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GENERATION OF PRERADICALS

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1. INTRODUCTION

This paper can be viewed as a continuation of the authors' previous investigations of preradicals [1]–[6] and is devoted to the methods of constructing preradicals from given fragments which seems to be useful in several directions, see e.g. [7]–[9]. It is shown that choosing arbitrary submodules of some modules gives rise to a preradical. The results obtained are used to investigate torsion submodules and ideals, in particular, conditions under which a given submodule N of a module M can be equal to its torsion part with respect to a preradical r having prescribed properties.

Throughout the paper, R always denotes an associative ring with unit and R-mod is the category of unital left R-modules. If $M \in R$-mod and $A, B \subseteq M$, $X, Y \subseteq R$ are subsets then $(A : B) = \{a \in R \mid aB \subseteq A\}$ and $(X : Y)_r = \{a \in R \mid Ya \subseteq X\}$.

Let $\mathscr{A}$ be a non-empty class of modules. We shall say that s is a function on $\mathscr{A}$ if s assigns to each module $A \in \mathscr{A}$ its submodule $s(A)$ (notice that no connection with morphisms is required). If t is another function on $\mathscr{A}$, we shall write $s \subseteq_{\mathscr{A}} t$ ($s =_{\mathscr{A}} t$) if $s(A) \subseteq t(A)$ ($s(A) = t(A)$) for each $A \in \mathscr{A}$. If s is a subfunctor of the identity functor on $\mathscr{A}$ regarded as a full subcategory of R-mod, s will be called an $\mathscr{A}$-subpreradical. In case $\mathscr{A} = R$-mod we shall omit the index at the inclusion sign and the prefix "$\mathscr{A}$-sub". Further, we shall denote by $\mathscr{I}(\mathscr{P})$ the class of all injective (projective) modules.

Now let us recall some basic facts and definitions concerning preradicals which will be used in the sequel (a systematic treatment of the topic can be found e.g. in [1] and [2]). We shall say that a non-empty class $\mathscr{A}$ of modules is

– hereditary if it is closed under submodules and isomorphic images,
– cohereditary if it is closed under homomorphic images,
– stable if for every $M \in \mathscr{A}$ there is an exact sequence $0 \to M \to Q \to K \to 0$ with $Q \in \mathscr{A} \cap \mathscr{I}$.

For every preradical r, we define $\mathscr{T}_r = \{M \in R\text{-mod} \mid r(M) = M\}$ and $\mathscr{F}_r = \{M \in R\text{-mod} \mid r(M) = 0\}$. Obviously $\mathscr{T}_r$ is a cohereditary class closed under direct sums and $\mathscr{F}_r$ is a hereditary class closed under direct products. A preradical s is said to be

- idempotent if $r(r(M)) = r(M)$ for all $M \in R$-mod,
- a radical if $r(M/r(M)) = 0$ for all $M \in R$-mod,
- hereditary if $r(N) = N \cap r(M)$ for every module M and its submodule N,
- superhereditary if it is hereditary and $\mathscr{T}_r$ is closed under direct products.

A preradical r is hereditary iff it is idempotent and $\mathscr{T}_r$ is hereditary. Further, if r is hereditary then $\mathscr{F}_r$ is stable. Conversely, if $\mathscr{F}_r$ is stable and r is a radical then it is hereditary. Similarly, r is cohereditary iff it is a radical and $\mathscr{F}_r$ is cohereditary.

If r is cohereditary and $I = r(R)$ then $r(M) = IM$ for all $M \in R$-mod, r is idempotent iff $I = I^2$, r is hereditary iff $x \in Ix$ for all $x \in I$ (i.e. R/I is flat as a right module). Conversely, if I is a left ideal and $r(M) = IM$ for all $M \in R$-mod then r is a cohereditary radical.

Now let r be a superhereditary preradical and $I = \bigcap K$, K running over all left ideals with $r(R/K) = R/K$. Then I is a two-sided ideal, $r(R/I) = R/I$, $r(M) = \{m \in M \mid Im = 0\}$ for all $M \in R$-mod and r is a radical iff $I = I^2$. Conversely, let I be a two-sided ideal and $r(M) = \{m \in M \mid Im = 0\}$ for every $M \in R$-mod. Then r is a superhereditary preradical, $r(R) = (0 : I)_r$ and $I = \bigcap K$, $K \subseteq R$ with $r(R/K) = R/K$.

Let r be an arbitrary preradical. For every $M \in R$-mod we define $\bar{r}(M) = \sum N$, $N \subseteq M$ with $r(N) = N$, $\tilde{r}(M) = \bigcap L$, $L \subseteq M$ with $r(M/L) = 0$, $h(r)(M) = M \cap r(E(M))$, where $E(M)$ denotes the injective hull of M, and $ch(r)(M) = r(R)M$. Then $\bar{r}(ch(r))$ is the largest idempotent preradical (cohereditary radical) contained in r and $\tilde{r}(h(r))$ is the least radical (hereditary preradical) containing r (cf. [1] and [2]).

We shall use the notation $M^{(K)}$ for the direct sum of copies of a module M over an index set K. Further, id and zer are preradicals with $\mathrm{id}(M) = M$ and $\mathrm{zer}(M) = 0$ for all $M \in R$-mod. Finally, let us recall that a submodule N of a module M is said to be characteristic if $f(N) \subseteq N$ for every $f \in \mathrm{Hom}(M, M)$.

