Svatoslav Staněk

Asymptotic properties of derivatives of central dispersions of the k-th kind for the differential equation $y'' = q(t)y$

Czechoslovak Mathematical Journal, Vol. 27 (1977), No. 4, 644–662

Persistent URL: http://dml.cz/dmlcz/101502

Czechoslovak Mathematical Journal, 27 (102) 1977, Praha

ASYMPTOTIC PROPERTIES OF DERIVATIVES OF CENTRAL DISPERSIONS OF THE k-TH KIND FOR THE DIFFERENTIAL EQUATION $y'' = q(t)\,y$

SVATOSLAV STANĚK, Olomouc

(Received November 7, 1975)

The distribution of zeros of solutions and the distribution of zeros of the first derivative of solutions of the differential equation (q) : $y'' = q(t)\,y$ may be successfully studied through central dispersions of the 1st, 2nd, 3rd and 4th kinds of (q). Sufficient conditions on p, q in order that the limits $(t \to \infty)$ of differences of central dispersions of all four kinds for the differential equations (p) : $y'' = p(t)\,y$ and (q) be equal to zero, were given in [9]. In [2], [3] and [8] sufficient conditions on p and q were shown that the limits $(t \to \infty)$ of derivatives of the differences of the central dispersions of the 1st kind for the differential equations (p) and (q) be equal to zero. These results are proved here under weaker assumptions and even for central dispersions of all four kinds of (p) and (q). The derivative of the basic dispersions of (q) may be utilized in investigating the asymptotic behaviour of solutions of (q) (see [1], [6] and [7]).

1. BASIC PROPERTIES

In what follows we shall be concerned with linear differential equations of the second order of the form

(q) $$y'' = q(t)\,y\,, \quad q \in C_I^0\,, \quad I = (a, \infty)\,.$$

Besides the assumption $q \in C_I^0$ we shall impose further assumptions on the function q if necessary. However, we shall alvays assume the equation (q) to be oscillatory for $t \to \infty$, that is to say, every nontrivial solution of (q) possesses infinitely many zeros in any interval of the form (t_0, ∞), $t_0 \in I$. The trivial solution is excluded from our considerations.

The fundamental concepts in this paper are those of the central dispersions of the first, second, third and fourth kinds of (q). Though their definitions and properties have been presented in [4], we list such properties that will needed in the sequel.

Let n be a positive integer, $x \in I$ and y a solution of (q) such that $y(x) = 0$. If $\varphi_n(x)$ $(\varphi_{-n}(x))$ is the n-th zero lying to the right (left) of the point x, then $\varphi_n(\varphi_{-n})$ is called the central dispersion of the 1st kind with the index n $(-n)$ of (q).

Let n be a positive integer, $q(t) < 0$ for $t \in I$; $x \in I$. Let y_1, y_2 be solutions of (q) such that $y_1(x) = 0$, $y_2'(x) = 0$. If $\psi_n(x)$ $(\psi_{-n}(x))$ $[\chi_n(x)\,(\chi_{-n}(x))$, $\omega_n(x)\,(\omega_{-n}(x))]$ is the n-th zero y_2' $[y_1', y_2]$ lying to the right (left) of the point x, then $\psi_n(\psi_{-n})$ $[\chi_n(\chi_{-n})$, $\omega_n(\omega_{-n})]$ is called the central dispersion of the 2nd [3rd, 4th] kind with the index n $(-n)$ of (q).

According to the assumption, (q) is oscillatory for $t \to \infty$ and thus the functions $\varphi_n, \psi_n, \chi_n, \omega_n$ are defined on I for every positive integer n. The functions $\varphi_{-n}, \psi_{-n}, \chi_{-n}, \omega_{-n}$ are generally defined on an interval $(a_1, \infty) \subset I$, where a_1 depends on the positive integer n and on the kind of the central dispersion. Particularly, $I^0({}^0I)$ will denote the domain of the function $\chi_{-1}(\omega_{-1})$. Instead of $\varphi_1, \psi_1, \chi_1, \omega_1$ we often write $\varphi, \psi, \chi, \omega$ only. It holds:

a) If for a positive integer n, $f^{[n]}$ denotes the n-th iteration of the function f, then

$$\varphi_n = \varphi^{[n]}, \quad \psi_n = \psi^{[n]}, \quad \chi_n = \chi \circ \varphi^{[n-1]}, \quad \omega_n = \omega \circ \psi^{[n-1]}. \tag{1}$$

b) If $q \in C_I^k$, then $\varphi_n \in C_I^{k+3}$. If moreover $q(t) < 0$ for $t \in I$, then all the remaining central dispersions belong to C_I^{k+1}.

c) Let $q(t) < 0$ for $t \in I$. Then

$$\chi \circ \omega = \psi, \quad \omega \circ \chi = \varphi, \quad \omega \circ \chi_{-1} = t, \quad \omega_{-1} \circ \chi_{-1} = \varphi_{-1}. \tag{2}$$

d) Let $q(t) < 0$ for $t \in I$. Then

$$\varphi'(t) = \frac{q(t_1)}{q(t_3)}, \quad \psi'(t) = \frac{q(t)}{q \circ \psi(t)} \cdot \frac{q(t_4)}{q(t_2)}, \quad \chi'(t) = \frac{q(t_1)}{q \circ \chi(t)}, \tag{3}$$

$$\omega'(t) = \frac{q(t)}{q(t_2)}, \quad \chi'_{-1}(t) = \frac{q(t_{-1})}{q \circ \chi_{-1}(t)}, \quad \varphi'_{-1}(t) = \frac{q(t_{-1})}{q(t_{-3})},$$

where $t_{-3}, t_{-1}, t_1, t_2, t_3, t_4$ are appropriate numbers, $t_{-3} < \chi_{-1}(t) < t_{-1} < t < < t_1 < \chi(t) < t_3 < \varphi(t)$, $t < t_2 < \omega(t) < t_4 < \psi(t)$.

e) Let $q(t) < 0$ for $t \in I$ and let u be a solution of (q). Then

$$\varphi'(t) = \begin{cases} \dfrac{u^2 \circ \varphi(t)}{u^2(t)} & \text{for} \quad u(t) \neq 0, \\[2ex] \dfrac{u'^2(t)}{u'^2 \circ \varphi(t)} & \text{for} \quad u(t) = 0, \end{cases} \tag{4}$$

$$\psi'(t) = \begin{cases} \dfrac{q(t)}{q \circ \psi(t)} \cdot \dfrac{u'^2 \circ \psi(t)}{u'^2(t)} & \text{for} \quad u'(t) \neq 0, \\[2ex] \dfrac{q(t)}{q \circ \psi(t)} \cdot \dfrac{u^2(t)}{u^2 \circ \psi(t)} & \text{for} \quad u'(t) = 0, \end{cases}$$

$$\chi'(t) = \begin{cases} -\dfrac{1}{q \circ \chi(t)} \cdot \dfrac{u'^2 \circ \chi(t)}{u^2(t)} & \text{for} \quad u(t) \neq 0\,, \\[2ex] -\dfrac{1}{q \circ \chi(t)} \cdot \dfrac{u'^2(t)}{u^2 \circ \chi(t)} & \text{for} \quad u(t) = 0\,, \end{cases}$$

$$\omega'(t) = \begin{cases} -q(t) \cdot \dfrac{u^2 \circ \omega(t)}{u'^2(t)} & \text{for} \quad u'(t) \neq 0\,, \\[2ex] -q(t) \cdot \dfrac{u^2(t)}{u'^2 \circ \omega(t)} & \text{for} \quad u'(t) = 0\,, \end{cases}$$

$$\chi'_{-1}(t) = \begin{cases} -\dfrac{1}{q \circ \chi_{-1}(t)} \cdot \dfrac{u'^2 \circ \chi_{-1}(t)}{u^2(t)} & \text{for} \quad u(t) \neq 0\,, \\[2ex] -\dfrac{1}{q \circ \chi_{-1}(t)} \cdot \dfrac{u'^2(t)}{u^2 \circ \chi_{-1}(t)} & \text{for} \quad u(t) = 0\,. \end{cases}$$

The proofs of the following two lemmas are in [5] p. 387.

Lemma 1. *Let y_1 be a solution of (q), $y_1(t) \neq 0$ for t on an interval $I_1, I_1 \in I$; $t_0 \in I_1$. Then the function y_2*

$$y_2(t) := y_1(t) \int_{t_0}^{t} \frac{\mathrm{d}s}{y_1^2(s)}\,, \quad t \in I_1$$

is a solution of (q) on I_1 and the solutions y_1, y_2 on I_1 are linearly independent.

Lemma 2. *Let y_1, y_2 be linearly independent solutions of (q), $y_1' y_2 - y_1 y_2' = 1$. Then the general solution of the nonhomogeneous equation*

$$y'' = q(t)\,y + h(t)\,, \quad q \in C_I^0\,, \quad h \in C_I^0$$

is given by the formula ($t_0 \in I$)

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t) + \int_{t_0}^{t} [y_1(t)\, y_2(s) - y_1(s)\, y_2(t)]\, h(s)\, \mathrm{d}s\,,$$

where c_1, c_2 are arbitrary constants.

