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EULERIAN POLAR GRAPHS

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The concept of a polar graph was introduced by F. ZÍTEK [1]. Its formal definition was given in [2]. For polar graphs see also [3].

In [2] a heteropolar path was defined. A heteropolar path in which no edges are repeated, is called a heteropolar trail. If $u_1 = u_n$ and the edges e_1, e_{n-1} are incident with different poles of u_1, then a heteropolar trail is called closed; otherwise it is called open (even if $u_1 = u_n$ and e_1, e_{n-1} are incident with the same pole of u_1).

A Eulerian trail in a polar graph G is a heteropolar trail which contains all edges of G. A Eulerian trail can be open or closed (according to the above definition). A Eulerian polar graph is a polar graph in which a closed Eulerian trail exists.

Lemma 1. *Let G be a finite polar graph. At any vertex of G let the poles belonging to it have the same non-zero degree. Then there exist closed heteropolar trails $T_1, \ldots, T_k$ in G such that each edge of G is contained exactly in one of them.*

Proof. We shall do the proof by induction. The least number of edges of a graph G satisfying the condition is equal to the number n of vertices of G (this is the case when the degree of each pole of G is 1). In this case G is a union of vertex-disjoint heteropolar circuits [3] and the assertion is true. Now let the number of edges be m and suppose that for any number of edges less than m the assertion holds. Choose a vertex v of G and form a heteropolar trail starting at v. After incoming into a vertex $w \neq v$ we choose an arbitrary not previously traversed edge incident with the pole of w other than that by which we came. This can be always done, because, when passing through w before, we have traversed the same number of edges at both poles of w. If we come into v by the same pole at which we have started, we proceed analogously. After a finite number of steps we come into v by the pole other than that by which we have started and thus we obtain a closed heterepolar trail T_1. After deleting all edges of T_1 from G and all resulting isolated vertices we obtain either a polar graph G' in which at any vertex the poles belonging to it have the same non-zero degree or an empty graph. If G' is empty, the proof is completed. If G' is nonempty, then it is a graph satisfying the assumptions of the lemma and having less edges than m. According to the induction assumption there exist closed heteropolar trails $T_2, \ldots, T_m$ in G' such that each edge of G' is contained exactly in one of them. Thus $T_1, T_2, \ldots, T_m$ satisfy the condition of the lemma for G and the assertion holds.

A polar graph G can be viewed as non-polar; we consider only vertices and edges of G (not poles) and the incidence only between vertices and edges. Therefore we say that a polar graph G is connected, if and only if it is connected when regarded as non-polar.

Theorem 1. *Let G be a finite connected polar graph. At any vertex of G let the poles belonging to it have the same degree. Then G is Eulerian.*

Proof. As G is connected, the degrees of all poles of G are non-zero and the assumption of Lemma 1 is satisfied. Therefore there exist trails $T_1, \ldots, T_k$ described in Lemma 1. We use the induction according to k. If $k = 1$, the proof is completed. Let $k = m > 1$ and assume that for $k = m - 1$ the assertion holds. Take the trail T_1. There exists at least one trail T_j $(2 \leqq j \leqq m)$ which has a common vertex v with T_1; otherwise the vertices and edges of T_1 would form a connected component of G and G would not be connected. Without loss of generality let $j = 2$. Now we traverse the trail T_1 starting and finishing at v and the we traverse T_2 again starting and finishing at v; we go out from v by the pole other than that by which we came when finishing the traversing of T_1. We have constructed a new closed heteropolar trail T_1'. Now the rails $T_1', T_3, \ldots, T_m$ are $m - 1$ closed heteropolar trails in G such that each edge of G is contained exactly in one of them and according to the induction assumption the assertion holds.

For a vertex v of a polar graph G we define $\delta(v)$ as the absolute value of the difference of degrees of poles of v. Further, let

$$\Delta(G) = \sum_{v \in V} \delta(v) .$$

We see that $\Delta(G) = 0$ if and only if G satisfies the condition of Theorem 1, i.e. if it is Eulerian. Further, we easily see that the converse of Theorem 1 holds. We have a corollary.

Corollary 1. *A finite polar graph G is Eulerian, if and only if it is connected and $\Delta(G) = 0$.*

We shall prove a lemma on the number $\Delta(G)$.