2. GENERATION OF PRERADICALS

Let $\mathscr{A}$ be a non-empty class of modules and s a function on $\mathscr{A}$. For every $M \in R$-mod we define

$$p^{(\mathscr{A},s)}(M) = \bigcap f^{-1}(s(A)), \quad f \in \mathrm{Hom}(M, A), \quad A \in \mathscr{A},$$

$$p_{(\mathscr{A},s)}(M) = \sum f(s(A)), \quad f \in \mathrm{Hom}(A, M), \quad A \in \mathscr{A}.$$

Proposition 2.1. *$p_{(\mathscr{A},s)}$ and $p^{(\mathscr{A},s)}$ are preradicals with $p^{(\mathscr{A},s)} \subseteq_{\mathscr{A}} s \subseteq_{\mathscr{A}} p_{(\mathscr{A},s)}$.*

Proof. Let $M, N \in R$-mod, $g \in \operatorname{Hom}(M, N)$, $x \in p_{(\mathscr{A},t)}(M)$ and $y \in p^{(\mathscr{A},s)}(M)$. There are $A_i \in \mathscr{A}$, $f_i \in \operatorname{Hom}(A_i, M)$ and $z_i \in s(A_i)$ with $x \in \sum f_i(z_i)$ so that $g(x) = \sum g f_i(z_i) \in p_{(\mathscr{A},s)}(N)$. Further, if $A \in \mathscr{A}$ and $h : N \to A$ are arbitrary then $hg \in \operatorname{Hom}(M, A)$, hence $h(g(y)) \in s(A)$ and $g(y) \in p^{(\mathscr{A},s)}(N)$. The rest is obvious.

Proposition 2.2. *Let $\mathscr{A}$ be a non-empty class of modules and s a function on $\mathscr{A}$. Then the following assertions are equivalent:*

(i) *s is an $\mathscr{A}$-subpreradical,*
(ii) $p^{(\mathscr{A},s)} =_{\mathscr{A}} s$,
(iii) $p_{(\mathscr{A},s)} =_{\mathscr{A}} s$.

Proof. (i) implies (iii). For all $A, B \in \mathscr{A}$ and $f \in \operatorname{Hom}(A, B)$ it is $f(s(A)) \subseteq s(B)$, so $p_{(\mathscr{A},s)}(B) \subseteq s(B)$.

(iii) implies (i). Let $A, B \in \mathscr{A}$ and $f \in \operatorname{Hom}(A, B)$. Then $f(s(A)) \subseteq p_{(\mathscr{A},s)}(B) = s(B)$. The equivalence of (i) and (ii) is proved similarly.

Proposition 2.3. *Let r, s be functions on a non-empty class $\mathscr{A}$ of modules. Then*
(i) *if $r \subseteq_{\mathscr{A}} s$ then $p_{(\mathscr{A},r)} \subseteq p_{(\mathscr{A},s)}$ and $p^{(\mathscr{A},r)} \subseteq p^{(\mathscr{A},s)}$,*
(ii) *if r, s are $\mathscr{A}$-subpreradicals and either $p_{(\mathscr{A},r)} \subseteq_{\mathscr{A}} p_{(\mathscr{A},s)}$ or $p^{(\mathscr{A},r)} \subseteq_{\mathscr{A}} p^{(\mathscr{A},s)}$ then $r \subseteq_{\mathscr{A}} s$.*

Proof. (i) is obvious and (ii) follows from Proposition 2.2.

Proposition 2.4. *Let $\emptyset \neq \mathscr{A} \subseteq \mathscr{B} \subseteq R$-mod and let s be a function on $\mathscr{B}$. Then $p_{(\mathscr{A},s)} \subseteq p_{(\mathscr{B},s)}$ and $p^{(\mathscr{B},s)} \subseteq p^{(\mathscr{A},s)}$.*

Proof. Obvious.

Proposition 2.5. *Let $\emptyset \neq \mathscr{A} \subseteq \mathscr{B} \subseteq R$-mod and let s be a function on $\mathscr{B}$. Then*

(i) *if s is a $\mathscr{B}$-subpreradical then $p_{(\mathscr{A},s)} \subseteq_{\mathscr{B}} s \subseteq_{\mathscr{B}} p^{(\mathscr{A},s)}$,*
(ii) *if either $p_{(\mathscr{B},s)} \subseteq_{\mathscr{A}} s$ or $s \subseteq_{\mathscr{A}} p^{(\mathscr{B},s)}$ then s is an $\mathscr{A}$-subpreradical.*

Proof. (i) Clearly, $p_{(\mathscr{A},s)} \subseteq p_{(\mathscr{B},s)} = s = p^{(\mathscr{B},s)} \subseteq p^{(\mathscr{A},s)}$ by 2.4 and 2.2.

(ii) We have either $p_{(\mathscr{A},s)} \subseteq p_{(\mathscr{B},s)} \subseteq_{\mathscr{A}} s$ or $s \subseteq_{\mathscr{A}} p^{(\mathscr{B},s)} \subseteq p^{(\mathscr{A},s)}$ by 2.4 and 2.1, 2.2 complete the proof.

From now till the end of this section we shall assume that $\emptyset \neq \mathscr{A} \subseteq R$-mod and s is a function on $\mathscr{A}$. Further, we shall denote $t = p_{(\mathscr{A},s)}$, $u = p^{(\mathscr{A},s)}$, $v = p^{(\mathscr{A},t)}$ and $w = p_{(\mathscr{A},u)}$.

