar SN-" or "heteropolar NS-". In the last two cases we must care for the ordering of the pair a, b.

The following propositions are easy to prove.

Proposition 1. *The relations of being homopolarly connected and heteropolarly connected are reflexive and symmetric, but in general not transitive.*

Figs. 1 and 2 show graphs in which the pairs a, b and b, c are respectively homopolarly or heteropolarly connected, but the pair a, c is not.

Proposition 2. *The relations of being SS-connected and NN-connected are equivalence relations.*

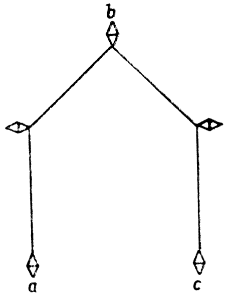

Fig. 1.

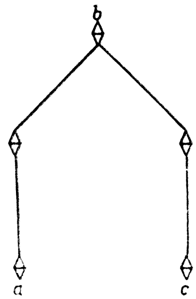

Fig. 2.

Fig. 3.

Proposition 3. *The relations of being heteropolarly SN- connected and heteropolarly NS-connected are reflexive and transitive, but not symmetric.*

Fig. 3 shows a graph in which a is SN-connected with b, but not b with a.

We could define also homopolar SN-paths, but we do not intend to study them in this paper.

The X-connectivity degree of the vertices a, b in G is the minimal number of vertices distinct from a, b which must be deleted from G in order that in the resulting graph the vertices a, b might not be X-connected. Analogously we define the X-edge-connectivity degree of a, b.

Now we shall prove some analoga of Menger's theorem [3].

Theorem 1. *Let a, b be two vertices of a polarized graph G. Then the maximal number of vertex-disjoint (up to a and b) heteropolar SN-paths from a into b in G is equal to the heteropolar SN-vertex-connectivity degree of a and b in G.*

Proof. To the graph G let us assign a directed non-polar graph G^* so that the vertex set of G^* is the vertex set of G and the edges of G^* are all SN-edges of G and each of these edges is directed so that the vertex with whose northern pole it is incident in G is its terminal vertex in G^*. By this transformation of G into G^*, each heteropolar SN-path becomes a directed path, therefore the maximal number of vertex-disjoint (up to a and b) heteropolar SN-paths from a into b in G is equal to the maximal number of vertex-disjoint (up to a and b) directed paths going from a into b in G^*. According to an analogon of Menger's theorem due to G. A. Dirac this is equal to the vertex connectivity degree of a and b in the directed graph G^*, i.e. to the minimal number of vertices distinct from a and b which must be deleted from G^* in order that there might not exist any directed path from a into b. But each of such sets of vertices (separating sets) is also a separating set in G (i.e. the set of vertices which must be deleted from G in order that there might not exist any SN-path in G from a into b) and vice versa, which is evident from the construction of G^*. Thus the assertion is true.

Theorem 2. *Let a, b be two vertices of a polarized graph G. Then the maximal number of edge-disjoint heteropolar SN-paths from a into b in G is equal to the heteropolar SN-edge-connectivity degree of a and b in G.*

Proof. We construct a graph G^* as in the proof of Theorem 1. Then the maximal number of edge-disjoint heteropolar SN-paths from a into b is equal to the maximal number of edge-disjoint directed paths from a into b in G^*. Using another analogon of Menger's theorem due also to G. A. Dirac we can prove that this is the minimal number of edges which must be deleted from G^* in order that there might not exist any directed path from a into b in G^*. From the construction of G^* it follows that this is equal to the minimal number of edges which must be deleted from G in order that there might not exist any heteropolar SN-path from a into b in the resulting graph, i.e. to the heteropolar SN-edge-connectivity degree of a and b in G.