Lemma 3. *Let $p(t) < q(t)$ for $t \in I$ and let $x \in I$. Further, let u, v be solutions of (p) and (q) respectively, such that $u(x) = v(x) = 0$, $u'(x) = v'(x) > 0$. Then there exist b, c, $b < x < c$ such that $u(t) \neq 0$ on $(b, x) \cup (x, c)$ and*

$$0 < u(t) < v(t) \quad \textit{for} \quad t \in (x, c)\,, \qquad 0 > u(t) > v(t) \quad \textit{for} \quad t \in (b, x)\,.$$

Proof. The first part of the statement of this Lemma (that is, the inequality $0 < u(t) < v(t)$ for $t \in (x, c)$) has been proved in [10] p. 136. The second part can be proved similarly.

Lemma 4. *Let* (p), (q) *be oscillatory for* $t \to \infty$ *equations,* $p(t) < q(t) < 0$ *for* $t \in I$. *Let* $\varphi, \psi, \chi, \omega$ $(\tilde{\varphi}, \tilde{\psi}, \tilde{\chi}, \tilde{\omega})$ *be the basic dispersions of the* 1*st,* 2*nd,* 3*rd,* 4*th kinds of* (q) ((p)), *and* χ_{-1}, ω_{-1} $(\tilde{\chi}_{-1}, \tilde{\omega}_{-1})$ *the central dispersions of the* 3*rd and* 4*th kinds with the index* -1 *of* (q) ((p)), *respectively. Then*

$$\varphi(t) > \tilde{\varphi}(t)\,, \quad \psi(t) > \tilde{\psi}(t)\,, \quad \chi(t) > \tilde{\chi}(t)\,, \quad \omega(t) > \tilde{\omega}(t)\,, \quad t \in I\,,$$
$$\chi_{-1}(t) < \tilde{\chi}_{-1}(t)\,, \quad t \in I^0\,,$$
$$\omega_{-1}(t) < \tilde{\omega}_{-1}(t)\,, \quad t \in {}^0I\,.$$

The proof is given in [9].

2. ASYMPTOTIC PROPERTIES OF THE FIRST DERIVATIVE OF THE CENTRAL DISPERSIONS OF THE k-TH KIND

In this and the next paragraphs we shall deal with equations (p), (q) under the following assumptions:

(i) there exist numbers m, M, $0 < m \leqq M$ such that

$$-M \leqq q(t) \leqq -m\,, \quad t \in I\,, \tag{5}$$

(ii) it holds

$$\lim_{t \to \infty} (p(t) - q(t)) = 0\,, \tag{6}$$

(iii) the first derivative of q is bounded on I.

Before presenting the main result of this paragraph, we shall introduce the notation: Let ν be an integer. Then $\varphi_\nu, \psi_\nu, \chi_\nu, \omega_\nu$ $(\tilde{\varphi}_\nu, \tilde{\psi}_\nu, \tilde{\chi}_\nu, \tilde{\omega}_\nu)$ denote the central dispersions of the 1st, 2nd, 3rd, 4th kinds with the index ν of (q) ((p)), respectively. For every constant ε, $|\varepsilon| < m$, $\varphi_\nu^\varepsilon, \psi_\nu^\varepsilon, \chi_\nu^\varepsilon, \omega_\nu^\varepsilon$ denote the central dispersions of the 1st, 2nd, 3rd, 4th kinds with the index ν of $(q + \varepsilon)$, respectively. Finally, $I^\varepsilon({}^\varepsilon I)$ will stand for the domain of the function $\chi_{-1}^\varepsilon(\omega_{-1}^\varepsilon)$ and $I^0({}^0I)$ for the domain of the function $\chi_{-1}(\omega_{-1})$.

The main result of this paragraph may be expressed by the following

Theorem 1. *Let the functions* p, q *satisfy the assumptions* (i), (ii) *and* (iii). *This implies*

$$\lim_{t \to \infty} (\varphi(t) - \tilde{\varphi}(t))' = 0\,, \tag{7}$$

(8) $$\lim_{t\to\infty} (\psi(t) - \tilde{\psi}(t))' = 0\,,$$

(9) $$\lim_{t\to\infty} (\chi(t) - \tilde{\chi}(t))' = 0\,,$$

(10) $$\lim_{t\to\infty} (\omega(t) - \tilde{\omega}(t))' = 0\,.$$

Remark. The fact (7) was proved in [2] under the assumption that $q(t)$ is a constant (<0) and in [3] where the assumption (i) is replaced by a weaker assumption $0 > -m \geqq q(t)$, $t \in I$, but instead of (iii) a stronger assumption $\lim_{t\to\infty} q'(t) = 0$ is used.

Before proving Theorem 1 we will show and prove a number of lemmas that will be of need therein. From now on we shall understand the functions p, q to satisfy the assumptions (i), (ii) and (iii) which, however, will explicitly be referred to in the assumptions of the theorems only.

Lemma 5. *The following inequalities hold*:

$$\frac{\pi}{\sqrt{M}} \leqq \varphi(t) - t \leqq \frac{\pi}{\sqrt{m}}\,, \quad \frac{\pi}{\sqrt{M}} \leqq \psi(t) - t \leqq \frac{\pi}{\sqrt{m}}\,, \quad \frac{\pi}{2\sqrt{M}} \leqq \chi(t) - t \leqq \frac{\pi}{2\sqrt{m}}\,,$$

$$\frac{\pi}{2\sqrt{M}} \leqq \omega(t) - t \leqq \frac{\pi}{2\sqrt{m}}\,, \quad t \in I\,,$$

$$\frac{\pi}{2\sqrt{M}} \leqq t - \chi_{-1}(t) \leqq \frac{\pi}{2\sqrt{m}}\,, \quad t \in I^0\,,$$

$$\frac{\pi}{2\sqrt{M}} \leqq t - \omega_{-1}(t) \leqq \frac{\pi}{2\sqrt{m}}\,, \quad t \in {}^0I\,.$$

The proof is given in [9].

Lemma 6. *For $t \in I^0 \cap I^\varepsilon$ we have the estimates*

$$|\varphi(t) - \varphi^\varepsilon(t)| \leqq \frac{|\varepsilon|\,(M + |\varepsilon|)}{(m - |\varepsilon|)^2} \cdot \frac{\pi}{\sqrt{m}}\,,$$

$$|\psi(t) - \psi^\varepsilon(t)| \leqq \frac{|\varepsilon|\, M(M + |\varepsilon|)}{(m - |\varepsilon|)^3} \cdot \frac{\pi}{\sqrt{m}}\,,$$

$$|\chi(t) - \chi^\varepsilon(t)| \leqq \frac{|\varepsilon|}{m - |\varepsilon|} \cdot \frac{\pi}{2\sqrt{m}}\,,$$

$$|\omega(t) - \omega^\varepsilon(t)| \leqq \frac{|\varepsilon|\, M}{(m - |\varepsilon|)^2} \cdot \frac{\pi}{2\sqrt{m}}\,,$$

$$|\chi_{-1}(t) - \chi^\varepsilon_{-1}(t)| \leqq \frac{|\varepsilon|}{m - |\varepsilon|} \cdot \frac{\pi}{2\sqrt{m}}\,.$$

Further, it holds

$$\lim_{t\to\infty}(\varphi(t) - \tilde{\varphi}(t)) = 0\,, \quad \lim_{t\to\infty}(\psi(t) - \tilde{\psi}(t)) = 0\,,$$

$$\lim_{t\to\infty}(\chi(t) - \tilde{\chi}(t)) = 0\,, \quad \lim_{t\to\infty}(\omega(t) - \tilde{\omega}(t)) = 0\,,$$

$$\lim_{t\to\infty}(\chi_{-1}(t) - \tilde{\chi}_{-1}(t)) = 0\,.$$

Proof. The lemma is presented as Theorem 1 and Theorem 2 in [9] except for $\lim_{t\to\infty}(\chi_{-1}(t) - \tilde{\chi}_{-1}(t)) = 0$, the proof of which proceeds analogously to those of the equalities in Theorem 1 in [9].

Lemma 7. *Let* $x \in I$, $|\vartheta| < \min(m, \pi m/2M\sqrt{M})$, $x + \vartheta \in I$. *Let* v *be a solution of* (q), $v(x + \vartheta) = 0$, $v \circ \chi(x + \vartheta) = 1$. *Then*

$$|v(t)| \leqq 1\,, \quad t \in [x + \vartheta,\ \chi(x) + \vartheta]\,.$$

Proof. First we have $0 \leqq v(t) \leqq 1$ on $[x + \vartheta, \varphi(x + \vartheta)]$. Let us assume first that $\vartheta > 0$. Then from the inequalities $\varphi(x + \vartheta) > \varphi(x) \geqq \chi(x) + \pi/2\sqrt{M} > \chi(x) + \vartheta$ we get $|v(t)| \leqq 1$ on $[x + \vartheta, \chi(x) + \vartheta]$. Let $\vartheta < 0$. From (3) and (5) we have (ξ is an appropriate number) $\varphi(x) - \varphi(x + \vartheta) = -\vartheta \,.\, \varphi'(\xi) \leqq -\vartheta(M/m) < \pi/2\sqrt{M}$. Making use of Lemma 5 we obtain $\varphi(x) - \varphi(x + \vartheta) < \omega \circ \chi(x) - \chi(x) = \varphi(x) - \chi(x)$ whence it follows $\varphi(x + \vartheta) > \chi(x) > \chi(x) + \vartheta$, hence $[x + \vartheta, \chi(x) + \vartheta] \subset [x + \vartheta, \varphi(x + \vartheta)]$ and Lemma 7 is proved.