Proposition 2.6. (i) $p_{(\mathscr{A},t)} = t \subseteq v$ *and* $w \subseteq u = p^{(\mathscr{A},u)}$,

(ii) $u \subseteq v$ *and* $w \subseteq t$,
(iii) *if s is an $\mathscr{A}$-subpreradical then equalities hold in* (ii),
(iv) $p_{(\mathscr{A},v)} = t$ *and* $p^{(\mathscr{A},w)} = u$.

Proof. (i) By Proposition 2.1, $u \subseteq_{\mathscr{A}} s \subseteq_{\mathscr{A}} t$, hence $p^{(\mathscr{A},u)} \subseteq u$ and $t \subseteq p_{(\mathscr{A},t)}$. The other inclusions hold due to Proposition 2.5. (ii) and (iii) follow from 2.1, 2.3 and 2.5, while (iv) is an immediate consequence of (i) and (iii).

Proposition 2.7. (i) *If $t(s(A)) = s(A)$ for all $A \in \mathscr{A}$ then t is idempotent,*

(ii) *t is idempotent iff $t(t(A)) = t(A)$ for all $A \in \mathscr{A}$,*
(iii) *if $u(A/s(A)) = 0$ for all $A \in \mathscr{A}$ then u is a radical,*
(iv) *u is a radical iff $u(A/u(A)) = 0$ for all $A \in \mathscr{A}$.*

Proof. (i) If $A \in \mathscr{A}$, $M \in R$-mod and $f \in \operatorname{Hom}(A, M)$ are arbitrary then $f(s(A)) = f(t(s(A))) \subseteq t(f(s(A))) \subseteq t(t(M))$ and so $t(M) \subseteq t(t(M))$.

(ii) follows from (i), since t is a preradical and $p_{(\mathscr{A},t)} = t$.

(iii) Let $M \in R$-mod and $x + u(M) \in u(M/u(M))$. For all $A \in \mathscr{A}$ and $f \in \operatorname{Hom}(M, A)$, f induces a homomorphism $g : M/u(M) \to A/s(A)$ and $g(x + u(M)) = 0$, so that $f(x) \in s(A)$. Thus $x \in u(M)$.

(iv) is an immediate consequence of (iii) and Proposition 2.6.

Proposition 2.8. (i) *If all $A \in \mathscr{A}$ are injective with respect to all natural embeddings $u(M) \to M$ then u is idempotent,*

(ii) *if $\mathscr{A} \subseteq \mathscr{I}$ then u is hereditary,*
(iii) *if all $A \in \mathscr{A}$ are projective with respect to all natural projections $M \to M/t(M)$ then t is a radical,*
(iv) *if $\mathscr{A} \subseteq \mathscr{P}$ then t is cohereditary.*

Proof. (i) If $x \in u(M)$, $A \in \mathscr{A}$ and $f \in \operatorname{Hom}(u(M), A)$ then $f(x) = g(x)$ for some $g \in \operatorname{Hom}(M, A)$, so $f(x) \in s(A)$ and $x \in u(u(M))$.

(ii) If $N \subseteq M$, $x \in N \cap u(M)$, $A \in \mathscr{A}$ and $f \in \operatorname{Hom}(N, A)$ then there is $g \in \operatorname{Hom}(M, A)$ with $f(x) = g(x)$ and hence $f(x) \in s(A)$. Thus $x \in u(N)$.

(iii) Let $M \in R$-mod, $A \in \mathscr{A}$ and $f \in \operatorname{Hom}(A, M/t(M))$. There is $g \in \operatorname{Hom}(A, M)$ with $f = pg$, p being the canonical projection. Hence $f(s(A)) = 0$ and $t(M/t(M)) = 0$.

(iv) Let $N \subseteq M$ and $x \in t(M/N)$. Then $x = \sum f_i(y_i) = \sum p(g_i(y_i)) = p(\sum g_i(y_i)) \in p(t(M)) = (t(M) + N)/N$, where p is the natural projection $M \to M/N$ and $f_i \in \operatorname{Hom}(A_i, M/N)$, $y_i \in s(A_i)$, $g_i \in \operatorname{Hom}(A_i, M)$ are suitably chosen.

Corollary 2.9. *Let r be a preradical. Then $h(r) = p^{(\mathscr{I},r)}$ and $ch(r) = p_{(\mathscr{P},r)}$.*

Proof. Let $M \in R$-mod and let $0 \to K \to P \xrightarrow{g} M \to 0$ be a projective presentation of M. Then $ch(r)(M) = g(r(P)) \subseteq a_{(\mathscr{P},r)}(M)$ and $p^{(\mathscr{I},r)}(M) \subseteq M \cap r(E(M)) = h(r)(M)$. On the other hand, $p^{(\mathscr{I},r)}$ is hereditary and $p_{(\mathscr{P},r)}$ is cohereditary by Proposition 2.8, while Proposition 2.5 yields $p_{(\mathscr{P},r)} \subseteq r \subseteq p^{(\mathscr{I},r)}$.

Now we shall introduce some notation. Let $\mathscr{A}$ be a class of modules, $M \in R$-mod and let N be a submodule of M. We shall denote $p_{\mathscr{A}} = p_{(\mathscr{A},\mathrm{id})}$, $p^{\mathscr{A}} = p^{(\mathscr{A},\mathrm{zer})}$, $p_M =$

$= p_{\{M\}}$ and $p^M = p^{\{M\}}$. Further, if $\mathscr{A} = \{M\}$ and $s(M) = N$ then we shall write $t_{(N \subseteq M)} = p_{(\mathscr{A},s)}$ and $t^{(N \subseteq M)} = p^{(\mathscr{A},s)}$.