Theorem 3. *Let a and b be two vertices of a polarized graph G. Then the maximal number of vertex-disjoint (up to a and b) SS-paths connecting a and b in G is equal to the SS-vertex-connectivity degree of a and b in G.*

Proof. To the graph G let us assign an undirected non-polar graph $\tilde{G}$ so that the vertex set of $\tilde{G}$ is the vertex set of G and the edges of $\tilde{G}$ are all SS-edges of G. Then any SS-path in G is a path in $\tilde{G}$ and vice versa. The maximal number of vertex-disjoint (up to a and b) SS-paths connecting a and b in G is therefore equal to the maximal number of vertex-disjoint (up to a and b) paths connecting a and b in $\tilde{G}$. Using Menger's theorem, we prove that this is equal to the vertex connectivity degree of a and b in $\tilde{G}$, which (as is evident from the construction of G) is equal to the SS-vertex-connectivity degree of a and b in G.

Theorem 4. *Let a, b be two vertices of a polarized graph G. Then the maximal number of edge-disjoint SS-paths connecting a and b in G is equal to the SS-edge-connectivity degree of a and b in G.*

Proof. We use again the graph $\tilde{G}$ from the proof of Theorem 3. The maximal number of edge-disjoint SS-paths connecting a and b in G is the equal to the maximal number of edge-disjoint paths connecting a and b in $\tilde{G}$. Using an analogon of Menger' number of edge-disjoint paths connecting a and b in $\tilde{G}$. Using an analogon of Menger's theorem due to A. KOTZIG [4], we prove that this is equal to the edge-connectivity degree of a and b in $\tilde{G}$, which is (as is evident from the construction of G) equal to the SS-edge-connectivity degree of a and b in G.

A quite analogous theorem can be proved for the NN-paths.

As we have seen, the analoga of Menger's theorem for polarized graphs can be easily deduced from the well-known results on non-polar graphs (directed or un-undirected). In the case of polar graphs the situation is more difficult.

Let us have two heteropolar paths P_1, P_2 connecting the same pair of vertices a, b of a polar graph G. They are called quasi-vertex-disjoint, if and only if, going along both of them from a to b and incoming into an arbitrary common vertex of P_1 and P_2 different from a and b, we income into distinct poles of this vertex. An example of such paths is in Fig. 4. P_1 and P_2 are called quasi-edge-disjoint, if and only if, going along both of them from a to b, no common edge of P_1 and P_2 is traversed in the same direction.

Remark. If P_1 and P_2 are quasi-vertex-disjoint, they are evidently also quasi-edge-disjoint.

Theorem 4. *Let a, b be two vertices of a polar graph G. Let the maximal number of pairwise quasi-vertex-disjoint heteropolar paths connecting a and b in G be r, let the heteropolar vertex-connectivity degree of a and b in G be s. Then*

$$s \leqq r \leqq 2s$$

and for any r, s satisfying this inequality, a polar graph G and vertices a, b in it can be found so that the assumption of the theorem is true.

Remark. We do not take poles of a and b into account.

