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MAXIMAL DEDEKIND COMPLETION OF AN ABELIAN LATTICE ORDERED GROUP

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For an archimedean lattice ordered group H we denote by $D(H)$ the Dekekind completion of H (cf. BIRKHOFF [1], Chap. XIII, § 13). Let K be an abelian lattice ordered group. EVERETT [5] defined an extension $M(K)$ of K that was constructed by means of Dedekind cuts of the lattice $(K; \leqq)$; we shall call $M(K)$ the maximal Dedekind completion of K. (In [5], $M(K)$ was said to be the Dedekind completion of K.) The generalized Dedekind completion $D_1(G)$ of a lattice ordered gıoup G has been defined in [8]; cf. also [9]. If G ıs archimedean, then both $M(G)$ and $D_1(G)$ coincide with the Dedekind completion $D(G)$.

A lattice ordered group K will be called M-complete if it is abelian and $M(K) = K$. In § 1 it will be shown that each lattice ordered group G possesses a largest convex M-complete l-subgroup $m(G)$. From this it follows that the class of all M-complete lattice ordered groups is a radical class [7].

In § 2 it will be proved that if an abelian lattice ordered group is a direct product of its l-sugroups B_i $(i \in I)$, then $M(G)$ is a direct product of its l-subgroups $M(B_i)$ $(i \in I)$. An analogous assertion is valid for direct sums.

A natural question arises what are the relations between $M(G)$ and $D_1(G)$ for an abelian lattice ordered group G and, in particular, when do the both completions $M(G)$ and $D_1(G)$ coincide. It will be shown in § 3 that $D_1(G)$ is always an l-subgroup of $M(G)$. In each lattice ordered group G there exists a largest archimedean convex l-subgroup $A(G)$ of G; this is said to be the archimedean kernel of G (cf. [8]). If G is an abelian lattice ordered group such that (i) G is linearly ordered, and (ii) $A(G) \neq \{0\}$, then $M(G) = D_1(G)$. If either (i) or (ii) fails to hold, then $M(G)$ need not coincide with $D_1(G)$. In Proposition 3.11 a necessary and sufficient condition is given for $M(G) = D_1(G)$ to be valid.

Some results on the relationship between G and $M(G)$ are established in § 4. E.g., it is shown that if G fulfils the condition

(F) each upper bounded disjoint subset of G is finite,

then $M(G)$ fulfils the condition (F) as well. CONRAD [3] has proved that if G satisfies

(F), then there exists a system $S = \{A_i\}$ $(i \in I)$ of convex linearly ordered subgroups of G such that G can be constructed from S by means of the operations of direct sums and lexicographic extensions; it will be shown that then $M(G)$ can be constructed in an analogous way from the system $S' = \{M(A_i)\}$ $(i \in I)$.

We shall use the standard notation for lattices and lattice ordered groups (cf. Birkhoff [1], Conrad [2], Fuchs [6]).

1. MAXIMAL DEDEKIND COMPLETION

Let L be a lattice. For $X \subseteq L$ we denote by X^u or X^l, respectively, the set of all upper and the set of all lower bounds of the set X in L. Let $d(L)$ be the system of all sets $(X^u)^l$, where X runs over the system of all nonempty upper bounded subsets of L. The system $d(L)$ is partially ordered by the set theoretical inclusion. Then $d(L)$ is a conditionally complete lattice. The lattice operations in $d(L)$ will be denoted by $\wedge$ and $\vee$. If $(X_i^u)^l$ $(i \in I)$ are subsets of L such that the system $\{(X_i^u)^l\}_{i \in I}$ S is lower bounded in L, then

$$\bigwedge_{i \in I} (X_i^u)^l = \bigcap_{i \in I} (X_i^u)^l ;$$

if the system S is upper bounded in L, then

$$\bigvee_{i \in I} (X_i^u)^l = ((\bigcup_{i \in I} (X_i^u)^l)^u)^l .$$

The mapping $\varphi : L \to d(L)$ defined by

$$\varphi(x) = (\{x\}^u)^l \quad \text{for each} \quad x \in L$$

is an isomorphism of L into $d(L)$. We shall identify x with $\varphi(x)$ for each $x \in L$. Then L is a sublattice of $d(L)$. If X_1 is a subset of L and x_1 is the supremum of X_1 in L, then x_1 is also the supremum of X_1 in $d(L)$; the analogous dual assertion is also valid.

Let $(G; \leqq; +)$ be an abelian lattice ordered group. Consider the lattice $(G; \leqq)$. We define a binary operation $+$ in $d(G)$ as follows: for $Y_1, Y_2 \in d(G)$ we set

$$Y_1 + Y_2 = (\{y_1 + y_2 : y_1 \in Y_1,\ y_2 \in Y_2\}^u)^l .$$

The following results 1.1–1.3 have been proved in [5].

1.1. Lemma. *$(d(G); +)$ is a semigroup. The element $0 \in G$ is a neutral element in $(d(G); +)$. If $a, b, c \in d(G)$, $a \leqq b$, then $c + a \leqq c + b$. The set G is a subsemigroup of $(d(G); +)$.*

1.2. Theorem. *Let $M(G)$ be the set of all elements of $d(G)$ having an inverse in the semigroup $d(G)$. Then*

(a) *$M(G)$ is a group with respect to the operation $+$;*

(b) *$M(G)$ is a sublattice of the lattice $d(G)$.*

From 1.1 and 1.2 it follows that $M(G)$ is a lattice ordered group. From the definition of $M(G)$ we obtain immediately that $M(G)$ is a maximal subsemigroup of $d(G)$ with respect to the property of being a group. G is obviously an l-subgroup of $M(G)$. We shall call $M(G)$ the maximal Dedekind completion of G. If $M(G) = G$, then G is said to be M-complete.

