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SELF-DERIVED POLAR GRAPHS

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In this paper we shall study polar and polarized graphs. These concepts were introduced by F. ZÍTEK [6] and their definitions are given in [3], [4] and [5].

For polar graphs we can define the derived graph ∂G analogously to the interchange graph for undirected graphs [1] and the derived graph for directed graphs [2]. The notation ∂G is taken from [2].

Let G be a polar graph. The derived graph of G is such a graph ∂G that there exists a bijective mapping φ of the edge set of G onto the vertex set of ∂G satisfying the following conditions:

(1) For any two edges e_1, e_2 of G the vertices $\varphi(e_1), \varphi(e_2)$ of ∂G are joined by an edge, if and only if e_1, e_2 are incident in G with different poles of the same vertex.

(2) If x, y, z are vertices of ∂G, then y and z are joined by edges with the same pole of x, if and only if $\varphi^{-1}(y), \varphi^{-1}(z)$ are incident with the same pole of a vertex of G and $\varphi^{-1}(x)$ is joined with the other pole of this vertex.

A polar graph G is called self-derived, if $\partial G \cong G$. In this case we can regard φ as a mapping of the edge set of G onto the vertex set of G.

For some purposes we may view a polar graph as non-polar, i.e. consider the incidence of vertices and edges without speaking about poles. Then we can speak about the cyclomatic number, connectivity etc. (Analogously as when we view a directed graph as undirected.)

A heteropolar edge of a polarized graph is an edge joining the southern pole of one vertex with the northern pole of another. A heteropolar path in a polar or polarized graph is a sequence $u_1, h_1, u_2, h_2, \ldots, u_{n-1}, h_{n-1}, u_n$, where $u_1, \ldots, u_n$ are vertices, $h_1, \ldots, h_{n-1}$ are edges, h_i joins u_i and u_{i+1} for $i = 1, \ldots, n-1$ and the edges h_i, h_{i+1} are incident with different poles of u_{i+1} for $i = 1, \ldots, n-2$.

Now we can prove a theorem.

Theorem. *A finite connected polar graph G is a self-derived polar graph, if and only if the following conditions are satisfied:*

(α) *G (viewed as non-polar) contains exactly one circuit.*

(β) *G can be polarized so that each edge of G is heteropolar and the northern pole of each vertex is incident exactly with one edge.*

Remark. To polarize a polar graph means to declare for every vertex of the graph one of the poles as northern, the other as southern [1].

Proof. Necessity. As $\partial G \cong G$, we may regard φ as a mapping of the edge set of G onto the vertex set of G. This mapping is bijective, therefore the number of edges of G must be equal to the number of vertices of G. The cyclomatic number of G is equal to 1 and G contains exactly one circuit. Thus the necessity of (α) is proved. For proving the necessity of (β), we first prove that if at a vertex u of G one pole is isolated, then the other is incident exactly with one edge. Both poles of a vertex of G cannot be isolated, because then u would be an isolated vertex and G would be disconnected. Let $\{u_1, \ldots, u_k\}$ be the set of all vertices of G which have isolated poles; let the degree of the non-isolated pole of u_i be r_i for $i = 1, \ldots, k$. Let $h_1^{(i)}, \ldots, h_{r_i}^{(i)}$ be the edges incident with the non-isolated pole of u_i. No edge h can be incident with both u_i, u_j for $i \neq j$, because $\varphi(h)$ would be an isolated vertex. Therefore the number of edges which are incident with vertices which have isolated poles is $\sum_{i=1}^{k} r_i$. For any of such edges e the vertex $\varphi(e)$ has an isolated pole. Therefore G has at least $\sum_{i=1}^{k} r_i$ vertices with isolated poles and thus $\sum_{i=1}^{k} r_i \leqq k$. As any r_i is a positive integer, this is possible only if $r_i = 1$ for $i = 1, \ldots, k$, which was to be proved.