Lemma 8. *Let* $x \in I^0$, $|\vartheta| < \min(m, \pi m/2M\sqrt{M})$, $x + \vartheta \in I^0$. *Let* v *be a solution of* (q), $v(x + \vartheta) = 0$, $v \circ \chi_{-1}(x + \vartheta) = -1$. *Then*

$$|v(t)| \leqq 1\,, \quad t \in [\chi_{-1}(x) + \vartheta,\ x + \vartheta] \cap I\,.$$

Proof. If $x \in \vartheta$ is in the domain of φ_{-1}, then on $[\varphi_{-1}(x + \vartheta), x + \vartheta]$ we have $|v(t)| \leqq 1$. If $x + \vartheta$ is not in the domain of φ_{-1}, then $|v(t)| \leqq 1$ on $(a, x + \vartheta]$. Hence, without any loss of generality φ_{-1} may be assumed to be defined in $x + \vartheta$. It is sufficient for the proof of the lemma to prove $\varphi_{-1}(x + \vartheta) < \chi_{-1}(x) + \vartheta$. Let $\vartheta > 0$. Then from (3) and (5) we have (ξ is an appropriate number) $\varphi_{-1}(x + \vartheta) - \varphi_{-1}(x) = \varphi'_{-1}(\xi)\,\vartheta < \vartheta(M/m) < \pi/2\sqrt{M}$ and therefrom making use of (2) and Lemma 5 $\varphi_{-1}(x + \vartheta) - \varphi_{-1}(x) < \chi_{-1}(x) - \omega_{-1} \circ \chi_{-1}(x) = \chi_{-1}(x) - \varphi_{-1}(x)$. Hence $\varphi_{-1}(x + \vartheta) < \chi_{-1}(x) < \chi_{-1}(x) + \vartheta$. Let now $\vartheta < 0$. Then $\varphi_{-1}(x + \vartheta) < \varphi_{-1}(x) \leqq \chi_{-1}(x) - \pi/2\sqrt{M} < \chi_{-1}(x) + \vartheta$.

Lemma 9. *Let* $x \in I$, $\varkappa = \pm 1$, $0 < \tau < 1/\sqrt{M}$, $x - \tau \in I$. *Let* v *be a solution of* (q), $v(x) = \varkappa$, $v'(x) = 0$. *Then for* $t \in [x - \tau, x + \tau]$

$$|v(t)| < 2\,, \quad |v'(t)| < 2\sqrt{M}\,.$$

Proof. The function v satisfies the equality

$$v(t) = \varkappa + \int_x^t \left[\int_x^s q(z)\, v(z)\, \mathrm{d}z \right] \mathrm{d}s\,. \tag{11}$$

Let us put $K = \max_{[x-\tau, x+\tau]} |v(t)|$ and let $t \in [x - \tau, x + \tau]$. From (11) and (5) we have then

$$|v(t)| \leqq 1 + MK\, \tfrac{1}{2}(t - x)^2 \leqq 1 + MK\, \tfrac{1}{2}\tau^2 < 1 + \tfrac{1}{2}K$$

and thus also $K < 1 + \frac{1}{2}K$ and therefrom $K < 2$. From (11) we get

$$v'(t) = \int_x^t q(z)\, v(z)\, \mathrm{d}z$$

so that $|v'(t)| \leqq 2M\tau < 2\sqrt{M}$.

Lemma 10.

$$\lim_{t \to \infty} (\chi(t) - \tilde{\chi}(t))' = 0\,, \tag{9}$$

$$\lim_{t \to \infty} (\chi_{-1}(t) - \tilde{\chi}_{-1}(t))' = 0\,. \tag{12}$$

Proof. Throughout the proof of Lemma 10 the number ε will satisfy $0 < \varepsilon < m$.

a) First of all we prove (9). We define functions $k, \tilde{k}, k_\varepsilon, k_{-\varepsilon}$ on I in terms of solutions of (p), (q), (q + ε), (q − ε) in the following way: Let t ($\in I$) be an arbitrary but fixed number. Then we associate functions $u, \tilde{u}, u_\varepsilon, u_{-\varepsilon}$ uniquely with the number t. These functions $u, \tilde{u}, u_\varepsilon, u_{-\varepsilon}$ are solutions of (q), (p), (q + ε), (q − ε), respectively, and satisfy the conditions $u(t) = \tilde{u}(t) = u_\varepsilon(t) = u_{-\varepsilon}(t) = 0$, $u \circ \chi(t) = \tilde{u} \circ \tilde{\chi}(t) = u_\varepsilon \circ \tilde{\chi}^\varepsilon(t) = u_{-\varepsilon} \circ \tilde{\chi}^{-\varepsilon}(t) = 1$. The values of the functions $k, \tilde{k}, k_\varepsilon, k_{-\varepsilon}$ at t are now defined by the formulas $k(t) = u'(t)$, $\tilde{k}(t) = \tilde{u}'(t)$, $k_\varepsilon(t) = u'_\varepsilon(t)$, $k_{-\varepsilon}(t) = u'_{-\varepsilon}(t)$. We have $k_\varepsilon(t) < k(t) < k_{-\varepsilon}(t)$, $t \in I$, which follows from the inequalities $\chi^\varepsilon(t) > \chi(t) > \chi^{-\varepsilon}(t)$ and may be deduced from Lemmas 3 and 4. Using the same type of reasoning we have $k_\varepsilon(t) < \tilde{k}(t) < k_{-\varepsilon}(t)$ for $t \in (b, \infty) \subset I$ where $|q(t) - p(t)| < \varepsilon$. We now prove that $\lim_{\varepsilon \downarrow 0} (k_\varepsilon(t) - k_{-\varepsilon}(t)) = 0$ uniformly on I. The assumption (6) then immediately gives

$$\lim_{t \to \infty} (k(t) - \tilde{k}(t)) = 0\,. \tag{13}$$

From Lemma 3 and from (5) we get

$$\sqrt{m} \leqq k(t) \leqq \sqrt{M}\,, \quad \sqrt{(m + \varepsilon)} \leqq k_{-\varepsilon}(t) \leqq \sqrt{(M + \varepsilon)}\,, \tag{14}$$
$$\sqrt{(m - \varepsilon)} \leqq k_\varepsilon(t) \leqq \sqrt{(M - \varepsilon)}\,, \quad t \in I\,.$$

Let $x \in I$ and let u, v be solutions of (q), $u(x) = 0$, $u \circ \chi(x) = 1$, $v(x) = -1/k(x)$,

$v'(x) = 0$. Obviously $k(x) = u'(x)$ and the Wronskian $w = uv' - u'v = 1$. The function u is increasing on $I_1 = [x, \chi(x)]$ and here $0 \leqq u(t) \leqq 1$. Since $v'(x) = 0$, $v(x) = -1/k(x)$, $v' \circ \chi(x) = 1$, v is an increasing function on I_1 and for all t $(\in I_1)$ where $v(t) \leqq 0$ we have by (14) $|v(t)| \leqq 1/k(x) < 1/\sqrt{m}$. Let us admit the existence of a number τ $(\in I_1)$ with $v(\tau) = 0$. From $1 = u(\tau)\, v'(\tau) - v(\tau)\, u'(\tau) = u(\tau)\, v'(\tau)$ we then obtain $v'(\tau) = 1/u(\tau)$. Now let us put on $I_2 = [\tau, \chi(x)]$ $\bar{v}(t) := u(t)\,.$ $.\int_\tau^t \mathrm{d}s/u^2(s)$. Following Lemma 1, $\bar{v}$ is a solution of (q) on I_2. On account of $\bar{v}(\tau) = 0$, $\bar{v}'(\tau) = 1/u(\tau)$ we get $v(t) = \bar{v}(t)$, $t \in I_2$. We utilize the last equality to estimate the absolute value of v on I_2. $y(t) := (k(x)/\sqrt{M}) \sin (t - x) \sqrt{M}$, $x \in I$ is a solution of the equation $y'' = -My$, $y(x) = 0$, $y'(x) = k(x)$ and therefore by Lemma 3 $y(t) \leqq$ $\leqq u(t)$ for $t \in (x, b)$ where $y(t) \neq 0$. Since by Lemma 5 $\tau - x = \omega(x) - x \geqq$ $\geqq \pi/2\sqrt{M}$ it is necessarily $u(\tau) \geqq k(x)/\sqrt{M}$ and on account of (14), $u(\tau) \geqq \sqrt{(m/M)}$. We have $\chi(x) - \tau < \chi(x) - x \leqq \pi/2\sqrt{m}$ and $u(t) \geqq u(\tau) \geqq \sqrt{(m/M)}$ for $t \in I_2$, $|u(t)| \leqq 1$ for $t \in I_1$ and therefore

$$v(t) = u(t) \int_\tau^t \frac{\mathrm{d}s}{u^2(s)} \leqq \frac{M}{m}(t - \tau) \leqq \frac{\pi M}{2m\sqrt{m}}, \quad t \in I_2\,.$$