For $a \in d(G)$ we denote

$$u(a) = \{g \in G : a \leqq g\}, \quad l(a) = \{g \in G : g \leqq a\},$$
$$v(a) = \{g_1 - g_2 : g_1 \in u(a),\ g_2 \in l(a)\}.$$

1.3. Theorem. *Let $a \in d(G)$. Then a belongs to $M(G)$ if and only if* $\inf v(a) = 0$ *holds in G.*

Let $x, y \in G$, $x \leqq y$. The set $[x, y] = \{z \in G : x \leqq z \leqq y\}$ is said to be an interval of G. The interval $[x, y]$ is nontrivial, if $x < y$. Let us consider the following condition for a nontrivial interval $[x, y]$ of G:

(m) If $X \neq \emptyset \neq Y$ are subsets of $[x, y]$ such that $x_1 \leqq y_1$ for each $x_1 \in X$ and each $y_1 \in Y$ and

$$\inf \{y_1 - x_1 : x_1 \in X,\ y_1 \in Y\} = 0\,, \tag{1}$$

then $\sup X$ exists in $[x, y]$.

1.4. Lemma. *G is M-complete if and only if each nontrivial interval of G fulfils the condition* (m). *If each nontrivial interval of G^+ fulfils* (m), *then G is M-complete.*

Proof. Let G be M-complete. Suppose that $[x, y]$ is a nontrivial interval of G and let X, Y be as in (m). Then X is upper bounded in G and hence $z = (X^u)^l$ belongs to $d(G)$. Clearly

$$X \subseteq l(z)\,, \quad Y \subseteq u(z)$$

and hence

$$\{y_1 - x_1 : x_1 \in X,\ y_1 \in Y\} \subseteq v(z)\,.$$

From this and from (1) we infer that $\inf v(z) = 0$ holds in G. Thus according to Theorem 1.3, z belongs to $M(G)$. Because G is M-complete, we have $z \in G$. In $M(G)$ the relation $\sup X = z$ is valid; thus $\sup X = z$ holds in G. Obviously $z \in [x, y]$. Hence $z = \sup X$ in $[x, y]$. Therefore $[x, y]$ fulfils (m).

Conversely, assume that each nontrivial interval of G fulfils the condition (m). If $G = \{0\}$, then clearly $M(G) = \{0\}$. Suppose that $G \neq \{0\}$ and let $a \in M(G)$. There are elements $x, y \in G$ with $x < a < y$. Put

$$X = l(a) \cap [x, y]\,, \quad Y = u(a) \cap [x, y]\,.$$

Then $x_1 \leqq y_1$ for each $x_1 \in X$ and each $y_1 \in Y$. If $g_1 \in u(a)$, $g_2 \in l(a)$, then $g_1 \wedge y \in$

$\in Y$, $g_2 \vee x \in X$ and

$$g_1 \wedge y - g_2 \vee x \leqq g_1 - g_2 .$$

From this and from inf $v(a) = 0$ we obtain that (1) is valid. Hence by (m), sup $X = x_1$ exists in G. Thus sup $X = x_1$ holds in $d(G)$. On the other hand, according to the construction of X we have sup $X = a$ in $d(G)$. Hence $a = x_1 \in G$ and thus $G = M(G)$.

Suppose that each nontrivial interval of G^+ fulfils (m) and let $I_1 = [a, b]$ be a nontrivial interval of G. Then $I_2 = [0, b - a]$ is a nontrivial interval of G^+ isomorphic with I_1; hence I_1 fulfils (m) and thus G is M-complete.

1.5. Lemma. *Let $x, y, z \in G$, $x < y < z$. Suppose that both intervals $[x, y]$ and $[y, z]$ fulfil the condition* (m). *Then $[x, z]$ fulfils* (m).

Proof. Let X, Y be nonempty subsets of the interval $[x, z]$ such that $x_1 \leqq y_1$ for each $x_1 \in X$ and each $y_1 \in Y$. Suppose that (1) is valid. Put

$$X' = \{x_1 \wedge y : x_1 \in X\}, \quad X'' = \{x_1 \vee y : x_1 \in X\}$$

and let Y', Y'' be defined analogously. X' and Y' are nonempty subsets of the interval $[x, y]$ and $x_1' \leqq y_1'$ for each $x_1' \in X'$ and each $y_1' \in Y'$. For $x_1 \in X$ and $y_1 \in Y$ we have $y_1 \wedge y - x_1 \wedge y \leqq y_1 - x_1$; hence from (1) we obtain

$$\inf \{y_1' - x_1' : y_1' \in Y', \ x_1' \in X'\} = 0 . \tag{2}$$

Because $[x, y]$ fulfils (m), there exists sup $X' = p$ in $[x, y]$. Similarly, by considering the subsets X'' and Y'' in $[y, z]$ we conclude that $q = \sup X''$ exists in $[y, z]$. Put $r = p + q - y$. For each $x_1 \in X$ and each $y_1 \in Y$ denote

$$x_1' = x_1 \wedge y, \quad x_1'' = x_1 \vee y,$$

$$y_1' = y_1 \wedge y, \quad y_1'' = y_1 \vee y .$$

Then $x_1 = x_1' + x_1'' - y \leqq r$ and $y_1 = y_1' + y_1'' - y \geqq r$ holds for each $x_1 \in X$ and each $y_1 \in Y$. Suppose that $r \neq \sup X$ in $[x, z]$; then there is $r_1 \in [x, z]$ such that $r_1 < r$ and $r_1 \leqq x_1$ for each $x_1 \in X$. Hence $0 < r - r_1 \leqq y_1 - x_1$ for each $x_1 \in X$ and each $y_1 \in Y$, which contradicts (1). Thus $[x, z]$ fulfils (m).