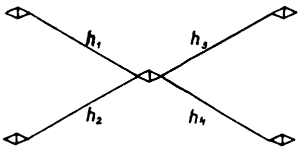


Fig. 1.

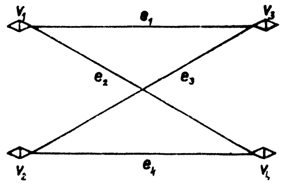
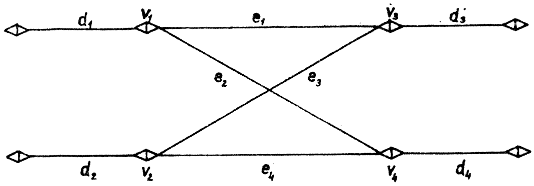


Fig. 2.

Now we shall prove that in G there exists no vertex each of whose poles is incident with more than one edge. Suppose that such a vertex u exists (Fig. 1). One pole of u is incident with edges h_1, h_2, the other with edges h_3, h_4. Denote $v_i = \varphi(h_i)$ for $i = 1, 2, 3, 4$; then G contains a subgraph in Fig. 2; the edges of this subgraph are denoted by e_1, e_2, e_3, e_4. As each of the vertices v_1, v_2, v_3, v_4 has a pole of a degree at least 2, it cannot have an isolated pole. (The degree of a pole is defined analogously to the degree of a vertex in a nonpolar graph.) Therefore let d_i be an edge incident with the pole of v_i which is not incident with any of the edges e_1, e_2, e_3, e_4. In Fig. 3 the situation is drawn, when these edges are pairwise distinct, but this is not necessary. When we consider the images of edges in φ, we obtain a subgraph in Fig. 4 or a subgraph obtained from it by identifying some of the vertices $\varphi(d_i)$. If d_1, d_2, d_3, d_4 are pairwise distinct, we obtain a circuit of the length 8; as the circuit in Fig. 2 had the length 4, we have two distinct circuits in G, which contradicts (α). If some of them coincide, we obtain at least two circuits (circuits of the length 2 are admitted) and this is the same contradiction.



Fig, 3.

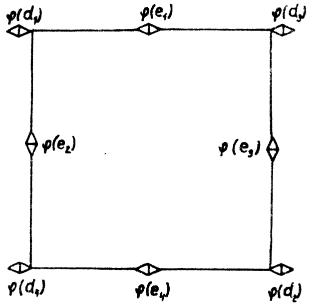


Fig. 4.

Now we shall prove the assertion about the polarisation of vertices. Until now we have proved that each vertex of G has at least one pole of the degree 1. It is easy to prove that (β) is equivalent to the following assertion: If two poles of G are terminal poles of a heteropolar path, then at least one of them has the degree 1.

Indeed, if a polarized graph contains only heteropolar edges, then from the two terminal poles of any heteropolar path one must be northern and the other southern and vice versa.

Therefore it suffices to prove the impossibility of the following cases:

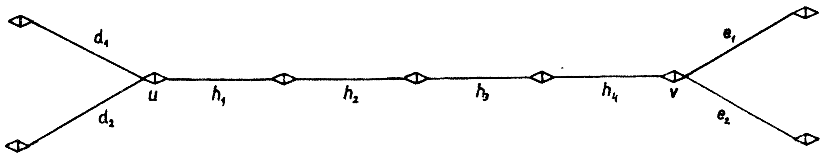


Fig. 5.

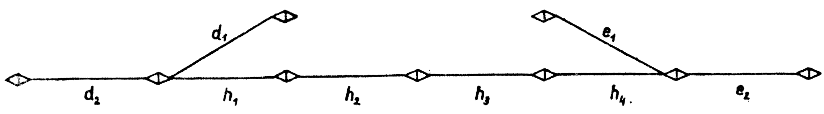


Fig. 6.