With respect to

$$\frac{1}{k(x)} \leqq \frac{1}{\sqrt{m}} \leqq \frac{\pi M}{2m\sqrt{m}}$$

we obtain

$$(15) \qquad |v(t)| \leqq \frac{\pi M}{2m\sqrt{m}}, \quad t \in I_1\,.$$

Let $u_{\pm\varepsilon}$ be solutions of $(\mathrm{q} \pm \varepsilon)$, $u_{\pm\varepsilon}(x) = 0$, $u'_{\pm\varepsilon}(x) = k_{\pm\varepsilon}(x)$. From Lemma 2 we have

$$u_\varepsilon(t) = l_1\, u(t) - \varepsilon \int_x^t u_\varepsilon(s) \left[u(t)\, v(s) - u(s)\, v(t)\right] \mathrm{d}s\,,$$

$$u(t) = l_2\, u_{-\varepsilon}(t) - \varepsilon l_2 \int_x^t u_{-\varepsilon}(s) \left[u(t)\, v(s) - u(s)\, v(t)\right] \mathrm{d}s\,, \quad t \in I$$

with $l_1 = k_\varepsilon(x)/k(x)$ (<1), $l_2 = k(x)/k_{-\varepsilon}(x)$ (<1). From the inequalities $0 \leqq u_\varepsilon(t) \leqq$ $\leqq 1$, $0 \leqq u(t) \leqq 1$, $t \in I_1$, $0 \leqq u_{-\varepsilon}(t) \leqq 1$, $t \in I_3$ $(= [x, \chi^{-\varepsilon}(x)] \subset I_1)$, Lemma 5, (14) and (15) we get

$$|u_\varepsilon(t) - l_1\, u(t)| \leqq \varepsilon \frac{\pi M}{m\sqrt{m}}(t - x) \leqq \varepsilon \frac{\pi^2 M}{2m^2}, \quad t \in I_1\,,$$

$$|u(t) - l_2\, u_{-\varepsilon}(t)| \leqq \varepsilon l_2 \frac{\pi M}{m\sqrt{m}}(t - x) \leqq \varepsilon \sqrt{\left(\frac{M}{m+\varepsilon}\right)} \cdot \frac{\pi M}{m\sqrt{m}}(\chi^{-\varepsilon}(x) - x) \leqq$$

$$\leqq \varepsilon \frac{\pi^2 M \sqrt{M}}{2m^2 \sqrt{(m + \varepsilon)}}, \quad t \in I_3\,.$$

Hence

$$\left|u_\varepsilon \circ \chi(x) - l_1\right| \leqq \varepsilon \frac{\pi^2 M}{2m^2}, \quad \left|u \circ \chi^{-\varepsilon}(x) - l_2\right| \leqq \varepsilon \frac{\pi^2 M \sqrt{M}}{2m^2 \sqrt{(m+\varepsilon)}}.$$

We now see that $u_\varepsilon'(t) = \int_{\chi^\varepsilon(x)}^t (q(s) + \varepsilon)\, u_\varepsilon(s)\, ds$, $u'(t) = \int_{\chi(x)}^t q(s)\, u(s)\, ds$ and thus the properties of the solutions u, u_ε, the inequality $\chi^\varepsilon(x) - x \leqq \pi/2\sqrt{(m-\varepsilon)}$ and Lemma 5 imply

$$\text{(16)} \qquad \left|u_\varepsilon'(t)\right| \leqq (M - \varepsilon)\,(\chi^\varepsilon(x) - t) \leqq \frac{\pi(M-\varepsilon)}{2\sqrt{(m-\varepsilon)}}, \quad t \in [x, \chi^\varepsilon(x)],$$

$$\left|u'(t)\right| \leqq M(\chi(x) - t) \leqq \frac{\pi M}{2\sqrt{m}}, \quad t \in I_1 .$$

By the mean value theorem there exist numbers ξ_1, ξ_2, $\xi_1 \in (\chi(x), \chi^\varepsilon(x))$, $\xi_2 \in (\chi^{-\varepsilon}(x), \chi(x))$ such that by virtue of (16) and Lemma 6 we have

$$\left|u_\varepsilon \circ \chi^\varepsilon(x) - u_\varepsilon \circ \chi(x)\right| = \left|u_\varepsilon'(\xi_1)\right| . \left|\chi^\varepsilon(x) - \chi(x)\right| \leqq \varepsilon \frac{\pi^2(M-\varepsilon)}{4(m-\varepsilon)\sqrt{(m(m-\varepsilon))}},$$

$$\left|u \circ \chi(x) - u \circ \chi^{-\varepsilon}(x)\right| = \left|u'(\xi_2)\right| . \left|\chi(x) - \chi^{-\varepsilon}(x)\right| \leqq \varepsilon \frac{\pi^2 M}{4m(m-\varepsilon)}.$$

So we have proved

$$\text{(17)} \qquad 1 - l_1 \leqq \left|u_\varepsilon \circ \chi^\varepsilon(x) - u_\varepsilon \circ \chi(x)\right| + \left|u_\varepsilon \circ \chi(x) - l_1\right| \leqq$$

$$\leqq \varepsilon \frac{\pi^2}{2}\left(\frac{M}{m^2} + \frac{M-\varepsilon}{2(m-\varepsilon)\sqrt{(m(m-\varepsilon))}}\right),$$

$$1 - l_2 \leqq \left|u \circ \chi(x) - u \circ \chi^{-\varepsilon}(x)\right| + \left|u \circ \chi^{-\varepsilon}(x) - l_2\right| \leqq$$

$$\leqq \varepsilon \frac{\pi^2 M}{2m}\left(\frac{1}{2(m-\varepsilon)} + \frac{\sqrt{M}}{m\sqrt{(m+\varepsilon)}}\right).$$

The formulas $l_1 = k_\varepsilon(x)/k(x)$, $l_2 = k(x)/k_{-\varepsilon}(x)$ together with (14) and (17) imply

$$0 < k_{-\varepsilon}(x) - k_\varepsilon(x) = (k_{-\varepsilon}(x) - k(x)) + (k(x) - k_\varepsilon(x)) = k_{-\varepsilon}(x)\,(1 - l_2) +$$

$$+ k(x)\,(1 - l_1) \leqq \varepsilon \frac{\pi^2 M \sqrt{(M+\varepsilon)}}{2m}\left(\frac{1}{2(m-\varepsilon)} + \frac{\sqrt{M}}{m\sqrt{(m+\varepsilon)}}\right) +$$

$$+ \varepsilon \frac{\pi^2 \sqrt{M}}{2}\left(\frac{M}{m^2} + \frac{M-\varepsilon}{2(m-\varepsilon)\sqrt{(m(m-\varepsilon))}}\right).$$

Therefore $\lim_{\varepsilon \downarrow 0} (k_\varepsilon(t) - k_{-\varepsilon}(t)) = 0$ uniformly on I and (13) is thus proved.

From (4) and from the properties of the functions k, $\tilde{k}$ we obtain

$$\chi'(t) - \tilde{\chi}'(t) = -\frac{k^2(t)}{q \circ \chi(t)} + \frac{\tilde{k}^2(t)}{p \circ \tilde{\chi}(t)}, \quad t \in I. \tag{18}$$

It holds further (η is an appropriate number)

$$\frac{\tilde{k}^2(t)}{p \circ \tilde{\chi}(t)} - \frac{k^2(t)}{q \circ \chi(t)} = \frac{\tilde{k}^2(t) \,.\, q \circ \chi(t) - k^2(t) \,.\, p \circ \tilde{\chi}(t)}{p \circ \tilde{\chi}(t) \,.\, q \circ \chi(t)},$$

$$\tilde{k}^2(t)\, q \circ \chi(t) - k^2(t)\, p \circ \tilde{\chi}(t) = k^2(t)\,[q \circ \tilde{\chi}(t) - p \circ \tilde{\chi}(t)] + + q \circ \tilde{\chi}(t)\,[\tilde{k}^2(t) - k^2(t)] + \tilde{k}^2(t)\,[q \circ \chi(t) - q \circ \tilde{\chi}(t)], \tag{19}$$

$$q \circ \chi(t) - q \circ \tilde{\chi}(t) = q'(\eta)\,(\chi(t) - \tilde{\chi}(t)). \tag{20}$$

According to the assumption (iii), q' is bounded on I and therefore in virtue of (5), (6), (13), (14) and of Lemma 6 the equality (9) follows from (18), (19) and (20).