1.6. Corollary. *Let $x, y \in G$, $0 < x$, $0 < y$. If both $[0, x]$ and $[0, y]$ fulfil* (m), *then $[0, x + y]$ fulfils* (m).

Let $m(G)$ be the set of all elements $g \in G$ such that either $g = 0$ or $g \neq 0$ and the interval $[0, |g|]$ fulfils (m).

1.7. Theorem. *For each abelian lattice ordered group G, $m(G)$ is a convex l-subgroup of G. Moreover, $m(G)$ is M-complete and $A \subseteq m(G)$ whenever A is an M-complete convex l-subgroup of G. For each automorphism φ of the lattice ordered group G we have $\varphi(m(G)) = m(G)$.*

Proof. If $g \in m(G)$, then clearly $-g \in m(G)$. Let $g_1, g_2 \in m(G)$, $g_1 \neq 0 \neq g_2$. Put $|g_1| + |g_2| = g$. According to 1.6, the interval $[0, g]$ fulfils (m). We have

$$0 < |g_1 + g_2| \leqq g ,$$

hence the interval $[0, |g_1 + g_2|]$ fulfils (m). Thus $g_1 + g_2 \in m(G)$. Hence $m(G)$ is a subgroup of G. It is obvious that $m(G)$ is upper-directed and convex, hence it is a convex l-subgroup of G. If $x, y \in m(G)$, $x < y$, then $0 < y - x \in m(G)$ and the intervals $[x, y]$, $[0, y - x]$ are isomorphic; hence $[x, y]$ fulfils the condition (m). Thus according to 1.4, $m(G)$ is M-complete. Let A be a convex l-subgroup of G and suppose that A is M-complete. Let $0 < a \in A$. By 1.4, the interval $[0, a]$ of G fulfils (m) and thus $a \in m(G)$. Therefore $A \subseteq m(G)$. Let φ be an automorphism of the lattice ordered group G. Then $\varphi(m(G))$ is isomorphic with $m(G)$, hence $\varphi(m(G))$ is M-complete and hence $\varphi(m(G)) \subseteq m(G)$. Similarly $\varphi^{-1}(m(G)) \subseteq m(G)$ and thus $\varphi(m(G)) = m(G)$.

1.7.1. Theorem. *Let H be a lattice ordered group. There exists a convex l-subgroup $m_1(H)$ of H with the following properties:*

(i) *$m_1(H)$ is M-complete;*

(ii) *if A is a convex l-subgroup of H such that A is M-complete, then $A \subseteq m_1(H)$.*

Proof. There exists a largest convex abelian l-subgroup $a(H)$ of H (cf. [8], Lemma 1.1). Put $a(H) = G$, $m_1(H) = m(G)$. Then $m_1(H)$ is abelian and by 1.7, $m_1(H)$ is M-complete. Let A be a convex l-subgroup of H such that A is M-complete. Thus $A \subseteq a(H)$ and hence according to 1.7 we have $A \subseteq m_1(H)$.

Let $\mathscr{K}$ be a class of lattice ordered groups. Suppose that $\mathscr{K}$ fulfils the following conditions:

(a) $\mathscr{K}$ is closed with respect to isomorphisms.

(b) If $A \in \mathscr{K}$ and B is a convex l-subgroup of A, then $B \in \mathscr{K}$.

(c) If $\{A_i\}$ is a system of convex l-subgroups of a lattice ordered group H such that each A_i belongs to $\mathscr{K}$, then $\bigvee A_i$ belongs to $\mathscr{K}$.

Under these assumptions $\mathscr{K}$ is said to be a radical class [7].

Let $\mathscr{M}$ be the class of all lattice ordered groups that are M-complete. $\mathscr{M}$ obviously fulfils (a). From 1.4 it follows that (b) is valid for $\mathscr{M}$. Let $\{A_i\}$ be a system of convex l-subgroups of a lattice ordered group H such that each A_i is M-complete. According to 1.7.1 we have $A_i \subseteq m_1(H)$ for each A_i and hence $\bigvee A_i \subseteq m_1(H)$. Thus by (b), $\bigvee A_i$ belongs to $\mathscr{M}$ and hence $\mathscr{M}$ fulfils (c). Therefore $\mathscr{M}$ is a radical class.

1.8. Lemma. *Let K be a lattice ordered group, $k \in K$, $X \subseteq K$, $Y \subseteq K$,* $\sup X = k = \inf Y$, $Z = \{y - x : x \in X,\ y \in Y\}$. *Then* $\inf Z = 0$.

Proof. Denote $X = \{x_i\}_{i \in I}$, $Y = \{y_j\}_{j \in J}$. Suppose that $\inf Z \neq 0$. Then there is $0 < t \in K$ such that $y_j - x_i \geqq t$ for each $i \in I$ and each $j \in J$. Hence $y_j \geqq t + x_i$

and thus

$$k < t + k = t + \bigvee_{i \in I} x_i = \bigvee_{i \in I} (t + x_i) \leqq \bigwedge_{j \in J} y_j = k ,$$

which is a contradiction.

