Bohdan Zelinka

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# A REMARK ON SYSTEMS OF MAXIMAL CLIQUES OF A GRAPH

BOHDAN ZELINKA, Liberec

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In [1] a concept of a $\tau$-covering of a set was defined and applied to the study of tolerances, i.e. reflexive and symmetric binary relations. The aim of this paper is to translate the results from [1] into the language of the graph theory.

Let $S$ be a non-empty set. A covering of $M$ by subsets is a system $\mathfrak{M}$ of subsets of $S$ such that $S \subseteq \bigcup_{M \in \mathfrak{M}} M$. A covering $\mathfrak{M}$ of the set $S$ by subsets is called a $\tau$-covering of $S$, if and only if

(i) for each $M_0 \in \mathfrak{M}$ and each $\mathfrak{N} \subseteq \mathfrak{M}$ the following implication holds:

$$M_0 \subseteq \bigcup_{M \in \mathfrak{N}} M \Rightarrow \bigcap_{M \in \mathfrak{N}} M \subseteq M_0 ;$$

(ii) if $R \subseteq S$ and $R$ is contained in no set from the covering, then $R$ contains a two-element subset of the same property.

**Theorem 1.** *Let $G$ be an undirected graph with the vertex set $V$, let $\mathfrak{M}$ be the system of vertex sets of all maximal cliques of $G$. Then $\mathfrak{M}$ is a $\tau$-covering of $V$. Conversely, if a set $V$ and its $\tau$-covering $\mathfrak{M}$ are given, then there exists a graph $G$ such that $V$ is its vertex set and $\mathfrak{M}$ is the system of vertex sets of all maximal cliques of $G$.*

Proof. Let $\mathfrak{M}$ be the system of vertex sets of all maximal cliques of $G$. Let $M_0 \in \mathfrak{M}$, $\mathfrak{N} \subseteq \mathfrak{M}$ and let $M_0 \subseteq \bigcup_{M \in \mathfrak{N}} M$. The set $M_0$ is the vertex set of a maximal clique $C_0$ of $G$. Suppose that $\bigcap_{M \in \mathfrak{N}} M \nsubseteq M_0$. Denote $N_0 = \bigcap_{M \in \mathfrak{N}} M$. Then $N_0 - M_0 \neq \emptyset$. Let $u \in N_0 - M_0$. If $v \in M_0$, then there exists a set $N(v) \in \mathfrak{N}$ such that $v \in N(v)$. As $N(v) \in \mathfrak{N}$, $u \in N_0$, we have $u \in N(v)$. The set $N(v)$ is the vertex set of a maximal clique of $G$, therefore $u$ and $v$ are joined by an edge. As $v$ was chosen arbitrarily, the vertex $u$ is joined with all vertices of $M_0$. Thus $M_0 \cup \{u\}$ is the vertex set of a clique $C_1$ containing $C_0$ as a proper subgraph, which is a contradiction with the assumption that $C_0$ is a maximal clique. Therefore (i) is fulfilled. Now let $R \subseteq V$ and let $R$ be contained in no set from $\mathfrak{M}$. This means that $R$ is contained in the vertex set of no maximal clique. The subgraph $G(R)$ of $G$ induced by $R$ is not a clique, because every clique is contained in a maximal clique. Thus $R$ contains two vertices which are not adjacent in $G$; this means that the set consisting of these two vertices is a subset of the vertex set of no clique and (ii) is fulfilled. The covering $\mathfrak{M}$ is a $\tau$-covering of $V$.

Now let a set $V$ and its $\tau$-covering $\mathfrak{M}$ be given. We join by an edge any two distinct elements of $V$ to which a set from $\mathfrak{M}$ exists containing both of them; thus we obtain a graph $G$. Each set from $\mathfrak{M}$ is the vertex set of a clique of $G$; it remains to prove that this clique is maximal. Suppose that there exists $M_1 \in \mathfrak{M}$ such that the clique whose vertex set is $M_1$ is not maximal. Then there exists a vertex $x \notin M_1$ which is joined by edges with all vertices of $M_1$. Let $y \in M_1$; as $y$ is joined with $x$, there exists a set $M(y) \in \mathfrak{M}$ such that $x \in M(y)$, $y \in M(y)$. Thus we can assign $M(y)$ to each $y \in M_1$. Evidently $M_1 \subseteq \bigcup_{y \in M_1} M(y)$. As $\mathfrak{M}$ is a $\tau$-covering, this implies $\bigcap_{y \in M_1} M(y) \subseteq M_1$. But $x \in M(y)$ for each $y \in M_i$, therefore $x \in \bigcap_{y \in M_1} M(y)$ and also $x \in M_1$, which is a contradiction.

**Theorem 2.** *Let $G$ be an undirected graph with the vertex set $V$, let $\mathfrak{M}$ be the system of all maximal independent sets of $G$. Then $\mathfrak{M}$ is a $\tau$-covering of $V$. Conversely, if a set $V$ and its $\tau$-covering $\mathfrak{M}$ are given, then there exists a graph $G$ such that $V$ is its vertex set and $\mathfrak{M}$ is the system of all maximal independent sets of $G$.*

Proof. Let $\bar{G}$ be the complement of $G$. Then each maximal independent set of $\bar{G}$ is the vertex set of a certain maximal clique in $G$ and vice versa. Thus the assertion follows from Theorem 1.

Remark. The word "maximal" means here always "maximal with respect to the set inclusion" and not "with the maximal number of elements".

As mentioned at the beginning, the $\tau$-coverings are of importance not only in these problems, but in all problems of finding the maximal subsets of a set with the property that any two elements of such a subset are in a certain symmetric binary relation. A particular case of a $\tau$-covering is a partition.

*Reference*

[1] *I. Chajda, J. Niederle, B. Zelinka:* On existence conditions for compatible tolerances. Czech. Math. J. *26* (1976), 304—311.

*Author's address:* 460 01 Liberec, Komenského 2, ČSSR (Vysoká škola strojní a textilní).