Proposition 2.10. (i) $p_{(\mathscr{A},s)} \subseteq p_{\{s(A) | A \in \mathscr{A}\}}$ *and* $p^{\{A/s(A) | A \in \mathscr{A}\}} \subseteq p^{(\mathscr{A},s)}$,

(ii) $p^{(\mathscr{A},s)}(M) = M$ *iff* $p_M \subseteq_{\mathscr{A}} s$,

(iii) $p_{(\mathscr{A},s)}(M) = 0$ *iff* $s \subseteq_{\mathscr{A}} p^M$,

(iv) *if* $Ry \in \mathscr{T}_t$ (*where* $t = p_{(\mathscr{A},s)}$) *for all* $A \in \mathscr{A}$ *and* $y \in s(A)$ *then* $\mathscr{F}_t$ *is stable and* t *is a hereditary radical,*

(v) *for every preradical* f, $\bar{r} = p_{\mathscr{T}_r}$ *and* $\tilde{r} = p^{\mathscr{T}_r}$.

Proof. (i) Denote $\mathscr{B} = \{s(A) \mid A \in \mathscr{A}\}$, $\mathscr{C} = \{A/s(A) \mid A \in \mathscr{A}\}$, $r = p_{\mathscr{B}}$ and $q = p^{\mathscr{C}}$. Obviously $q \subseteq_{\mathscr{A}} s \subseteq_{\mathscr{A}} r$ and hence Propositions 2.3 and 2.5 yield $p_{(\mathscr{A},s)} \subseteq p_{(\mathscr{A},r)} \subseteq r$ and $q \subseteq p^{(\mathscr{A},q)} \subseteq p^{(\mathscr{A},s)}$.

(ii) and (iii) follow immediately from the definitions.

(iv) Let $F \in \mathscr{F}_t$, $A \in \mathscr{A}$ and $f \in \operatorname{Hom}(A, E(F))$. If $f(s(A)) \neq 0$ then there are $0 \neq x \in F$ and $y \in s(A)$ with $f(y) = x$. However, $Rx = f(Ry) \subseteq t(Rx) \subseteq t(F) = 0$, a contradiction. Hence $E(F) \in \mathscr{F}_t$ and $\mathscr{F}_t$ is stable. Since $\mathscr{F}_t = \mathscr{F}_{\bar{t}}$ and $\bar{t}$ is a radical, t is hereditary.

(v) is obvious.

Examples 2.11. (i) Let $\mathscr{A} = \{M, N\}$, where $M \cong N$, and $s(M) = M$, $s(N) = 0$ and $r(M) = 0$, $r(N) = N$. Then $p_{(\mathscr{A},s)} = p_{(\mathscr{A},r)} = p_{\mathscr{A}} = p_M = p_N$ and $p^{(\mathscr{A},s)} = p^{(\mathscr{A},r)} = p^{\mathscr{A}} = a^M = p^N$.

(ii) Let P and Q be a generator and a cogenerator of R-mod respectively, $\mathscr{A} = \{P, Q\}$ and $s(P) = P$, $s(Q) = Q$. Then $p^{(\mathscr{A},s)} = \text{zer}$ and $p_{(\mathscr{A},s)} = \text{id}$.

(iii) If Q is the additive group of rationals and Z is the ring of integers then $t_{(Z \subseteq Q)} \neq \text{id} = p_Z$ and $t^{(2Z \subseteq Z)} \neq p^{Z/2Z}$.

(iv) Let $M \cong N$ and $0 \neq T \subseteq M$ with $\operatorname{Hom}(M, T) = 0$. Define $\mathscr{A} = \{M, N\}$, $s(M) = T$ and $s(N) = N$. Then $p_{(\mathscr{A},s)} = p_N$ is idempotent and $p_{(\mathscr{A},s)}(s(M)) = 0 \neq s(M)$.

(v) Let $M \cong N$ and $T \subsetneqq M$ with $\operatorname{Hom}(M/T, M) = 0$. If $\mathscr{A} = \{M, N\}$ and $s(M) = T$, $s(N) = 0$, then $p^{(\mathscr{A},s)} = p^N$ is a radical and $p^{(\mathscr{A},s)}(M/s(M)) = M/T \neq 0$.

3. TORSION SUBMODULES AND IDEALS

Throughout this section we shall always assume without mentioning it explicitly that $M \in R$-mod, N is a submodule of M and I is a left ideal of R.

Proposition 3.1. *The following assertions are equivalent:*

(i) *N is a characteristic submodule of M,*

(ii) *there is a preradical r with $r(M) = N$.*

In this case, $t^{(N\subseteq M)}$ is the largest and $t_{(N\subseteq M)}$ the least among the preradicals s with $s(M) = N$.

Proof follows immediately from Propositions 2.1, 2.2 and 2.5.

We shall say that N is

— *i-characteristic* if $r(M) = N$ for an idempotent preradical r,
— *r-characteristic* if $r(M) = N$ for a radical r,
— *h-characteristic* if $r(M) = N$ for a hereditary preradical r,
— *ch-characteristic* if $r(M) = N$ for a cohereditary radical r,
— *ir-characteristic* if $r(M) = N$ for an idempotent radical r,
— *hr-characteristic* if $r(M) = N$ for a hereditary radical r.