1.9. Lemma. *$M(G)$ is M-complete.*

Proof. Denote $M(G) = H$ and let $h_0 \in M(H)$. There are sets $X, Y \subseteq H$ such that $\sup X = h_0 = \inf Y$ holds in $M(H)$. Put $X = \{x_i\}$, $Y = \{y_j\}$. For each x_i there is a subset X_i of G such that $x_i = \sup X_i$ is valid in H. Similarly, for each y_j there is a subset Y_j of G such that $y_j = \inf Y_j$ holds in H. Denote $X' = \bigcup X_i$, $Y' = \bigcup Y_j$. Then

$$\sup X' = h_0 = \inf Y'$$

is valid in $M(H)$. Put $Z' = \{y' - x' : y' \in Y',\ x' \in X'\}$. According to 1.8 we have $\inf Z' = 0$ in $M(H)$, and hence $\inf Z' = 0$ in G. Hence in view of 1.3 there is $h_1 \in H$ such that

$$(*) \qquad \sup X' = h_1 = \inf Y'$$

is valid in H; thus $(*)$ holds in $M(H)$ as well and therefore $h_1 = h_0$. Hence $h_0 \in H$ and so $M(H) = H$.

1.10. Theorem. *Let G be an abelian lattice ordered group. Let H be a lattice ordered group fulfilling the conditions*

(a) *H is M-complete;*

(b) *G is an l-subgroup of H;*

(c) *for each $h \in H$ there are sets $X \subseteq G$, $Y \subseteq G$ such that $\sup X = h = \inf Y$.*

Then there exists an isomorphism φ of $M(G)$ onto H such that $\varphi(g) = g$ for each $g \in G$.

Proof. Let $a \in M(G)$. There exists a subset $X_1 \neq \emptyset$ of G such that $\sup X_1 = a$ holds in $M(G)$. The set X_1 is upper bounded in G. Put $Y = X_1^u$, $X = (X_1^u)^l$ in G. Then we have in $M(G)$ the relations

$$\sup X = a = \inf Y .$$

Put $Z = \{y - x : x \in X,\ y \in Y\}$. According to Lemma 1.8,

$$(2) \qquad \inf Z = 0$$

holds in $M(G)$. Hence (2) is valid in G.

Denote $Y_2 = X^u$, $X_2 = Y_2^l$, where the symbols u and l are taken with respect to H. Put $Z_2 = \{y_2 - x_2 : x_2 \in X_2,\ y_2 \in Y_2\}$. From (2) we obtain

$$(3) \qquad \inf Z_2 = 0 .$$

Hence according to 1.3 there exists $h \in M(H)$ such that

$$\sup X_2 = h = \inf Y_2 .$$

Because H is M-complete, we have $h \in H$. We put $\varphi(a) = h$. Clearly $\varphi(g) = g$ for each $g \in G$. The element h is correctly defined (i.e., it is uniquely defined by the element a); namely, if X'_1 is another subset of G with $\sup X'_1 = a$ in $M(G)$, then $(X'_1)^u = X^u_1$ in G. From the definition of the operations $+$, $\wedge$ and $\vee$ in $M(G)$ it follows that φ is a homomorphism of the lattice ordered group $M(G)$ into H.

Suppose that $\varphi(a) = 0$ for some $a \in M(G)$, $a \neq 0$. Then $\varphi(|a|) = 0$. There is $0 < g \in G$ with $g \leqq |a|$; thus $\varphi(g) = 0$. This is a contradiction, since $\varphi(g) = g$. Hence φ is an isomorphism of $M(G)$ into H.

Let $h \in H$. Put $X' = \{g \in G : g \leqq h\}$, $Y' = \{g \in G : h \leqq g\}$, $Z' = \{y' - x' : : x' \in X', y' \in Y'\}$. From (c) we obtain

$$\sup X' = h = \inf Y' .$$

Hence according to 1.8 we have

$$\inf Z' = 0 \tag{4}$$

in H. From this and from (b) it follows that (4) holds in G as well. Thus by 1.3, there exists $a \in M(G)$ with $\sup X' = a$ and then $\varphi(a) = h$. Hence φ is onto. This completes the proof.

According to 1.2 and 1.9 the lattice ordered group $H = M(G)$ fulfils the conditions (a) and (b). From the construction of $M(G)$ it follows that it satisfies also the condition (c). Hence these conditions may serve as an intrinsic definition of the maximal Dedekind completion of G.

From the definition of $M(G)$ and from the definition of the Dedekind closure of an archimedean lattice ordered group it follows immediately that if G is archimedean, then $M(G)$ coincides with the Dedekind closure $D(G)$ of G.

The lattice ordered group $M(G)$ is said to be the M-closure of G.