b) Let us pass to the proof of the equality (12). Let $\tilde{I}^\varepsilon$ be equal to the intersection of the domains of the functions χ^ε_{-1}, $\tilde{\chi}_{-1}$. We define functions $f, \tilde{f}, f_\varepsilon, f_{-\varepsilon}$ on $\tilde{I}^\varepsilon$ by means of solutions of (q), (p), (q + ε), (q − ε) in this way: Let t ($\in \tilde{I}^\varepsilon$) be an arbitrary but fixed number. Then we associate functions u, $\tilde{u}$, u_ε, $u_{-\varepsilon}$ uniquely with the number t. These functions are solutions of (q), (p), (q + ε), (q − ε), respectively, and satisfy the conditions $u(t) = \tilde{u}(t) = u_\varepsilon(t) = u_{-\varepsilon}(t) = 0$, $u \circ \chi_{-1}(t) = \tilde{u} \circ \tilde{\chi}_{-1}(t) = u_\varepsilon \circ \chi^\varepsilon_{-1}(t) = u_{-\varepsilon} \circ \chi^{-\varepsilon}_{-1}(t) = -1$. The values of the functions $f, \tilde{f}, f_\varepsilon, f_{-\varepsilon}$ at t are now defined by the formulas $f(t) = u'(t)$, $\tilde{f}(t) = \tilde{u}'(t)$, $f_\varepsilon(t) = u_\varepsilon'(t)$, $f_{-\varepsilon}(t) = u'_{-\varepsilon}(t)$. We have $f_\varepsilon(t) < f(t) < f_{-\varepsilon}(t)$, $t \in \tilde{I}^\varepsilon$ which follows from the inequalities $\chi^\varepsilon_{-1}(t) < \chi_{-1}(t) < \chi^{-\varepsilon}_{-1}(t)$, $t \in \tilde{I}^\varepsilon$ which can be obtained from Lemmas 3 and 4. By a similar reasoning we have $f_\varepsilon(t) < \tilde{f}(t) < f_{-\varepsilon}(t)$ for $t \in (b, \infty) \subset I$ such that $|q(t) - p(t)| < \varepsilon$. In the sequel we shall prove that $\lim_{\varepsilon \downarrow 0} (f_\varepsilon(t) - f_{-\varepsilon}(t)) = 0$ uniformly on I. Then (6) immediately implies

$$\lim_{t \to \infty} (f(t) - \tilde{f}(t)) = 0. \tag{21}$$

From Lemma 3 and from (5) we get

$$\sqrt{m} \leqq f(t) \leqq \sqrt{M}, \quad \sqrt{(m + \varepsilon)} \leqq f_{-\varepsilon}(t) \leqq \sqrt{(M + \varepsilon)}, \quad \sqrt{(m - \varepsilon)} \leqq f_\varepsilon(t) \leqq \sqrt{(M - \varepsilon)}, \quad t \in I^\varepsilon. \tag{22}$$

Let $x \in \tilde{I}^\varepsilon$, u, v be solutions of (q), $u(x) = 0$, $u \circ \chi_{-1}(x) = -1$, $v(x) = -1/f(x)$, $v'(x) = 0$. Obviously $u'(x) = f(x)$ and the Wronskian $w = uv' - vu' = 1$. The function u is increasing on $I_1 = [\chi_{-1}(x), x]$ and thus we have $-1 \leqq u(t) \leqq 0$ on this interval. Since $v'(x) = 0$, $v(x) = -1/f(x)$, $v' \circ \chi_{-1}(x) = -1$, v is necessarily a decreasing function on I_1 and for all $t \in I_1$, where $v(t) < 0$, we have with respect to (22) $|v(t)| \leqq 1/f(x) \leqq 1/\sqrt{m}$. Let us admit the existence of a number $\tau \in I_1$ such that $v(\tau) = 0$. The equalities $1 = u(\tau)\, v'(\tau) - u'(\tau)\, v(\tau) = u(\tau)\, v'(\tau)$ yield $v'(\tau) = 1/u(\tau)$. We define the function $\bar{v}$ on $I_2 = [\chi_{-1}(x), \tau]$ by the relation $\bar{v}(t) := u(t) \int_\tau^t \mathrm{d}s/u^2(s)$. By Lemma 1 $\bar{v}$ is a solution of (q) on I_2. Since $\bar{v}(\tau) = 0$, $\bar{v}'(\tau) = 1/u(\tau)$ it is $v(t) = \bar{v}(t)$, $t \in I_2$. Analogously to part a) of the proof, the last equality will be utilized to estimate the absolute value v on I_2. $y(t) := (f(x)/\sqrt{M}) \sin (t - x)\sqrt{M}$, $t \in I$ is a solution of $y'' = -My$, $y(x) = 0$, $y'(x) = f(x)$. Thus according to Lemma 3 $0 > y(t) \geqq u(t)$ for all $t \in (c, x)$, where $y(t) \neq 0$. Since by Lemma 5 $x - \tau = x - \omega_{-1}(x) \geqq \pi/2\sqrt{M}$ we have $|u(\tau)| \geqq f(x)/\sqrt{M}$ and making use of (22) we get $|u(\tau)| \geqq \sqrt{(m/M)}$. Hence

$$|v(t)| \leqq |u(t)| \int_t^\tau \frac{\mathrm{d}s}{u^2(s)} \leqq \frac{M}{m}(\tau - t) \leqq \frac{\pi M}{2m\sqrt{m}}, \quad t \in I_2,$$

and

$$\frac{1}{f(x)} \leqq \frac{1}{\sqrt{m}} \leqq \frac{\pi M}{2m\sqrt{m}}$$

then yields

$$|v(t)| \leqq \frac{\pi M}{2m\sqrt{m}}, \quad t \in I_1. \tag{23}$$

Let $u_{\pm\varepsilon}$ be solutions of (q $\pm$ ε), $u_{\pm\varepsilon}(x) = 0$, $u'_{\pm\varepsilon}(x) = f_{\pm\varepsilon}(x)$. From Lemma 2 we have

$$u_\varepsilon(t) = k_1\, u(t) - \varepsilon \int_x^t u_\varepsilon(s) \left[u(t)\, v(s) - u(s)\, v(t)\right] \mathrm{d}s, \quad t \in I,$$

$$u(t) = k_2\, u_{-\varepsilon}(t) - \varepsilon k_2 \int_x^t u_{-\varepsilon}(s) \left[u(t)\, v(s) - u(s)\, v(t)\right] \mathrm{d}s, \quad t \in I,$$

where $k_1 = f_\varepsilon(x)/f(x)$ (<1), $k_2 = f(x)/f_{-\varepsilon}(x)$ (<1). With respect to the inequalities $0 \geqq u_\varepsilon(t) \geqq -1$, $0 \geqq u(t) \geqq -1$ for $t \in I_1$, $0 \geqq u_{-\varepsilon}(t) \geqq -1$ for $t \in I_3$ $(= [\chi_{-1}^{-\varepsilon}(x), x] \subset I_1)$ and to Lemma 5, (22), (23) we get

$$|u_\varepsilon(t) - k_1\, u(t)| \leqq \varepsilon \frac{\pi M}{m\sqrt{m}}(t - x) \leqq \varepsilon \frac{\pi^2 M}{2m^2}, \quad t \in I_1,$$

$$|u(t) - k_2\, u_{-\varepsilon}(t)| \leqq \varepsilon k_2 \frac{\pi M}{m\sqrt{m}}(t - x) \leqq \varepsilon \sqrt{\left(\frac{M}{m + \varepsilon}\right)} \frac{\pi M}{m\sqrt{m}}(x - \chi_{-1}^{-\varepsilon}(x)) \leqq$$

$$\leqq \varepsilon \frac{\pi^2 M\sqrt{M}}{2m^2\sqrt{(m + \varepsilon)}}, \quad t \in I_3.$$

Herefrom

$$|u_\varepsilon \circ \chi_{-1}(x) + k_1| \leqq \varepsilon \frac{\pi^2 M}{2m^2}, \quad |u \circ \chi_{-1}^{-\varepsilon}(x) + k_2| \leqq \varepsilon \frac{\pi^2 M\sqrt{M}}{2m^2\sqrt{(m + \varepsilon)}}.$$

It holds $u'_\varepsilon(t) = \int_{\varepsilon\chi_{-1}(x)}^t (q(s) + \varepsilon)\, u_\varepsilon(s)\, ds$, $u'(t) = \int_{\chi_{-1}(x)}^t q(s)\, u(s)\, ds$ and consequently from the properties of the functions u, u_ε and from Lemma 5 and the inequality $x - \chi^\varepsilon_{-1}(x) \leqq \pi/2\sqrt{(m - \varepsilon)}$ we have

$$(24)\qquad |u'_\varepsilon(t)| \leqq (M - \varepsilon)(t - \chi^\varepsilon_{-1}(x)) \leqq \frac{\pi(M - \varepsilon)}{2\sqrt{(m - \varepsilon)}}, \quad t \in I_3,$$

$$|u'(t)| \leqq M(t - \chi_{-1}(x)) \leqq \frac{\pi M}{2\sqrt{m}}, \quad t \in I_1.$$

By the mean value theorem there exist numbers ξ_1, ξ_2, $\xi_1 \in (\chi^\varepsilon_{-1}(x), \chi_{-1}(x))$, $\xi_2 \in \in (\chi_{-1}(x), \chi^\varepsilon_{-1}(x))$ such that with regard to (24) and Lemma 6, we see that the following estimates hold:

$$|u_\varepsilon \circ \chi^\varepsilon_{-1}(x) - u_\varepsilon \circ \chi_{-1}(x)| = |u'_\varepsilon(\xi_1)| \cdot |\chi^\varepsilon_{-1}(x) - \chi_{-1}(x)| \leqq \varepsilon \frac{\pi^2(M - \varepsilon)}{4(m - \varepsilon)\sqrt{(m(m - \varepsilon))}},$$

$$|u \circ \chi_{-1}(x) - u \circ \chi^\varepsilon_{-1}(x)| = |u'(\xi_2)| \cdot |\chi_{-1}(x) - \chi^{-\varepsilon}_{-1}(x)| \leqq \varepsilon \frac{\pi^2 M}{4m(m - \varepsilon)},$$

and therefore

$$(25)\qquad |k_1 - 1| \leqq |k_1 + u_\varepsilon \circ \chi_{-1}(x)| + |u_\varepsilon \circ \chi_{-1}(x) - u_\varepsilon \circ \chi^\varepsilon_{-1}(x)| \leqq$$

$$\leqq \varepsilon \frac{\pi^2}{2}\left(\frac{M}{m^2} + \frac{M - \varepsilon}{2(m - \varepsilon)\sqrt{(m(m - \varepsilon))}}\right),$$

$$|k_2 - 1| \leqq |k_2 + u \circ \chi^{-\varepsilon}_{-1}(x)| + |u \circ \chi_{-1}(x) - u \circ \chi^{-\varepsilon}_{-1}(x)| \leqq$$

$$\leqq \varepsilon \frac{\pi^2 M}{2m}\left(\frac{1}{2(m - \varepsilon)} + \frac{\sqrt{M}}{m\sqrt{(m + \varepsilon)}}\right).$$