(1) The vertices u and v are joined by a heteropolar path and in both u and v the pole not incident with an edge of this path has a degree different from one (Fig. 5).

(2) The vertices u and v are joined by a heteropolar path and at both u and v the pole incident with an edge of this path has a degree different from one (Fig. 6).

First, consider the case (1). Assume that this case occurs and from all pairs of vertices satisfying (1) choose such a pair u, v for which the mentioned path P has the minimal length; let this length be l. Let $h_1, \ldots, h_l$ be the edges of P so that h_i, h_{i+1} have a common terminal vertex for $i = 1, \ldots, l - 1$, h_1 is incident with u, h_l is incident with v. Let d_1, d_2 (or e_1, e_2) be two edges incident with the pole of u (or v respectively) which is not incident with h_1 (or h_l respectively). If $l = 1$, then one pole of $\varphi(h_1)$ is joined by edges with $\varphi(d_1)$, $\varphi(d_2)$ and the other with $\varphi(e_1)$, $\varphi(e_2)$,

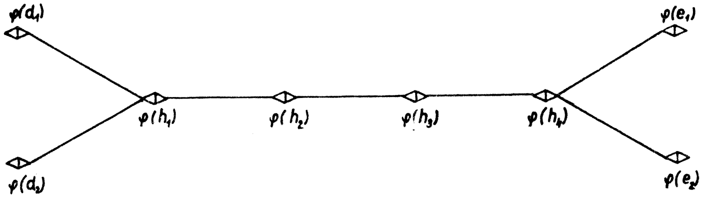


Fig. 7.

which is the case which was proved to be impossible. If $l > 1$, then $\varphi(h_1)$ and $\varphi(h_l)$ are joined by a heteropolar path (Fig. 7) of the length $l - 1$ (its internal vertices are $\varphi(h_2), \ldots, \varphi(h_{l-1})$) and the pole of $\varphi(h_1)$ (or $\varphi(h_l)$) not incident with an edge of this path is joined with $\varphi(d_1)$, $\varphi(d_2)$ (or $\varphi(e_1)$, $\varphi(e_2)$ respectively), which contradicts the minimality of l. Therefore (1) is impossible.

Investigate (2) and consider such a pair u, v for which the mentioned path P is simple (no vertex is repeated in it) and has the maximal length among all such paths; as G is finite, such a pair must exist. Let the length of P be denoted again by l. Also vertices and edges of P are denoted in the same way as in the preceding case. Further, let d_1 (or e_1) be an edge which is incident with the same pole of u (or v respectively) as the edge h_1 (or h_l respectively). Let d_2 (or e_2) be an edge which is incident with the other pole of u (or v respectively) than h_1 (or h_l respectively). Such an edge exists,

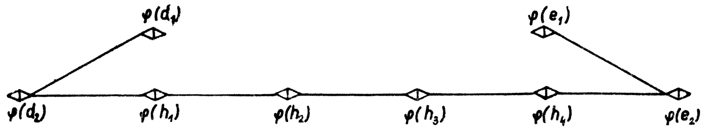


Fig. 8.

because this pole cannot be isolated (the other pole has a degree at least 2). As the vertices of P are pairwise distinct, so are its edges. If it were $d_2 = e_2$, then this edge would form a heteropolar path between u and v for which (1) would occur. Thus $d_2 \neq e_2$. The vertices $\varphi(d_2), \varphi(h_1), \ldots, \varphi(h_l), \varphi(e_2)$ are vertices of a heteropolar path P' of the length $l + 1$ joining $\varphi(d_2)$ and $\varphi(e_2)$; according to the above argument P' is simple (Fig. 8). The edge joining $\varphi(d_2)$ with $\varphi(d_1)$ is incident with the same pole of $\varphi(d_2)$ as the edge joining $\varphi(d_2)$ with $\varphi(h_1)$. The edge joining $\varphi(e_2)$ with $\varphi(e_1)$ is incident with the same pole of $\varphi(e_2)$ as the edge joining $\varphi(e_2)$ with $\varphi(h_l)$. Therefore P' is a simple path of the length $l + 1$, for which (2) occurs, which contradicts the maximality of l. Thus the necessity is proved.