Proposition 3.2. *The following assertions are equivalent:*

(i) *N is i-characteristic,*
(ii) *there is a preradical s with $s(M) = s(N) = N$,*
(iii) $\operatorname{Im} f \subseteq N$ *for all* $f \in \operatorname{Hom}(N, M)$,
(iv) $p_N(M) = N$,
(v) $p_N \subseteq \bar{t}^{(N\subseteq M)}$,
(vi) $\bar{t}^{(N\subseteq M)}(M) = N$.

In this case, $\bar{t}^{(N\subseteq M)}$ is the largest and p_N the least among the idempotent preradicals s with $s(M) = N$.

Proof. Obviously (i) implies (ii) and (vi) implies (i).

(ii) implies (iii). If $f \in \operatorname{Hom}(N, M)$ then $f(N) = f(s(N)) \subseteq s(M) = N$.

(iii) implies (iv). We have $N = p_N(N) \subseteq p_N(M) = \sum_{f \in \operatorname{Hom}(N,M)} f(N) \subseteq N$.

(iv) implies (v). If $p_N(M) = N$ then Proposition 2.5 and the idempotence of p_N yield $p_N \subseteq \bar{t}^{(N\subseteq M)}$.

(v) implies (vi). We have $\bar{t}^{(N\subseteq M)}(M) \subseteq N \subseteq p_N(M) \subseteq \bar{t}^{(N\subseteq M)}(M)$.

Now, if s is an idempotent preradical with $s(M) = N$ then $s(N) = N$. Hence Proposition 2.5 and the idempotence of s implies $p_N \subseteq s \subseteq \bar{t}^{(N\subseteq M)}$.

Proposition 3.3. *The following assertions are equivalent:*

(i) *N is r-characteristic,*
(ii) *there is a preradical s with $s(M) = N$ and $s(M/N) = 0$,*
(iii) $N \subseteq \operatorname{Ker} f$ *for all* $f \in \operatorname{Hom}(M, M/N)$,
(iv) $p^{M/N}(M) = N$,
(v) $\tilde{t}_{(N\subseteq M)} \subseteq p^{M/N}$,
(vi) $\tilde{t}_{(N\subseteq M)}(M) = N$.

In this case, $p^{M/N}$ is the largest and $\tilde{t}_{(N\subseteq M)}$ the least among the radicals s with $s(M) = N$.

Proof. Obviously (i) implies (ii) and (vi) implies (i).

(ii) implies (iii). If $f \in \mathrm{Hom}(M, M/N)$ then $f(N) = f(s(M)) \subseteq s(M/N) = 0$.

(iii) implies (iv). Obviously, $p^{M/N}(M) \subseteq N$ and $N \subseteq p^{M/N}(M)$ by (iii).

(iv) implies (v). We have $t_{(N \subseteq M)} \subseteq p^{M/N}$ by (iv) and Proposition 2.5 and consequently $\bar{t}_{(N \subseteq M)} \subseteq p^{M/N}$, $p^{M/N}$ being a radical.

(v) implis (vi). We have $\bar{t}_{(N \subseteq M)}(M) \subseteq p^{M/N}(M) \subseteq N \subseteq t_{(N \subseteq M)}(M) \subseteq \bar{t}_{(N \subseteq M)}(M)$. Finally, if s is a radical with $s(M) = N$ then $s(M/N) = 0$ and a repeated application of Proposition 2.5 completes the proof.

Proposition 3.4. *The following assertions are equivalent:*

(i) *N is h-characteristic,*

(ii) *there is a preradical s such that $s(M) = N$ and $s(A) = A$ whenever $A \subseteq N^{(K)}$ is a submodule and K is a finite set,*

(iii) $\mathrm{Im} f \subseteq N$ *whenever* $f \in \mathrm{Hom}(Rn, M)$, $n \in N^{(K)}$ *and K is a finite set,*

(iv) *for every finite subset $T \subseteq N$ and $m \in M \setminus N$ there exists $a \in R$ with $aT = 0$ and $am \neq 0$,*

(v) $h(p_N)(M) = N$,

(vi) $h(t_{(N \subseteq M)})(M) = N$.

In this case, $h(p_N) = h(t_{(N \subseteq M)})$ is the least among the hereditary preradicals s with $s(M) = N$.

Proof. Obviously (i) implies (ii), (ii) implies (iii) and (vi) implies (i). (iii) implies (iv). If there were $T = \{t_1, \ldots, t_k\} \subseteq N$ and $m \in M \setminus N$ such that $am = 0$ whenever $a \in (0 : T)$, then we could define a homomorphism $g : Rn \to M$ by sending $n = (t_1, \ldots, t_k) \in N^k$ onto M, a contradiction with (iii).

(iv) implies (v). Obviously $N = p_N(N) \subseteq p_N(M) \subseteq h(p_N)(M)$. Conversely, let $m \in h(p_N)(M)$. Since $h(p_N)(M) = M \cap p_N(E(M))$, there are $f_i \in \mathrm{Hom}(N, E(M))$ and $t_i \in N$, $i = 1, 2, \ldots, k$, with $m = \sum f_i(t_i)$. Hence (iv) implies $m \in N$.

(v) implies (vi). Using Propositions 2.1 and 2.10 (i) we get $N \subseteq t_{(N \subseteq M)}(M) \subseteq h(t_{(N \subseteq M)}(M) \subseteq h(p_N)(M) = N$.