1.11. Lemma. *Let G_1 be a convex l-subgroup of G. Let H be as in* 1.10 *and let*

$$H_1 = \{h \in H \text{: there are } g, g' \in G_1 \text{ with } g \leqq h \leqq g'\} .$$

Then

(i) *H_1 is a convex l-subgroup of H;*

(ii) *H_1 is the maximal Dedekind closure of G_1.*

Proof. Let $h_i \in H_1$ $(i = 1, 2)$. There are $g_i, g'_i \in G$ with $g_i \leqq h_i \leqq g'_i$ $(i = 1, 2)$. Then $-g'_i \leqq -h_i \leqq -g'_i$ $(i = 1, 2)$, $g_1 + g_2 \leqq h_1 + h_2 \leqq g'_1 + g'_2$, hence H_1 is a subgroup of H. Since $g_1 \wedge g_2 \leqq x \leqq g'_1 \vee g'_2$ holds for $x = h_1 \wedge h_2$ and for $x = h_1 \vee h_2$, H_1 is a sublattice of H. It is obvious that H_1 is a convex subset of H. Thus (i) is valid.

In order to prove (ii) we have to verify that the conditions (a), (b) and (c) from 1.10 are fulfilled with H and G replaced by H_1 and G_1. Since H_1 is a convex l-subgroup of H by (i), it follows from 1.4 that H_1 is M-complete, thus (a) holds. The condition (b) is obviously valid. Let $h \in H_1$. Since $h \in H$, there are $X, Y \subseteq G$ such that (c) is valid. Put $X_1 = X \cap G_1$, $Y_1 = Y \cap G_1$. From (c) and from the definition of H_1 it follows that $\sup X_1 = h = \inf Y_1$. Hence (ii) holds.

2. DIRECT DECOMPOSITIONS

For the notions and notation concerning direct decompositions of a lattice ordered group cf., e.g., [9], § 2. The notion of a completely subdirect product of lattice ordered groups has been introduced by Šik [10]. Let G be an abelian lattice ordered group and let H be the M-completion of G.

2.1. Lemma. *Let B be a direct factor of G. If G is M-complete, then B is M-complete as well.*

Proof. From Lemma 1.4 it follows that each convex l-subgroup of an M-complete lattice ordered group is M-complete. Since B is a convex l-subgroup of G, the M-completeness of G implies that B is M-complete.

2.2. Proposition. *Let $G = \prod_{i \in I}^0 B_i$. Then G is M-complete if and only if all B_i are M-complete.*

Proof. The assertion "only if" follows from 2.1. Suppose that all B_i are M-complete and let $[a, b]$ be a nontrivial interval of G. Let X, Y be nonempty subsets of $[a, b]$ such that $x \leqq y$ for each $x \in X$ and each $y \in Y$. Assume that (1) is valid. For each $i \in I$ let $a_i = a(B_i)$, $X_i = X(B_i)$ and let b_i, Y_i have the analogous meanings. For each $x \in X$, $y \in Y$, $i \in I$ we have

$$0 \leqq y(B_i) - x(B_i) = (y - x)(B_i) \leqq y - x .$$

Hence from (1) we obtain that

$$(1') \qquad \inf\{y_1 - x_1 : y_1 \in Y_i,\ x_1 \in X_i\} = 0$$

is valid for each $i \in I$. Because B_i is M-complete, $z_i = \sup X_i$ exists B_i for each $i \in I$. Let $z \in G$ be such that $z(B_i) = z_i$ for each $i \in I$. It is easy to verify that $z = \sup X$ is valid in G. Hence in view of 1.4, G is M-complete.

For $X \subseteq G$ we denote

$$c(X) = \{y \in H : x_1 \leqq y \leqq x_2 \quad \text{for some} \quad x_1, x_2 \in X\} .$$

If X is a convex l-subgroup of G, then $c(X)$ is a convex l-subgroup of H.

2.3. Lemma. *Let B be a direct factor of G. Then $c(B)$ is a direct factor of H. For each $g \in G$, $g(B) = g(c(B))$.*

Proof. Let $0 < h \in H$. There are subsets $X, Y \subseteq G^+$ such that $\sup X = h = \inf Y$ holds in H. Hence (1) is valid in G (cf. Lemma 1.8). Put $X_1 = X(B)$, $Y_1 = Y(B)$. From (1) it follows that

$$(1'') \qquad \inf\{y_1 - x_1 : x_1 \in X_1,\ y_1 \in Y_1\} = 0$$

is valid in G. Hence there is $h_1 \in H$ such that $\sup X_1 = h_1 = \inf Y_1$. Clearly $h_1 \in c(B)$ and $0 \leqq h_1 \leqq h$. Assume that there is $t \in c(B)$ such that $h_1 < t \leqq h$. We have $0 < -h_1 + t \in c(B)$, $-h_1 + t \in H$ and hence there is $0 < v \in B$ such that $v \leqq -h_1 + t$. Hence $h_1 + v \leqq h$ and therefore $x_1 + v \leqq y_1$ for each $x_1 \in X_1$ and each $y_1 \in Y_1$, which contradicts (1″). Thus h_1 is the greatest element of the set $\{z \in c(B) : 0 \leqq z \leqq h\}$. From this it follows that $c(B)$ is a direct factor of H and that $h_1 = h(c(B))$.

Now suppose that the element $0 < h$ belongs to G. Then we may suppose that $h \in X$ and hence $h(B)$ is the greatest element of X_1. This implies that $h_1 = \sup X_1 = h(B)$, hence $h(c(B)) = h(B)$. If g is an arbitrary element of G, then there are $g_1, g_2 \in G^+$ with $g = g_1 - g_2$; then $g(B) = g_1(B) - g_2(B) = g_1(c(B)) - g_2(c(B)) = g(c(B))$.