From the formulas $k_1 = f_\varepsilon(x)/f(x)$, $k_2 = f(x)/f_{-\varepsilon}(x)$, (22) and (25) it follows

$$|f_\varepsilon(x) - f_{-\varepsilon}(x)| \leqq |f_\varepsilon(x) - f(x)| + |f(x) - f_{-\varepsilon}(x)| = f(x)\,|1 - k_1| +$$

$$+ f_{-\varepsilon}(x)\,|1 - k_2| \leqq \varepsilon \frac{\pi^2\sqrt{M}}{2}\left(\frac{M}{m^2} + \frac{M - \varepsilon}{2(m - \varepsilon)\sqrt{(m(m - \varepsilon))}}\right) +$$

$$+ \varepsilon \frac{\pi^2 M\sqrt{(M + \varepsilon)}}{2m}\left(\frac{1}{2(m - \varepsilon)} + \frac{\sqrt{M}}{m\sqrt{(m + \varepsilon)}}\right).$$

Thus $\lim_{\varepsilon \downarrow 0} (f_\varepsilon(t) - f_{-\varepsilon}(t)) = 0$ uniformly on $\tilde{I}^\varepsilon$ and the proof of the equality (21) is complete.

From (4) and the properties of the functions $f, \tilde{f}$ we have

$$\chi'_{-1}(t) - \tilde{\chi}'_{-1}(t) = -\frac{f^2(t)}{q \circ \chi_{-1}(t)} + \frac{\tilde{f}^2(t)}{p \circ \tilde{\chi}_{-1}(t)}, \tag{26}$$

and further (η is an appropriate number)

$$\frac{\tilde{f}^2(t)}{p \circ \tilde{\chi}_{-1}(t)} - \frac{f^2(t)}{q \circ \chi_{-1}(t)} = \frac{\tilde{f}^2(t) \cdot q \circ \chi_{-1}(t) - f^2(t) \cdot p \circ \tilde{\chi}_{-1}(t)}{p \circ \tilde{\chi}_{-1}(t) \cdot q \circ \chi_{-1}(t)},$$

$$\begin{aligned} \tilde{f}^2(t)\, q \circ \chi_{-1}(t) - f^2(t)\, p \circ \tilde{\chi}_{-1}(t) &= f^2(t)\,[q \circ \tilde{\chi}_{-1}(t) - p \circ \tilde{\chi}_{-1}(t)] + \\ &+ [\tilde{f}^2(t) - f^2(t)]\, q \circ \tilde{\chi}_{-1}(t) + \tilde{f}^2(t)\,[q \circ \chi_{-1}(t) - q \circ \tilde{\chi}_{-1}(t)], \end{aligned} \tag{27}$$

$$q \circ \chi_{-1}(t) - q \circ \tilde{\chi}_{-1}(t) = q'(\eta)\,(\chi_{-1}(t) - \tilde{\chi}_{-1}(t)). \tag{28}$$

According to the assumption (iii) the function q' is bounded on I and therefore with respect to (5), (6), (21), (22) and to Lemma 6 we obtain from (26), (27) and (28) the equality (12).

Lemma 11. *χ'' is bounded on I.*

Proof. Following the assumption (iii) $|q'(t)| \leqq L$, $t \in I$, and from (4) it follows that χ has the second derivative on I. Let k be the function defined on I in the same way as in part a) of the proof of Lemma 10. From the formula $\chi'(t) = -k^2(t)/q \circ \chi(t)$ and (5) it follows then that k has the derivative on I. To prove the statement it is sufficient to show that this derivative is bounded on I. Let

$$|\vartheta| < \min\left(m, \frac{m}{(m + M)\sqrt{M}}\right).$$

The boundedness of the function k' on I immediately follows from

$$|k(t) - k(t + \vartheta)| \leqq |\vartheta|\left(2\,\frac{M(m + M)}{m} + \frac{\pi^2 ML}{2m^2}\sqrt{M}\right), \quad t \in I, \quad t + \vartheta \in I, \tag{29}$$

which we shall prove now.

Let $x \in I$ and $x + \vartheta \in I$ and let u, z be solutions of (q), $u(x) = 0$, $u \circ \chi(x) = 1$, $z(x + \vartheta) = 0$, $z \circ \chi(x + \vartheta) = 1$. Obviously $k(x) = u'(x)$, $k(x + \vartheta) = z'(x + \vartheta)$. For all $t + I$ where $t \in \vartheta \in I$ we define $\bar{z}$ by the relation $\bar{z}(t) := z(t + \vartheta)$. We have then $\bar{z}(x) = 0$ and it follows

$$\bar{z}''(t) = q(t)\,\bar{z}(t) + (q(t + \vartheta) - q(t))\,\bar{z}(t).$$

From the latter formula and from Lemma 2 we now get (v is a solution of (q), $v(x) = -1/k(x)$, $v'(x) = 0$)

$$\bar{z}(t) = \frac{k(x + \vartheta)}{k(x)} u(t) - \int_x^t \bar{z}(s) [q(s + \vartheta) - q(s)] [u(t) v(s) - u(s) v(t)] \, ds .$$

Hence

$$(30) \quad \bar{z} \circ \chi(x) = \frac{k(x + \vartheta)}{k(x)} - \int_x^{\chi(x)} \bar{z}(s) [q(s + \vartheta) - q(s)] [u(t) v(s) - u(s) v(t)] \, ds .$$

Moreover $|\vartheta| < m/(m + M) \sqrt{M} < \pi m/2M \sqrt{M}$ and therefore by Lemma 7 we have $|\bar{z}(t)| \leqq 1$, $t \in I_1$ $(= [x, \chi(x)])$. From the latter inequality and the inequality $|u(t)| \leqq 1$ holding on I_1 and from Lemma 5 and (15) we obtain ($|q(s + \vartheta) - q(s)| \leqq |\vartheta| L$)

$$(31) \quad \left| \int_x^{\chi(x)} \bar{z}(s) [q(s + \vartheta) - q(s)] [u(t) v(s) - u(s) v(t)] \, ds \right| \leqq$$
$$\leqq 2L \frac{|\vartheta| \, \pi M}{2m \sqrt{m}} (\chi(x) - x) \leqq |\vartheta| \frac{\pi^2 ML}{2m^2} .$$

By the mean value theorem there exists a number η on the interval with the end points $\chi(x)$, $\chi(x + \vartheta) - \vartheta$ and a number ξ such that

$$(32) \quad |1 - \bar{z} \circ \chi(x)| = |\bar{z}[\chi(x + \vartheta) - \vartheta] - \bar{z} \circ \chi(x)| =$$
$$= |\bar{z}'(\eta)| \, . \, |\chi(x + \vartheta) - \chi(x) - \vartheta| \leqq |\bar{z}'(\eta)| \, (\chi'(\xi) + 1) \, |\vartheta| .$$

Taking account of (5) and (3) we have

$$(33) \quad |\chi(x + \vartheta) - \chi(x) - \vartheta| \leqq (\chi'(\xi) + 1) \, |\vartheta| \leqq \left(1 + \frac{M}{m}\right) |\vartheta| < \frac{1}{\sqrt{M}}$$

and therefore by Lemma 9 $|\bar{z}'(\eta)| \leqq 2 \sqrt{M}$. From this inequality and from (32) and (33) we get

$$(34) \quad |1 - \bar{z} \circ \chi(x)| \leqq 2 |\vartheta| \frac{m + M}{m} \sqrt{M} .$$

Using (30) we have

$$k(x + \vartheta) - k(x) = (\bar{z} \circ \chi(x) - 1) \, k(x) + k(x) \, .$$
$$. \int_x^{\chi(x)} \bar{z}(s) [q(s + \vartheta) - q(s)] [u(t) v(s) - u(s) v(t)] \, ds ,$$

and from (14), (31) and (34) we come finally to

$$|k(x + \vartheta) - k(x)| \leqq |\vartheta| \left(2 \frac{M(m + M)}{m} + \frac{\pi^2 ML}{2m^2} \sqrt{M}\right) .$$

Lemma 12. χ''_{-1} *is bounded on* I^0.