Proof. Let G' be a polarized graph obtained from G by the following procedure. If the vertices of G are $v_1, \ldots, v_n$, then the vertices of G' are $v'_1, \ldots, v'_n, v''_1, \ldots, v''_n$. If we have two vertices v_i, v_j of G, let the poles of v_i be $p_i^{(1)}, p_i^{(2)}$, let the poles of v_j be $p_j^{(1)}, p_j^{(2)}$. If $p_i^{(1)}$ is joined with $p_j^{(1)}$ by an edge, then the northern pole of v'_i is joined with the southern pole of v'_j and the southern pole of v''_i is joined with the northern pole of v''_j. If $p_i^{(1)}$ is joined with $p_j^{(2)}$ by an edge in G, then the northern pole of v'_i is joined with the southern pole of v''_j and the southern pole of v''_i is joined with the northern pole of v'_j. If $p_i^{(2)}$ is joined with $p_j^{(1)}$ by an edge in G, then the southern pole of v'_i is joined with the northern pole of v''_j and the northern pole of v''_i is joined with the southern pole of v'_j. If $p_i^{(2)}$ is joined with $p_j^{(2)}$ by an edge in G, then the southern pole of v'_i is joined with the northern pole of v'_j and the northern pole of v''_i is joined with the southern pole of v''_j in G'. Now to any heteropolar path P connecting a and b in G, a heteropolar NS-path P' in G' corresponds; if P contains an edge v_iv_j and going from a to b along P we meet v_i before v_j, then P' contains that one from the two corresponding edges in G' which joins the northern pole of v'_i or v''_i with the southern pole of v'_j or v''_j; this edge is determined uniquely for any i and j, therefore the assigning of paths P' in G' to paths P in G is uniquely determined. Now let us have two heteropolar paths P_1, P_2 in G connecting a and b. Let P_1, P_2 be quasi-vertex-disjoint. If they are vertex-disjoint, then evidently the corresponding paths P'_1, P'_2 in G' will be vertex-disjoint. Thus let P_1, P_2 have a common vertex v_i. Let the vertex of P_1 (or of P_2) immediately preceding v_i when going along this path from a to b v_j (or v_k, respectively). Let the vertex of P_1 (or of P_2) immediately following v_i be v_l (or v_m, respectively). Let the poles of v_i be $p_i^{(1)}$, $p_i^{(2)}$; without loss of generality let the edge v_jv_i of P_1 be incident with $p_i^{(1)}$. Then v_iv_l must be incident with $p_i^{(2)}$, because P_1 is heteropolar. As P_1, P_2 are quasi-vertex-disjoint, v_kv_i must be incident with $p_i^{(2)}$ and v_iv_m with $p_i^{(1)}$. Then P'_1 contains the edge joining the northern pole of v'_j or v''_j with the southern pole of v'_i as well as the edge joining the northern pole of v'_i with the southern pole of v'_l or v''_l. The path P'_2 contains the edge joining the northern pole of v'_k or v''_k with the southern pole of v''_i as well as the edge joining the northern pole of v''_i with the southern pole of v'_m or v''_m. Therefore the common vertex v_i of P_1 and P_2 is assigned different vertices v'_i, v''_i in the paths P'_1 and P'_2. As any v_i is assigned either v'_i, or v''_i (exclusive "or"), we see that P'_1 and P'_2 are vertex disjoint (up to a and b). If P_1 and P_2 are not quasi-vertex-disjoint, we can prove in an analogous way that P'_1 and P'_2 are not vertex-disjoint (their common vertex "traversed in the same direction" is assigned the same vertex in G' in both P'_1, P'_2). Therefore the systems of pairwise quasi-vertex-disjoint heteropolar paths connecting a and b in G are assigned systems of pairwise vertex-disjoint (up to a and b) heteropolar NS-paths in G' connecting the pair a', a'' with the pair b', b'' (these vertices are those corresponding to a and b in G'). Now we identify the two vertices a', a'' corresponding to a in G' so that the northern poles of both of them are identified and so are their southern poles; the resulting vertex will be denoted also by a. The same will be done for b. The maximal number of pairwise quasi-vertex-disjoint heteropolar paths connecting a and b in G is equal to the maximal number of pairwise vertex-disjoint (up to a and b) heteropolar NS-paths connecting a and b in G'. According to Theorem 1 this number is equal to the minimal number of vertices which must be deleted from G' in order that there might exist no heteropolar NS-path connecting a and b in the resulting graph. Let such a separating set be S'.