Finally, if s is a hereditary preradical with $s(M) = N$ then $t_{(N \subseteq M)} \subseteq s$ by Proposition 2.5 and $p_N \subseteq s$ by Proposition 3.2. Therefore $h(t_{(N \subseteq M)}) \subseteq s$ and $h(p_N) \subseteq s$ and it remains to use (v) and (vi).

Proposition 3.5. *Denote* $K = \bigcap_{m \in M} (N : m)$. *The the following assertions are equivalent:*

(i) *N is ch-characteristic,*

(ii) *there is a two-sided ideal L with $LM = N$,*

(iii) *there is a left ideal L with $LM = N$,*

(iv) *there is a preradical s with $s(M) = N$ and $s(M/KM) = 0$,*

(v) $N \subseteq \mathrm{Ker} f$ *for all* $f \in \mathrm{Hom}(M, M/KM)$,

(vi) $ch(p^{M/N})(M) = N$,
(vii) $ch(t^{(N \subseteq M)})(M) = N$.
In this case, $ch(p^{M/N}) = ch(t^{(N \subseteq M)})$ *is the largest among the cohereditary radicals* s *with* $s(M) = N$.

Proof. The implications (i) implies (ii), (ii) implies (iii), (iv) implies (v), and (vii) implies (i) are obvious.

(iii) implies (iv). We have $N = LM = LRM \subseteq KM \subseteq N$.

(v) implies (vi). Obviously, $p^{M/N}(R) = K$ and $N = KM$ by (v).

(vi) implies (vii). We have $N = ch(p^{M/N})(M) \subseteq ch(t^{(N \subseteq M)})(M) \subseteq t^{(N \subseteq M)}(M) \subseteq$ $\subseteq N$ by Propositions 2.1 and 2,10(i).

Finally, if s is a cohereditary radical with $s(M) = N$ then an application of Propositions 2.5 and 3.3 yields $s \subseteq ch(p^{M/N}) = ch(t^{(N \subseteq M)})$.

Proposition 3.6. *The following assertions are equivalent*:

(i) *N is ir-characteristic*,
(ii) *there is a preradical s with* $s(M) = s(N) = N$ *and* $s(M/N) = 0$,
(iii) $\operatorname{Hom}(N, M/N) = 0$,
(iv) $\tilde{p}_N(M) = N$,
(v) $\bar{p}^{M/N}(M) = N$.

In this case, $\tilde{p}_N$ *is the least and* $\bar{p}^{M/N}$ *the largest among the idempotent radicals* s *with* $s(M) = N$.

Proof. Obviously (i) implies (ii) and (ii) implies (iii).

(iii) implies (iv). If (iii) holds then obviously $p_N(M/N) = 0$, so $\tilde{p}_N(M/N) = 0$ and $N = p_N(N) = \tilde{p}_N(N) \subseteq \tilde{p}_N(M) \subseteq N$.

(iv) implies (i). Since p_N is idempotent, $\tilde{p}_N$ is an idempotent radical. The implications (iii) implies (v) and (v) implies (i) can be proved similarly and the last assertion follows from Propositions 3.2 and 3.3.

Corollary 3.7. *N is ir-characteristic provided at least one of the following conditions holds*:

(i) *N is a characteristic direct summand*,
(ii) *N is i-characteristic and* $\operatorname{Ext}(N, N) = 0$,
(iii) *N is r-characteristic and* $\operatorname{Ext}(M/N, M/N) = 0$.

Proof. (i) follows immediately from Proposition 3.6.

(ii) If $f \in \operatorname{Hom}(N, M/N)$ then $\operatorname{Ext}(N, N) = 0$ yields the existence of $h \in$ $\in \operatorname{Hom}(N, M)$ with $f = ph$, where p is the canonical projection $M \to M/N$. Now Proposition 3.2 gives $f = ph = 0$.

(iii) Let $f \in \operatorname{Hom}(N, M/N)$. There is $g \in \operatorname{Hom}(M, M/N)$ with $g \mid N = f$. By Proposition 3.3, $N \subseteq \operatorname{Ker} g$ and so $f = 0$.

Proposition 3.8. *The following assertions are equivalent:*

(i) *N is hr-characteristic,*
(ii) *there is a preradical s with $s(M) = s(N) = N$ and $s(E(M/N)) = 0$,*
(iii) $\operatorname{Hom}(N, E(M/N)) = 0$,
(iv) *for all $n \in N$ and $m \in M \setminus N$ there is $a \in R$ with $an = 0$ and $am \notin N$,*
(v) $\operatorname{Hom}(Rn, M/N) = 0$ *for all $n \in N$,*
(vi) $\tilde{h}(p_N)(M) = N$,
(vii) $\tilde{h}(t_{(N \subseteq M)})(M) = N$.

In this case, $p_{\mathscr{A}}$ is the largest and $\tilde{h}(p_N) = \tilde{h}(t_{(N \subseteq M)})$ the least among the hereditary radicals s with $s(M) = N$, where $\mathscr{A}$ is the class of all $T \in R$-mod such that $p^{M/N}(S) = S$ for every submodule S of T.

Proof. The implications (i) implies (ii), (ii) implies (iii) and (iv) implies (v) are trivial.

(iii) implies (iv). If $(0 : n) \subseteq (N : m)$ for some $n \in N$ and $m \in M \setminus N$ then the homomorphism $f : Rn \to M/N$ given by $an \mapsto am + N$ can be extended to $g : N \to E(M/N)$ and hence $f = 0$, a contradiction.