Proof. According to the assumption (iii) $|q'(t)| \leqq L$, $t \in I$, and from (4) it follows that χ_{-1} has the second derivative on I^0. Let f be the function defined in the same way as in part b) of the proof in Lemma 10. Then from the formula $\chi'_{-1}(t) = -f^2(t)/q \circ \chi_{-1}(t)$ and from (5) we see that f has the derivative on I^0 and for the proof of the statement of the lemma it suffices to show that f' is bounded on I^0. Let $|\vartheta| < \min(m, m/(m+M)\sqrt{M})$. The boundedness of f' on I^0 follows from the inequality

$$(35)\quad |f(t) - f(t+\vartheta)| \leqq |\vartheta| \left(2M \frac{m+M}{m} + \frac{\pi M^2 L}{2m^2} \sqrt{M}\right), \quad t \in I^0, \quad t + \vartheta \in I^0,$$

which we shall prove. Let $x \in I^0$, $x + \vartheta \in I^0$ and let u, z be solutions of (q), $u(x) = 0$, $u \circ \chi_{-1}(x) = -1$, $z(x+\vartheta) = 0$, $z \circ \chi_{-1}(x+\vartheta) = -1$. This gives $u'(x) = f(x)$, $z'(x+\vartheta) = f(x+\vartheta)$. We define a function $\bar{z}$ by the relation $\bar{z}(t) := z(t+\vartheta)$ for all $t \in I^0$ where $t + \vartheta \in I^0$. We have $\bar{z}(x) = 0$ and

$$\bar{z}''(t) = q(t)\,\bar{z}(t) + (q(t+\vartheta) - q(t))\,\bar{z}(t).$$

Herefrom and from Lemma 2 we get (v is a solution of (q), $v(x) = -1/f(x)$, $v'(x) = 0$)

$$\bar{z}(t) = \frac{f(x+\vartheta)}{f(x)}\,u(t) - \int_x^t \bar{z}(s)\,[q(s+\vartheta) - q(s)]\,[u(t)\,v(s) - u(s)\,v(t)]\,\mathrm{d}s.$$

Hence

$$(36)\quad \bar{z} \circ \chi_{-1}(x) = -\frac{f(x+\vartheta)}{f(x)} + \int_{\chi_{-1}(x)}^{x} \bar{z}(s)\,[q(s+\vartheta) - q(s)]\,[u(t)\,v(s) - u(s)\,v(t)]\,\mathrm{d}s.$$

Since $|\vartheta| < m/(m+M)\sqrt{M} < \pi m/2M\sqrt{M}$, we have by Lemma 8 $|\bar{z}(t)| \leqq 1$, $t \in I_1$ $(= [\chi_{-1}(x), x])$. Further $|u(t)| \leqq 1$ on I_1 and therefore by Lemma 5 and (23) we obtain ($|q(s+\vartheta) - q(s)| \leqq L|\vartheta|$)

$$(37)\quad \left|\int_{\chi_{-1}(x)}^{x} \bar{z}(s)\,[q(s+\vartheta) - q(s)]\,[u(t)\,v(s) - u(s)\,v(t)]\,\mathrm{d}s\right| \leqq$$

$$\leqq 2|\vartheta| \frac{\pi L M}{2m\sqrt{m}}(x - \chi_{-1}(x)) \leqq |\vartheta| \frac{\pi^2 L M}{2m^2}.$$

By the mean value theorem there exist a number η in the interval with the end points $\chi_{-1}(x)$, $\chi_{-1}(x+\vartheta) - \vartheta$ and a number ξ such that

$$|1 + \bar{z} \circ \chi_{-1}(x)| = |\bar{z} \circ \chi_{-1}(x) - \bar{z}[\chi_{-1}(x+\vartheta) - \vartheta]| =$$

$$= |\bar{z}'(\eta)| \cdot |\chi_{-1}(x+\vartheta) - \chi_{-1}(x) - \vartheta| \leqq |\bar{z}'(\eta)|\,(1 + \chi'_{-1}(\xi))\,|\vartheta|.$$

With respect to (5) and (3) we get

$$\left|\chi_{-1}(x + \vartheta) - \chi_{-1}(x) - \vartheta\right| \leqq (1 + \chi'_{-1}(\xi))\,|\vartheta| \leqq \left(1 + \frac{M}{m}\right)|\vartheta| < \frac{1}{\sqrt{M}},$$

by Lemma 9 it is $|\bar{z}'(\eta)| < 2\sqrt{M}$ and therefore

$$(38)\qquad \left|1 + \bar{z} \circ \chi_{-1}(x)\right| \leqq 2\sqrt{M} \cdot \frac{m + M}{m}\,|\vartheta|\,.$$

Using (36) we have

$$f(x + \vartheta) - f(x) = -(1 + \bar{z} \circ \chi_{-1}(x))\,f(x) + f(x)\,.$$
$$.\int_{\chi_{-1}(x)}^{x} \bar{z}(s)\,[q(s + \vartheta) - q(s)]\,[u(t)\,v(s) - u(s)\,v(t)]\,\mathrm{d}s$$

and (22), (37) and (38) then finally imply

$$|f(x + \vartheta) - f(x)| \leqq |\vartheta| \left(2M\,\frac{m + M}{m} + \sqrt{M} \cdot \frac{\pi^2 ML}{2m^2}\right).$$

Corollary 1. *ω'' is bounded on I.*

Proof. From (2) we have $\omega(t) = \chi_{-1}^{-1}(t)$, $t \in I$ (χ_{-1}^{-1} will here and in the sequel denote the function inverse to the function χ_{-1}). The proof of Corollary 1 now follows from the formulas

$$\omega'(t) = \frac{1}{\chi'_{-1} \circ \chi_{-1}^{-1}(t)}, \quad \omega''(t) = -\frac{\chi''_{-1} \circ \chi_{-1}^{-1}(t)}{[\chi'_{-1} \circ \chi_{-1}^{-1}(t)]^3},$$

(5), (3) and Lemma 12.

We now proceed to the proof of Theorem 1. Formula (9) has been proved in Lemma 10. The remaining formulas will be proved in the order (10), (7) and (8).

From (2) it follows $\omega(t) = \chi_{-1}^{-1}(t)$, $\tilde{\omega}(t) = \tilde{\chi}_{-1}^{-1}(t)$ and therefore

$$(\omega(t) - \tilde{\omega}(t))' = (\chi_{-1}^{-1}(t) - \tilde{\chi}_{-1}^{-1}(t))' = \frac{1}{\chi'_{-1} \circ \chi_{-1}^{-1}(t)} - \frac{1}{\tilde{\chi}'_{-1} \circ \tilde{\chi}_{-1}^{-1}(t)} =$$
$$= \frac{\tilde{\chi}'_{-1} \circ \tilde{\chi}_{-1}^{-1}(t) - \chi'_{-1} \circ \chi_{-1}^{-1}(t)}{\chi'_{-1} \circ \chi_{-1}^{-1}(t) \cdot \tilde{\chi}'_{-1} \circ \tilde{\chi}_{-1}^{-1}(t)}\,.$$

From (3) with respect to (5) and (6) we can deduce that $\chi'_{-1} \circ \chi_{-1}^{-1}(t) \,.\, \tilde{\chi}'_{-1} \circ \chi_{-1}^{-1}(t)$ has lower and upper positive bounds for sufficiently large t. From

$$\tilde{\chi}'_{-1} \circ \tilde{\chi}_{-1}^{-1}(t) - \chi'_{-1} \circ \chi_{-1}^{-1}(t) =$$
$$= (\tilde{\chi}'_{-1} \circ \chi_{-1}^{-1}(t) - \chi'_{-1} \circ \tilde{\chi}_{-1}^{-1}(t)) + (\chi'_{-1} \circ \tilde{\chi}_{-1}^{-1}(t) - \chi'_{-1} \circ \chi_{-1}^{-1}(t)) =$$
$$= (\tilde{\chi}'_{-1} \circ \tilde{\chi}_{-1}^{-1}(t) - \chi'_{-1} \circ \tilde{\chi}_{-1}^{-1}(t)) + \chi''_{-1}(\xi)\,(\tilde{\chi}_{-1}^{-1}(t) - \chi_{-1}^{-1}(t))\,,$$

where ξ is an appropriate number, from Lemmas 10 and 12 and from $\lim_{t\to\infty} (\chi_{-1}^{-1}(t) - \tilde{\chi}_{-1}^{-1}(t)) = 0$, which can be deduced from Lemma 6 and (6), we get (10).

The equality (7) follows from (5), (6), (2), (3), (9), (10), from Corollary 1 and Lemma 6 and from the following chain of equalities

$$(\varphi(t) - \tilde{\varphi}(t))' = (\omega \circ \chi(t) - \tilde{\omega} \circ \tilde{\chi}(t))' = \chi'(t)\,\omega' \circ \chi(t) - \tilde{\chi}'(t)\,\tilde{\omega}' \circ \tilde{\chi}(t) =$$
$$= (\omega' \circ \tilde{\chi}(t) - \tilde{\omega}' \circ \tilde{\chi}(t))\,\tilde{\chi}'(t) + \omega' \circ \tilde{\chi}(t)\,(\chi(t) - \tilde{\chi}(t))' +$$
$$+ \chi'(t)\,(\omega' \circ \chi(t) - \omega' \circ \tilde{\chi}(t)) = (\omega' \circ \tilde{\chi}(t) - \tilde{\omega}' \circ \tilde{\chi}(t))\,\tilde{\chi}'(t) +$$
$$+ \omega' \circ \tilde{\chi}(t)\,(\chi(t) - \tilde{\chi}(t))' + \chi'(t)\,\omega''(\xi)\,(\chi(t) - \tilde{\chi}(t)),$$

where ξ is an appropriate number.