2.4. Lemma. *Let B be a convex l-subgroup of G. Then $c(B)$ is the M-closure of B.*

Proof. We have to verify that the conditions (a), (b) and (c) from 1.10 are valid if we replace G and H by B and $c(B)$. The conditions (a) and (b) are obviously fulfilled. Let $h \in c(B)$. There are elements $b_1, b_2 \in B$ with $b_1 \leqq h \leqq b_2$. Further, there are subsets X, Y of G such that $\sup X = h = \inf Y$. Put

$$X_1 = \{x \vee b_1 : x \in X\}, \quad Y_1 = \{y \wedge b_2 : y \in Y\}.$$

Then $X_1, Y_1 \subseteq B$ and $\sup X_1 = h = \inf Y_1$ holds in $c(B)$. Hence (c) is valid.

2.5. Theorem. *Let G be a completely subdirect product of lattice ordered groups B_i $(i \in I)$. Then $M(G)$ is a completely subdirect product of lattice ordered groups $M(B_i)$ $(i \in I)$.*

Proof. As above, we put $M(G) = H$. According to 2.4 we have $M(B_i) = c(B_i)$ for each $i \in I$. By 2.3, each $c(B_i)$ is a direct factor in H. If $i, j \in I$, $i \neq j$, then $B_i \cap B_j = \{0\}$; this implies immediately that $c(B_i) \cap c(B_j) = \{0\}$. Thus it suffices to verify that for each $h \in H$ we have

$$h = \bigvee_{i \in I} h_i,$$

where $h_i = h(c(B_i))$. Clearly $h_i \leqq h$ for each $i \in I$. Assume that there is $f \in H$ with $f < h$ such that $h_i \leqq f$ for each $i \in I$. There is $0 < g \in G$ with $g \leqq -f + h$. We have

$g_i = g(B_i) > 0$ for some $i \in I$. Hence

$$h_i < h_i + g_i \leqq f + g < h\,,$$

which is a contradiction, because $h_i + g_i \in c(B_i)$ and h_i is the greatest element of the set $\{z \in c(B_i) : 0 \leqq z \leqq h\}$.

2.5.1. Corollary. (*Cf.* ČERNÁK [4].) *Let G be an archimedean lattice ordered group. Suppose that G is a completely subdirect product of lattice ordered groups B_i $(i \in I)$. Then $D(G)$ is a completely subdirect product of lattice ordered groups $D(G_i)$ $(i \in I)$.*

2.6. Lemma. *Let $G = \prod^0 G_i$ $(i \in I)$ and let X, Y be nonempty subsets of G such that $x \leqq y$ for each $x \in X$ and each $y \in Y$. For $i \in I$ let X_i and Y_i be as in 2.2. Suppose that $(1')$ holds for each $i \in I$. Then (1) is valid.*

Proof. Assume that (1) fails to hold. Then there is $0 < g \in G$ such that $y - x \geqq g$ for each $x \in X$ and each $y \in Y$. There is $i \in I$ with $g_i = g(B_i) > 0$. Then $y_i - x_i \geqq g_i$ for each $y_i \in Y_i$ and each $x_i \in X_i$, which is a contradiction.

2.7. Theorem. *Let $G = \prod^0 G_i$ $(i \in I)$. Then $M(G) = \prod^0 M(G_i)$ $(i \in I)$.*

Proof. We shall use the same notation as in 2.6. In view of 2.5 it suffices to prove that if $0 \leqq h^i \in c(B_i)$ for each $i \in I$, then $\bigvee_{i \in I} h^i$ exists in H.

Let $0 \leqq h^i \in c(B_i)$ for each $i \in I$. Then for each $i \in I$ there are nonempty subsets $X^i, Y^i \subseteq B_i$ such that $\sup X^i = h^i = \inf Y^i$ is valid in $c(B_i)$. Let X be the set of all elements $x \in G$ such that $x(B_i) \in X^i$ holds for each $i \in I$ and let Y be defined analogously. Then $X \neq \emptyset \neq Y$. According to 2.6, the condition (1) holds for X, Y and hence there is $h \in H$ such that $\sup X = h = \inf Y$. Put $h_i = h(c(B_i))$ for each $i \in I$. Let i be a fixed element of I and choose $x_i \in X^i$. There exists $x \in X$ such that $x_i = x(B_i)$, hence $x_i \leqq x \leqq h$. From this we obtain $x_i = x_i(c(B_i)) \leqq h(c(B_i)) = h_i$. Thus $h^i = \sup X^i \leqq h_i$. Analogously, by considering the elements $y_i \in Y^i$, we get $h_i \leqq h^i$; thus $h^i = h_i$. By the same method as in 2.5 we can now prove that

$$\bigvee h^i = h\,.$$

2.8. Proposition. *Let $G = \sum^0 B_i$ $(i \in I)$. Then $H = \sum^0 M(B_i)$ $(i \in I)$.*

Proof. According to 2.3 and 2.4, each $M(B_i)$ is a direct factor of H. If $i, j \in I$, $i \neq j$, then $B_i \cap B_j = \{0\}$ and from this we infer that $M(B_i) \cap M(B_j) = \{0\}$. Now in view of 2.5 we have only to verify that for each $x_0 \in H$, the set

$$I(x_0) = \{i \in I : x_0(M(B_i)) \neq 0\}$$

is finite. Since each element of H is a difference of positive elements, it suffices to consider the case $x_0 > 0$. There is $y \in G$ with $x_0 < y$. According to 2.3 we have $y(M(B_i)) = y(B_i)$, hence

$$I(y) = \{i \in I : y(B_i) \neq 0\} ;$$

thus $I(y)$ is finite. For each $i \in I$,

$$0 \leqq x_0(M(B_i)) \leqq y(M(B_i)) ;$$

therefore the set $I(x_0)$ is finite.