(v) implies (vi). Let $\mathscr{B} = \{Rn \mid n \in N\}$. By Proposition 2.10(iv), $\tilde{p}_{\mathscr{B}}$ is a hereditary radical. Further, $N = p_{\mathscr{B}}(N) \subseteq \tilde{p}_{\mathscr{B}}(N) \subseteq N$, so $p_N \subseteq \tilde{p}_{\mathscr{B}}$, and consequently $\tilde{h}(p_N) \subseteq \tilde{p}_{\mathscr{B}}$. On the other hand, (v) yields $p_{\mathscr{B}}(M/N) = 0$, so $\tilde{p}_{\mathscr{B}}(M/N) = 0$. Thus $\tilde{h}(p_N)(M/N) = 0$ and hence $\tilde{h}(p_N)(M) \subseteq N \subseteq p_N(M) \subseteq \tilde{h}(p_N)(M)$.

(vi) implies (vii). This follows immediately from the obvious equality $p_N(E(M)) = t_{(N \subseteq M)}(E(M))$.

(vii) implies (i). It is well-known that if r is hereditary then $\tilde{r}$ is a hereditary radical (see e.g. [2]).

Now let s be a hereditary radical with $s(M| = N$ and denote $r = p^{M/N}$. With respect to Propositions 3.3 and 3.4, $\tilde{h}(p_N) = \tilde{h}(t_{(N \subseteq M)}) \subseteq s \subseteq r$. Hence $\mathscr{T}_s \subseteq \mathscr{A}$ (since s is hereditary), and so $s = \bar{s} = p_{\mathscr{T}_s} \subseteq p_{\mathscr{A}}$. Thus it remains to prove that $p_{\mathscr{A}}$ is a hereditary radical and $p_{\mathscr{A}}(M) = N$. Obviously, $\mathscr{A}$ is a hereditary class and $\mathscr{A} \subseteq \mathscr{T}_{p_{\mathscr{A}}} \subseteq \mathscr{T}_{\tilde{p}_{\mathscr{A}}}$. Further, $\bar{r}$ is a radical (see [1]) and Proposition 2.10 (iv) implies that $\tilde{p}_{\mathscr{A}}$ is hereditary. Since $\mathscr{A} \subseteq \mathscr{T}_r$, it is $p_{\mathscr{A}}(M) \subseteq r(M) \subseteq N$ and $\tilde{p}_{\mathscr{A}} \subseteq \tilde{p}_{\mathscr{T}_r} = \tilde{\bar{r}} = \bar{r} \subseteq r$, so that $\mathscr{T}_{\tilde{p}_{\mathscr{A}}} \subseteq \mathscr{T}_r$. As $\tilde{p}_{\mathscr{A}}$ is hereditary, $\mathscr{T}_{\tilde{p}_{\mathscr{A}}}$ is hereditary and therefore $\mathscr{T}_{\tilde{p}_{\mathscr{A}}} \subseteq \mathscr{A}$. Thus $\mathscr{A} = \mathscr{T}_{p_{\mathscr{A}}} = \mathscr{T}_{\tilde{p}_{\mathscr{A}}}$. Since both $p_{\mathscr{A}}$ and $\tilde{p}_{\mathscr{A}}$ are idempotent, we conclude that $p_{\mathscr{A}} = \tilde{p}_{\mathscr{A}}$ is a hereditary radical. Finally, (v) yields $r(T) = T$ for all $T \subseteq N$, hence $N \in \mathscr{A}$, so $N \subseteq p_{\mathscr{A}}(M)$ and the proof is complete.

Corollary 3.9. *Let N be h-characteristic and* $\operatorname{Ext}(Rn, N) = 0$ *for all $n \in N$. Then N is hr-characteristic.*

Proof. Let $n \in N$ and $f \in \operatorname{Hom}(Rn, M/N)$. Since $\operatorname{Ext}(Rn, N) = 0$, there is $h \in$

$\in \operatorname{Hom}(Rn, M)$ with $f = ph$ (p being the canonical projection $M \to M/N$). However, N is h-characteristic so that $\operatorname{Im} h \subseteq N = \operatorname{Ker} p$ and consequently $f = 0$.

Proposition 3.10. (i) *If I is projective and $I = r(R)$ for an idempotent preradical r then I is idempotent,*

(ii) *if Ra is projective for all $a \in I$ and $I = r(R)$ for a hereditary preradical r then $x \in Ix$ for all $x \in I$,*

(iii) *if I is maximal and $I = r(R)$ for an idempotent radical r then I is idempotent,*

(iv) *if $I = r(R) = Rx \subseteq xR$ for an idempotent radical r and $x \in R$ then I is hr-characteristics.*

Proof. (i) Since r is idempotent and I is projective, $I = r(R) = r(r(R)) = r(I) = = r(R)\, I = I^2$.

(ii) If $x \in I$ then $Rx = r(Rx) = IRx = Ix$.

(iii) Obviously, I/I^2 is a vector space over R/I. Hence if $I \neq I^2$ then $\operatorname{Hom}(I/I^2, R/I) \neq 0$ and consequently $\operatorname{Hom}(I, R/I) \neq 0$, a contradiction with Proposition 3.6.