The equality (8) follows from (5), (6), (3), (9), (10), from Lemmas 6 and 11 and from the chain of equalities

$$(\psi(t) - \tilde{\psi}(t))' = (\chi \circ \omega(t) - \tilde{\chi} \circ \tilde{\omega}(t))' = \omega'(t)\,\chi' \circ \omega(t) - \tilde{\omega}'(t)\,\tilde{\chi}' \circ \tilde{\omega}(t) =$$
$$= (\chi' \circ \tilde{\omega}(t) - \tilde{\chi}' \circ \tilde{\omega}(t))\,\tilde{\omega}'(t) + \chi' \circ \tilde{\omega}(t)\,(\omega(t) - \tilde{\omega}(t))' +$$
$$+ \omega'(t)\,(\chi' \circ \omega(t) - \chi' \circ \tilde{\omega}(t)) = (\chi' \circ \tilde{\omega}(t) - \tilde{\chi}' \circ \tilde{\omega}(t))\,\tilde{\omega}'(t) +$$
$$+ \chi' \circ \tilde{\omega}(t)\,(\omega(t) - \tilde{\omega}(t))' + \omega'(t)\,\chi''(\xi)\,(\omega(t) - \tilde{\omega}(t)),$$

where ξ is an appropriate number.

Corollary 2. *For every positive integer n we have*:

$$\lim_{t\to\infty} (\varphi_n(t) - \tilde{\varphi}_n(t))' = 0, \quad \lim_{t\to\infty} (\psi_n(t) - \tilde{\psi}_n(t))' = 0,$$
$$\lim_{t\to\infty} (\chi_n(t) - \tilde{\chi}_n(t))' = 0, \quad \lim_{t\to\infty} (\omega_n(t) - \tilde{\omega}_n(t))' = 0.$$

Proof. As the proofs of all four equalities given in Corollary 2 are much the same, we will carry out only the first of them, that is, $\lim_{t\to\infty} (\varphi_n(t) - \tilde{\varphi}_n(t))' = 0$ by the mathematical induction. The statement for $n = 1$ follows from Theorem 1. Let $\lim_{t\to\infty} (\varphi_k(t) - \tilde{\varphi}_k(t))' = 0$, $k \geqq 1$. Then (1) yields $\varphi_{k+1}(t) = \varphi \circ \varphi_k(t)$, $\tilde{\varphi}_{k+1}(t) = \tilde{\varphi} \circ \tilde{\varphi}_k(t)$, $t \in I$. Therefore $(\tilde{\varphi}_{k+1}(t) - \varphi_{k+1}(t))' = \tilde{\varphi}_k'(t)\,\tilde{\varphi}' \circ \tilde{\varphi}_k(t) - \varphi_k'(t)\,\varphi' \circ \varphi_k(t) = \tilde{\varphi}_k'(t)\,(\tilde{\varphi}' \circ \tilde{\varphi}_k(t) - \varphi' \circ \tilde{\varphi}_k(t)) + \varphi' \circ \tilde{\varphi}_k(t)\,(\tilde{\varphi}_k(t) - \varphi_k(t))' + \varphi_k'(t)\,(\varphi' \circ \tilde{\varphi}_k(t) - \varphi' \circ \varphi_k(t)) = \tilde{\varphi}_k'(t)\,(\tilde{\varphi}' \circ \tilde{\varphi}_k(t) - \varphi' \circ \tilde{\varphi}_k(t)) + \varphi' \circ \tilde{\varphi}_k(t)\,(\tilde{\varphi}_k(t) - \varphi_k(t))' + \varphi_k'(t)\,.\,\varphi''(\eta)\,(\tilde{\varphi}_k(t) - \varphi_k(t))$, where η is an appropriate number. Next we have $\varphi(t) = \omega \circ \chi(t)$ from (2) and therefore it follows from Lemma 11 and from Corollary 1 that φ'' is bounded on I. From Corollary of Theorem 1 in [9] $(\lim_{t\to\infty} (\varphi_k(t) - \tilde{\varphi}_k(t)) = 0)$, from (1), (3), (5), from Theorem 1 and the assumption $\lim_{t\to\infty} (\varphi_k(t) - \tilde{\varphi}_k(t))' = 0$ we come to $\lim_{t\to\infty} (\varphi_{k+1}(t) - \tilde{\varphi}_{k+1}(t))' = 0$.

Theorem 2. *Let the functions p, q fulfil the assumptions* (i), (ii), (iii). *Then*

$$\lim_{t\to\infty} (\varphi(t) - \tilde{\varphi}(t))^{(j)} = 0\,, \quad j = 0, 1, 2, 3\,.$$

Moreover, let n be a positive integer, $p \in C_I^n$, $q \in C_I^n$, $\lim_{t\to\infty} (p(t) - q(t))^{(j)} = 0$ *for* $j = 0, 1, 2, \ldots, n$, let *q have the derivative of the* $(n + 1)$*st order on I and let* $q^{(k)}$ *be bounded on I for* $k = 1, 2, \ldots, n + 1$. *Then*

$$\lim_{t\to\infty} (\varphi(t) - \tilde{\varphi}(t))^{(j)} = 0\,, \quad j = 0, 1, 2, \ldots, n + 3\,.$$

Proof. According to Lemma 6 and Theorem 1 we have for $j = 0, 1$ $\lim_{t\to\infty} (\varphi(t) - \tilde{\varphi}(t))^{(j)} = 0$. The proof of the remaining part of the statement of Theorem 2 proceeds analogously to that of Theorem in [8].

Corollary 3. *Let the functions p, q fulfil the assumption* (i), (ii), (iii). *Then*

$$\lim_{t\to\infty} (\varphi_i(t) - \tilde{\varphi}_i(t))^{(j)} = 0\,, \quad j = 0, 1, 2, 3\,, \quad i = 1, 2, 3, \ldots\,.$$

Moreover, let n be a positive integer, $p \in C_I^n$, $q \in C_I^n$, $\lim_{t\to\infty} (p(t) - q(t))^{(j)} = 0$ *for* $j = 0, 1, 2, \ldots, n$ *and let q have the derivative of the* $(n+1)$*st order on I and* $q^{(k)}$ *be bounded on I for* $k = 1, 2, \ldots, n + 1$. *Then*

$$\lim_{t\to\infty} (\varphi_i(t) - \tilde{\varphi}_i(t))^{(j)} = 0\,, \quad j = 0, 1, \ldots, n + 3\,, \quad i = 1, 2, 3, \ldots\,.$$

Proof. For the proof we use Corollary 1 and proceed analogously to the proof of Corollary of Theorem in [8].

Remark. The statement of Theorem 2 may be deduced from [2] where $q(t) =$ = constant (<0) is assumed and from [3] where the assumption (i) is replaced by a weaker assumption $0 \geqq -m \geqq q(t)$, $t \in I$, but instead of (iii) a stronger assumption $\lim_{t\to\infty} q'(t) = 0$ is introduced. The statement of Theorem 2 was also proved in [8] under the additional assumptions that all solutions of (q) are bounded on I and $\lim_{t\to\infty} (\varphi(t) - \tilde{\varphi}(t))' = 0$.

References

[1] *Bartušek, M.:* Connection between asymptotic properties and zeros of solutions of $y'' = q(t)\,y$. Arch. Math. 3, scripta fac. sci. nat. UJEP Brunensis, VIII : 113—124, 1972.

[2] *Bartušek, M.:* On asymptotic properties and distribution of zeros of solutions of $y'' = q(t)\,y$. Acta F.R.N. Univ. Comenianae Math. *XXXII* (1975), 69—86.

[3] *Bartušek, M.:* On asymptotic behaviour of central dispersions of linear differential equations of the second order. Časopis pro pěstování matematiky, *100* (1975), 255—260.
[4] *Borůvka, O.:* Linear Differential Transformations of the Second Order. The English Universities Press Ltd, London 1971.
[5] *Hartman, P.:* Ordinary differential equations. (in Russian), Moscow 1970.
[6] *Neuman, F.:* A Role of Abel's Equation in the Stability Theory of Differential Equations. Aequat. Math. *6* (1971), 66—70.
[7] *Neuman, F.:* Distribution of zeros of solutions of $y'' = q(t)\,y$ in relation to their behaviour in large. Studia Sci. Math. Hungar. *8* (1973), 177—185.
[8] *Staněk, S.:* Asymptotic properties of dispersions of the differential equations $y'' = q(t)\,y$. Arch. Math. *2*, scripta fac. sci. nat. UJEP Brunensis XI: 85—98, 1975.
[9] *Staněk, S.:* On asymptotic properties of central dispersions of the k-th kind of $y'' = q(t)\,y$ with $k = 1, 2, 3, 4$. Arch. Math. 2, scripta fac. sci. nat. UJEP Brunensis *XII*, 87—98, 1976.
[10] *Tricomi, F. G.:* Differential equations. (in Russian), Moscow 1968.

Author's address: 772 00 Olomouc, Gottwaldova 15, ČSSR (Přírodovědecká fakulta UP).