The minimal separating set S (with minimal possible cardinality) between a and b in G must satisfy $|S| \leqq |S'| \leqq 2|S|$. The inequality $|S'| < |S|$ cannot hold, because the set of vertices to which the poles of S' correspond is evidently a separating set in G of the cardinality $|S'|$. Also $2|S| < |S'|$ cannot hold, because the set of all poles of S is evidently a separating set between a and b in G of the cardinality $2|S|$. But now $|S| = s$ and, as mentioned above, $|S'| = r$, q.e.d. Fig. 5 shows the case when $s = 1$, $r = 1$, Fig. 6 shows the case $s = 1$, $r = 2$. If any r, s are given so that $s \leqq r \leqq$ $\leqq 2s$, we can take $2s - r$ copies of the graph from Fig. 5 and $r - s$ copies of the graph from Fig. 6 and identify the vertices denoted by a in all of them and the vertices denoted by b in all of them. We obtain a graph in which the maximal number of pairwise quasi-vertex-disjoint heteropolar paths connecting a and b is r while the heteropolar vertex-connectivity degree of a and b is s.

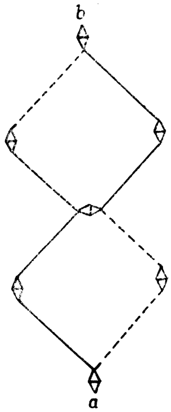

Fig. 4.

Fig. 5.

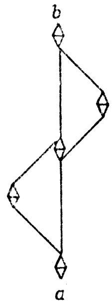

Fig. 6.

Fig. 7.

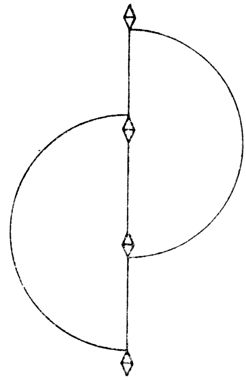

Fig. 8.

Theorem 5. *Let a, b be two vertices of a polar graph G. Let the maximal number of pairwise quasi-edge-disjoint heteropolar paths connecting a and b in G be ϱ, let the heteropolar edge-connectivity degree of a and b in G be σ. Then*

$$\sigma \leqq \varrho \leqq 2\sigma$$

and for any ϱ and σ satisfying this inequality such a graph and vertices a, b exist.

Proof. We shall use the graph G' from the proof of Theorem 4. Let P_1, P_2 be two heteropolar paths connecting a and b in G which are quasi-edge-disjoint. Let v_iv_j be their common edge. If going along P_1 from a to b we come to v_i before v_j, then going along P_2 from a to b we must come first to v_j. Let (without loss of generality) this edge join the pole $p_i^{(1)}$ of v_i with the pole $p_j^{(1)}$ of v_j. Then this edge is assigned in P'_1 the edge joining the northern pole of v'_i with the southern pole of v'_j and in P'_2 the edge joining the northern pole of v'_j with the southern pole of v'_i, i.e., distinct edges corresponds to this edge in P'_1 and P'_2. Thus, analogously to the proof of Theorem 4, we can prove that P'_1 and P'_2 are edge-disjoint. If P_1 and P_2 are not quasi-edge-disjoint, we can easily prove that P'_1 and P'_2 are not edge-disjoint (the common edge of P_1 and P_2 traversed in the same direction when going along both from a to b is assigned the same edge in both P'_1 and P'_2). Further we can proceed analogously as in the proof of Theorem 4; instead of vertices we consider edges. The figures corresponding to Figs. 5 and 6 in this case are Figs. 7 and 8.

Analogous theorems for vertex-disjoint or edge-disjoint (without "quasi-") paths do not hold, as counterexamples in Figs. 9 and 10 show. In the graph in Fig. 9 there exist no two vertex-disjoint (up to a and b) heteropolar paths connecting a and b, but after deleting an arbitrary vertex (distinct from a and b) the vertices a and b remain heteropolarly connected. An edge analogon of this case is in Fig. 10.

In all figures, vertices of polar graphs are drawn as magnetic needles; the poles of such a needle denote the poles of a vertex. The vertices of polarized graphs are drawn also as magnetic needles and the northern poles of vertices are black.

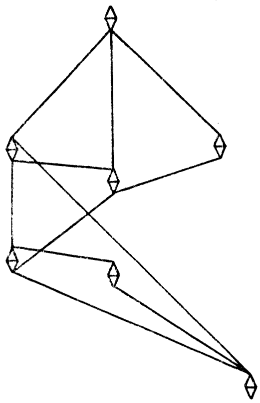

Fig. 9.