3. THE ARCHIMEDEAN KERNEL

Let G be a lattice ordered group. An element $0 < g \in G$ is called archimedean in G if for each $0 < x \in G$ there is a positive integer n such that $nx \nless g$. A lattice ordered group G is archimedean if and only if all its strictly positive elements are archimedean in G. Let $A(G)$ be the set of all elements $g \in G$ such that either $g = 0$ or $|g|$ is archimedean in G. It has been proved in [8] that $A(G)$ is a closed l-ideal of G; it is said to be the archimedean kernel of G.

Assume that G is abelian and let H be the maximal Dedekind completion of G.

3.1. Proposition. *The archimedean kernel of H is the set of all elements $h \in H$ with the property ihat $|h| = \sup Z$ for a subset $Z \subseteq A(G)$.*

Proof. Let $h \in H$, $|h| = \sup Z$ for some $Z \subseteq A(G)$, $h \neq 0$. Without loss of generality we can suppose that $0 < z$ for each $z \in Z$. Assume that there is $z \in Z$ that fails to be archimedean in H. Hence there is $0 < x \in H$ such that $nx \leqq z$ for each positive integer n. There exists $0 < g \in G$ with $g \leqq x$; hence $ng \leqq z$ for each positive integer n, which is a contradiction, because $z \in A(G)$. Thus $Z \subseteq A(H)$; since $A(H)$ is a closed l-ideal in H, we infer that $|h| \in A(H)$. This implies $h \in A(H)$.

Conversely, let $h \in A(H)$, $h \neq 0$. Then $0 < |h| \in A(H)$. There exists a subset $X \subseteq G^+$ with $\sup X = |h|$. If some $0 \neq x \in X$ is not archimedean in G, then $|h|$ fails to be archimedean in H, which is a contradiction. Thus $X \subseteq A(G)$.

3.1.1. Corollary. $c(A(G)) \subseteq A(H)$.

3.2. Proposition. *Let G be a linearly ordered group. Then $c(A(G)) = A(H)$.*

Proof. If $A(G) = \{0\}$, then it follows from 3.1 that $A(H) = \{0\}$. Suppose that $A(G) \neq \{0\}$. Hence according to 3.1 we have $A(H) \neq \{0\}$. Let $0 < h \in A(H)$ and assume that h does not belong to $c(A(G))$. Hence $a < h$ for each $a \in A(G)$. Let $h_1 \in A(H)$, $h_1 \geqq 0$. According to 3.1 there is $Z \subseteq A(G)$ with $\sup Z = h_1$. Hence $h_1 \leqq h$ and therefore h is the greatest element of $A(H)$. But no lattice ordered group distinct from $\{0\}$ can have a greatest element and thus we arrive at a contradiction.

Now we can ask whether $c(A(G)) = A(H)$ is valid for each abelian lattice ordered group G. The following example shows that the answer is negative.

3.3. Example. Let N be the set of all positive integers and let A be the additive group of all integers with the natural linear order. For each $n \in N$ let

$$G_n = B_n \circ C_n$$

(the symbol $\circ$ denoting the lexicographic product, cf. [6]) with $B_n = C_n = A$ for each $n \in N$. Put

$$G_0 = \prod_{n\in N} G_n .$$

The elements $g \in G_0$ will be written as pairs $g = (g_1, g_2)$, where

$$g(n) = (g_1(n), g_2(n)), \quad g_1(n) \in B_n, \quad g_2(n) \in C_n .$$

Denote

$$s_2(g) = \{n \in N : g_2(n) \neq 0\} .$$

Let G be the set of all elements $g \in G_0$ fulfilling the following conditions:

(i) the set $s_2(g)$ is finite;

(ii) there exists $n_0 \in N$ such that $g_1(n_1) = g_1(n_2)$ for each pair $n_1, n_2 \in N$ with $n_0 \leqq n_1 \leqq n_2$.

Then G is an l-subgroup of the lattice ordered group G_0. Clearly

$$A(G) = \{g \in G : g_1(n) = 0 \quad \text{for each} \quad n \in N\} .$$

We denote by g^0 the element of G_0 satisfying $g_1^0(n) = 0$ and $g_2^0(n) = 1$ for each $n \in N$. Put

$$X = \{g \in G : g < g^0\} , \quad Y = \{g \in G : g > g^0\} ,$$
$$Z = \{y - x : x \in X, \; y \in Y\} .$$

We have $\emptyset \neq Z \subset G^+$. Assume that there is $0 < u \in G$ such that $u \leqq z$ for each $z \in Z$. Then there is $n_0 \in N$ with $u(n_0) > 0$.

Define $x, y \in G$ as follows:

$x_1(n) = 0$ for each $n \in N$,

$x_2(n_0) = 1$ and $x_2(n) = 0$ for each $n \in N$, $n \neq n_0$;

$y_1(n_0) = 0$ and $y_1(n) = 1$ for each $n \in N$, $n \neq n_0$;

$y_2(n_0) = 1$ and $y_2(n) = 0$ for each $n \in N$, $n \neq n_0$.