Lemma 2. *For any finite polar graph G the number $\Delta(G)$ is even.*

Proof. For any vertex v of G let $\sigma(v)$ be the sum of degrees of poles of v. Let

$$\Sigma(G) = \sum_{v \in V} \sigma(v) .$$

The number $\Sigma(G)$ is even, because the number of edges of G is evidently equal to $\frac{1}{2}\Sigma(G)$ and this number must be an integer. Now let $v \in V$, $\varkappa(v) = \{p_1, p_2\}$, let ϱ_1, ϱ_2 be the degrees of p_1, p_2 respectively and, without loss of generality, let $\varrho_1 \geqq \varrho_2$. We have

$$\sigma(v) = \varrho_1 + \varrho_2\,, \quad \delta(v) = \varrho_1 - \varrho_2\,.$$

Therefore

$$\delta(v) = \sigma(v) - 2\varrho_2$$

and, as ϱ_2 is an integer, $\delta(v)$ has the same parity as $\sigma(v)$ for each v and therefore also $\Delta(G)$ and $\Sigma(G)$ have the same parity. As $\Sigma(G)$ is even, $\Delta(G)$ is also even.

Now we shall prove another theorem.

Theorem 2. *A connected polar graph G can be covered by k and not less pairwise edge-disjoint open heteropolar trails ($k > 0$), if and only if $\Delta(G) = 2k$.*

Proof. Let $\Delta(G) = 2k > 0$. Let v be a vertex of G with $\delta(v) > 0$. To G we add $[\delta(v)/2]$ new vertices and $2[\delta(v)/2]$ edges joining the pole of v with the smaller degree with both poles of each of the new vertices. We do this for any vertex v of G with $\delta(v) > 0$ and denote the resulting graph by G'. The number $2[\delta(v)/2]$ is equal to $\delta(v)$ for $\delta(v)$ even and $\delta(v) - 1$ for $\delta(v)$ odd. Therefore if $\delta(v)$ in G is even, then in G' it is 0, if $\delta(v)$ in G is odd, in G' it is 1. As $\Delta(G')$ is even, the number of vertices v of G' with $\delta(v) = 1$ is even. We divide the set of these vertices into disjoint pairs and for any of these pairs we add a new vertex and join the pole with the smaller degree of each vertex of such a pair by an edge with this new vertex; these edges are incident with different poles of this new vertex. The resulting graph will be denoted by G''. We have $\Delta(G'') = 0$ and therefore there exists a closed Eulerian trail T in G''. The graph G'' was obtained from G by adding k pairwise edge-disjoint heteropolar trails of the length 2 such that the union of no two of them is a heteropolar trail; this is easy to see from the construction. Therefore, by deleting the edges and the inner vertices of these trails from T we obtain k open heteropolar trails such that any of the remaining edges is contained exactly in one of them. But these remaining edges are exactly all edges of G and the assertion is true. Now assume that there exist $l < k$ such trails in G, let these trails be $T_1, \ldots, T_l$. Let v_j, w_j be respectively the initial and the terminal vertices of T_j for $j = 1, \ldots, l$. We add l new vertices $x_1, \ldots, x_l$ and join one of the poles of x_j with the pole of w_j with which the terminal edge of T_j is not incident, while the other pole of x_j is joined with the pole of v_{j+1} with which the initial edge of T_{j+1} is not incident; here $j + 1$ is taken modulo l. The graph thus obtained will be denoted by G'''. In G''' there exists a closed Eulerian trail; we obtain it traversing T_1, then going through x_1 to v_2, traversing T_2, going through x_2 to v_3, etc. and finally traversing T_l and going through x_l to v_1. This means that $\Delta(G''') = 0$. But the number of edges of G''' is equal to the number of edges of G plus $2l$, therefore $\Delta(G''') \geqq \Delta(G) - 2l$, which implies $\Delta(G) \leqq 2l$, which is a contradiction with $\Delta(G) = 2k$, $l < k$. Therefore if $\Delta(G) = 2k > 0$, there exist k and not less trails with the required property. Now if there exist k and not less such trails, we can prove $\Delta(G) \leqq \leqq 2k$ by the above procedure (where l is replaced by k). If $\Delta(G) = 2l < 2k$, there would exist l such trails provided $l > 0$, or a closed Eulerian trail if $l = 0$, which would be a contradiction.

Corollary 2. *An open Eulerian trail in a finite connected polar graph G exists if and only if $\Delta(G) = 0$ or $\Delta(G) = 2$. In the former case this trail is obtained from a closed Eulerian trail by deleting one edge. In the latter case, if there are two vertices v for which $\delta(v) = 1$, then these vertices are respectively the initial and the terminal ones of this trail; if there is one vertex v for which $\delta(v) = 2$, this vertex is both the initial and the terminal one of this trail and both the initial and the terminal edges of the trail are incident with the same pole of this vertex.*

References

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