Sufficiency. Assume that we have polarized G according to (β). For each edge h define $\varphi(h)$ as the vertex whose northern pole is incident with h. As any edge is heteropolar, this vertex is uniquely determined. As the northern pole of any vertex is in-

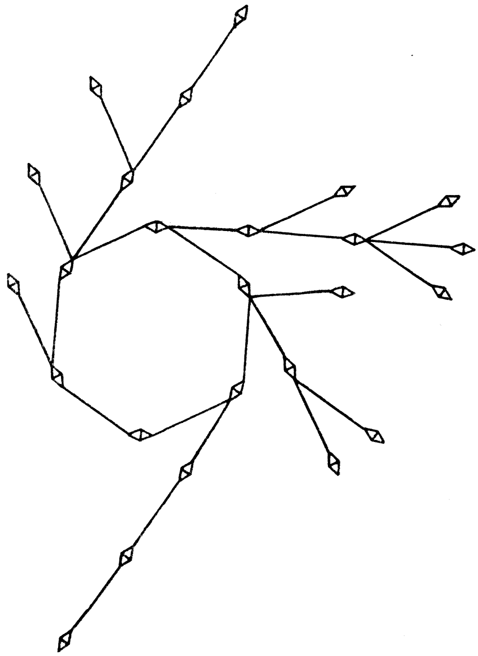


Fig. 9.

cident exactly with one edge, φ is an injection. As G contains exactly one circuit, its cyclomatic number is equal to 1 and its number of edges is equal to its number of vertices; therefore φ is also surjective and thus it is a bijection. Let two edges e_1, e_2 be incident in G with different poles of the same vertex v; without loss of generality suppose that e_1 is incident with the northern pole of v. Then $\varphi(e_1) = v$. The vertex $\varphi(e_2)$ is the terminal vertex w of e_2 different from v. The vertices $\varphi(e_1) = v$, $\varphi(e_2) = w$ are joined by the edge e_2. On the other hand, let two vertices v, w be joined by an edge e; without loss of generality let e be incident with the southern pole of v and with the northern pole of w. Then $\varphi^{-1}(w) = e$ and $\varphi^{-1}(v)$ is the edge e' incident with the northern pole of v. The edges e, e' are incident with different poles of v. We have proved that φ satisfies (1). Now let x, y, z be vertices of G and let y and z be joined by edges h_1, h_2 with the same pole of x. If $y = z$, then also $\varphi^{-1}(y) = \varphi^{-1}(z)$ and $\varphi^{-1}(x)$, $\varphi^{-1}(y)$ must be incident with different poles of a certain vertex according to (1). If $y \neq z$, then the pole of x with which h_1, h_2 are incident must be southern, because it is incident with two distinct edges. As all edges are heteropolar, the edge h_1 (or h_2) is incident with the northern pole of y (or z respectively) and we have $\varphi^{-1}(y) = h_1$, $\varphi^{-1}(z) = h_2$. These two edges are incident with the same pole of x, namely the southern. $\varphi^{-1}(x)$ must be incident with the northern pole of x. Now let the edges h_1, h_2 be incident with one pole of a vertex a and let h_3 be incident with the other pole of a. Then h_1, h_2 are incident with the southern pole of a and $\varphi(h_1)$ (or $\varphi(h_2)$) is the terminal vertex of h_1 (or h_2 respectively) different from a; moreover $\varphi(h_3) = a$. We see that $\varphi(h_1)$, $\varphi(h_2)$ are joined by edges with the same pole of $\varphi(h_3)$. We have proved that φ satisfies (2). Therefore G is self-derived. An example of a self-derived polar graph is in Fig. 9.

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