(iv) If $0 \neq f \in \operatorname{Hom}(Ra, R/I)$ for some $a \in I$ then $(0 : a) \subseteq (0 : f(a))$. However, $a = xy$ for some $y \in R$, so $(0 : x) \subseteq (0 : a)$ and $\operatorname{Hom}(Rx, R/I) \neq 0$ which contradicts Proposition 3.6.

For the sake of completeness we present the following well-known assertion.

Proposition 3.11. *The following assertions are equivalent:*

(i) *I is hr-characteristic,*

(ii) $I = (0 : E(R/I))$,

(iii) $I \,.\, E(R/I) = 0$.

Proof. Since $(0 : E(R/I)) \subseteq I$ always holds, (ii) is equivalent to (iii). Now suppose that $I \,.\, E(R/I) = 0$ and $f \in \operatorname{Hom}(I, E(R/I))$. Then f can be extended to $g \in \in \operatorname{Hom}(R, E(R/I))$ and $f(a) = ag(1) = 0$ for all $a \in I$. Thus I is hr-characteristic by Proposition 3.8. Conversely, if $I = r(R)$ for a hereditary radical r then $I \,.\, E(R/I) \subseteq \subseteq r(E(R/I)) = 0$

Proposition 3.12. *The following assertions are equivalent:*

(i) *there is a superhereditary preradical r with $r(R) = I$,*

(ii) *there is a subset $X \subseteq R$ such that $I = (0 : X)_r$,*

(iii) *for every $a \in R \setminus I$ there is $b \in R$ with $bI = 0$ and $ba \neq 0$,*

(iv) $I = (0 : (0 : I))_r$.

Moreover, if $I \subseteq (0 : (0 : Y))_r$ for a finite subset $Y \subseteq I$ then the above conditions are equivalent to

(v) *I is h-characteristic.*

Proof. The equivalence of (i)–(iv) is clear (with respect to the correspondence between superhereditary preradicals and two-sided ideals). If there is a finite subset $Y \subseteq I$ with $I \subseteq (0 : (0 : Y))_r$ then (v) implies (iii) by Proposition 3.4(iv).

Proposition 3.13. *The followinq assertions are equivalent:*

(i) *there is a superhereditary preradical r with $I = r(R)$ and $r(R/I) = 0$,*
(ii) *for every $a \in R \setminus I$ there is $b \in R$ with $bI = 0$ and $ba \notin I$,*
(iii) $I = (I : (0 : I))_r$.
Moreover, if $I \subseteq (0 : Y))_r$ for a finite subset $Y \subseteq I$ then the above conditions are equivalent to
(iv) *I is hr-characteristic.*

Proof. As above, the equivalence of (i)–(iii) can be easily derived.

(ii) implies (iv) by Proposition 3.8. If $I \subseteq (0 : (0 : Y))_r$ where $Y = \{y_1, \ldots, y_k\} \subseteq I$ and $a \in R \setminus I$, then, with respect to Proposition 3.8 (iv), $b_1 y_1 = 0$ and $b_1 a \notin I$ for some $b_1 \in R$. Using induction, we obtain a sequence $b_1, \ldots, b_k$ such that $b_j \ldots b_1 y_j = 0$ and $b_j \ldots b_1 a \notin I$ for all $j = 1, \ldots, k$. Thus $b_k \ldots b_1 I = 0$ and (ii) holds.

Proposition 3.14. *There is a finite subset $Y \subseteq I$ with $I \subseteq (0 : (0 : Y))_r$, provided at least one of the following conditions holds:*

(i) *I is two-sided and finitely generated as a right ideal,*
(ii) *R satisfies the maximal condition for right annihilators,*
(iii) *R satisfies the minimal condition for left annihilators.*

Proof. (i) is obvious.

(ii) Let $a_1 \in I$ be arbitrary. If $I \nsubseteq (0 : (0 : a_1))_r$ then there is $a_2 \in I$ with $a_2 \notin (0 : (0 : a_1))_r$, i.e. $(0 : (0 : a_1))_r \subsetneqq (0 : (0 : \{a_1, a_2\}))_r$. Now we may proceed by induction and use the maximal condition.

(iii) We shall prove that this condition is equivalent to (ii). Indeed, let $(0 : X_1) \supseteq (0 : X_2) \supseteq \ldots$, where X_i are subsets of R. Then $(0 : (0 : X_1))_r \subseteq (0 : (0 : X_2))_r \subseteq \ldots$. If $(0 : (0 : X_n))_r = (0 : (0 : X_{n+1}))_r$ for some $n \geqq 1$ then $X_{n+1} \subseteq (0 : (0 : X_n))_r$, hence $(0 : X_n) X_{n+1} = 0$ and $(0 : X_n) \subseteq (0 : X_{n+1})$. The converse can be treated similarly.

Proposition 3.15. *If I contains no non-zero nilpotent two-sided ideal then $I = (0 : (0 : I))_r$ iff $I = (I : (0 : I))_r$.*

Proof. Let $I = (0 : (0 : I))_r$ and $a \in (I : (0 : I))_r$. Then $(0 : I) aR$ is a nilpotent two-sided ideal contained in I. Hence $(0 : I) a = 0$ and the proof may be considered complete, the converse implication being trivial.

Corollary 3.16. *Let R be a semiprime ring with the maximal condition for right annihilators. Then the following assertions are equivalent*:

(i) *I is hr-characteristic*,

(ii) *I is h-characteristic*,

(iii) *there is a subset $X \subseteq R$ with $I = (0 : X)_r$.*

Proof. Follows immediately from Propositions 3.12, 3.13, 3.14 and 3.15.

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