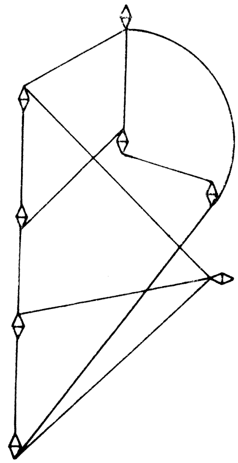

Fig. 10.

Theorem 6. *Let a, b be two vertices of a polar graph G. Let the maximal number of pairwise pole-disjoint homopolar paths connecting a and b in G be r', let the homopolar vertex-connectivity degree of a and b in G be s'. Then*

$$s' \leqq r' \leqq 2s'$$

and for any r' and s' satisfying this inequality, such a graph and vertices a, b exist.

Remark. Two paths connecting a and b are called pole-disjoint, if and only if they have no common pole except for poles of a and b.

Proof. We consider the so-called pole graph of G and denote it by $P(G)$. The vertices of $P(G)$ are poles of G, the edges of $P(G)$ are edges of G, the incidence is preserved. (The graph $P(G)$ is non-polar.) In $P(G)$ we identify the two vertices corresponding to the poles of a denoting the resulting vertex again by a; we do the same for b. Then there is a one-to-one correspondence between homopolar paths in G joining a and b and paths in $P(G)$ joining a and b so that the corresponding paths have the same edges. To two vertex-disjoint paths in $P(G)$ two pole-disjoint paths in G correspond and vice versa. In $P(G)$ we can use Menger's theorem. Therefore the maximal number of pairwise pole-disjoint homopolar paths connecting a and b in G is equal to the minimal possible cardinality of a separating set between a and b

in $P(G)$. Let the separating set of this minimal cardinality be S'. If S is the separating set between a and b in G of the minimal cardinality, then analogously as in the proof of Theorem 4 we can prove $|S| \leqq |S'| \leqq 2|S|$, from which the assertion follows. The example of a graph with $r' = 1$, $s' = 1$ is in Fig. 11, with $r' = 1$, $s' = 2$ in Fig. 12. We can combine these graphs as the graphs in Figs. 5 and 6 and so we can prove the second part of the theorem analogously to Theorem 4.

Fig. 11.

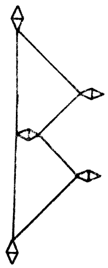

Fig. 12.

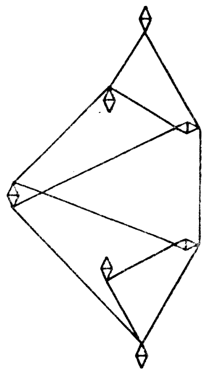

Fig. 13.

Theorem 7. *Let a, b be two vertices of a polar graph G. Then the maximal number of pairwise edge-disjoint homopolar paths connecting a and b in G is equal to the homopolar edge-connectivity degree of a and b in G.*

Proof. If we use again the pole graph $P(G)$, we can prove this theorem by an immediate application of the analogon of Menger's theorem due to A. Kotzig [4]. (The edges of G and $P(G)$ are the same.)

Fig. 13 shows that an analogon of Theorem 6 for vertex-disjoint homopolar paths instead of pole-disjoint ones does not hold. In this graph there exist no two vertex-disjoint homopolar paths connecting a and b, but after deleting an arbitrary vertex the vertices a and b remain homopolarly connected.

The heteropolar connectivity can have an importance in the applications of the graph theory in the railway traffic. If a polar graph represents a railway network, its vertices are railway stations and poles of a vertex denote the sides from which trains come to the station, then two vertices are heteropolarly connected if and only if from the station corresponding to one of them a train can go without changing the direction of its motion into the station corresponding to the other vertex.

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