Then $x \in X$, $y \in Y$ and hence $u \leqq y - x$. Thus

$$u(n_0) \leqq (y - x)(n_0) = 0 ,$$

which is a contradiction. Therefore

$$\inf Z = 0 .$$

Hence according to 1.3, the element

$$h = \sup X = \inf Y$$

exists in $M(G) = H$. Since $X \subset A(G)$, it follows from 3.1 that h belongs to $A(H)$.

$A(G)$ is a complete lattice ordered group. This implies that $c(A(G)) = A(G)$.

A subset $\emptyset \neq S$ of G such that $0 < s$ for each $s \in S$ and $s_1 \wedge s_2 = 0$ for any two distinct elements of S is said to be disjoint. Each subset of $A(G)$ that is upper bounded in $A(G)$ is finite. There exists an infinite disjoint subset of X. Since h is an upper bound of X, the element h cannot belong to $A(G)$ and thus $h \notin c(A(G))$. Therefore $c(A(G)) \neq A(H)$.

Let us recall the notion of the generalized Dedeking completion of a lattice ordered group G that was introduced in [8] (without assuming the commutativity of G).

Let $D_1(G)$ be a lattice ordered group fulfilling the following conditions:

(i) G is an l-subgroup of $D_1(G)$.

(ii) $D(A(G))$ is an l-ideal in $D_1(G)$.

(iii) If $x \in G$ and X is a nonempty subset of $x + A(G)$ such that X is upper bounded in $x + A(G)$, then there is $x_0 \in D_1(G)$ with $\sup X = x_0$.

(iv) For each $x_0 \in D_1(G)$ there exist $x \in G$ and $X \subseteq x + A(G)$ such that X is upper bounded in $x + A(G)$ and $x_0 = \sup X$.

Under these conditions $D_1(G)$ is said to be a generalized Dedekind completion of G. The following results have been obtained in [8]:

(a) Each lattice ordered group possesses a generalized Dedekind completion.

(b) If $D_1(G)$ and $D_2(G)$ are generalized Dedekind completions of G, then there is an isomorphism φ of $D_1(G)$ onto $D_2(G)$ such that $\varphi(g) = g$ for each $g \in G$.

Again, let G be abelian and let H be as above. Denote $G_1 = \{h \in H :$ there is $x \in G$ and there are $g_1, g_2 \in x + A(G)$ with $g_1 \leqq h \leqq g_2\}$.

3.4. Lemma. *G_1 is an l-subgroup of H.*

Proof. Let $h_1, h_2 \in G_1$. There are elements $x, y, g_i \in G$ $(i = 1, 2, 3, 4)$ such that $g_1, g_2 \in x + A(G)$, $g_3, g_4 \in y + A(G)$, $g_1 \leqq h_1 \leqq g_2$, $g_3 \leqq h_2 \leqq g_4$. Then $-g_2 \leqq -h_1 \leqq -g_1$ and $-g_1, -g_2$ belong to $-x + A(G)$. Let $f \in \{\wedge, \vee, +\}$. We have

$$g_1 f g_3 \leqq h_1 f h_2 \leqq g_2 f g_4$$

and both $g_1 f g_3$, $g_2 f g_4$ belong to $(x f y) + A(G)$. Hence G_1 is an l-subgroup of H.

3.5. Proposition. *G_1 is a generalized Dedekind completion of G.*

Proof. According to 3.4, the condition (i) is valid. From the construction of H it follows immediately that $c(A(G))$ is the Dedekind completion of $A(G)$, hence (ii) holds.

Let x and X be as in (iii), $X = \{x_i\}$. Denote $X_1 = \{x_i - x\}$. Then $X_1 \subseteq A(G)$. There is $y \in x + A(G)$ such that y is an upper bound of X. Thus $y - x$ is an upper bound of X_1 in $A(G)$. Hence $\sup X_1 = x_1$ exists in $D(A(G)) = c(A(G))$. Put $x_0 = x_1 + x$. We have

$$x_0 = \sup \{x_i - x\} + x = \sup \{x_i\} = \sup X$$

in H and clearly $x_i \leqq x_0 \leqq y$ for each $x_i \in X$; thus $x_0 \in G_1$. Hence (iii) is valid.

Let $x_0 \in G_1$. Then there are elements $x, g_i, g_2 \in G$ such that $g_i \in x + A(G)$ $(i = 1, 2)$ and $g_1 \leqq x_0 \leqq g_2$. At the same time we have $x_0 \in H$, hence there is a subset $X \subseteq G$ with $X \neq \emptyset$ such that

$$\sup X = x_0$$

holds in H. Put $X = \{x_i\}$, $Y = \{(x_i \vee g_1) \wedge g_2\}$. Then $\emptyset \neq Y \subseteq x + A(G)$ and

$$\sup Y = x_0$$

is valid in G_1. Therefore the condition (iv) holds. Hence G_1 is a generalized Dedekind completion of G.

In [8], the question was proposed what relations exist between the maximal Dedekind completion $M(G)$ and the generalized Dedekind completion $D_1(G)$ of G. From 3.4 and 3.5 it follows that $D_1(G)$ is an l-subgroup of $M(G)$. The following example shows that $D_1(G)$ does not, in general, coincide with $M(G)$.

3.6. Example. There exists a linearly ordered group G such that $D_1(G) = G \neq M(G)$.

Let N and A be as in 3.3. For each $n \in N$ let $A_n = A$ and consider the lexicographic product

$$G_0 = \Gamma_{n \in N} A_n$$

(cf. [6]). For $g \in G_0$ put $s(g) = \{n \in N : g(n) \neq 0\}$. Let G be the set of all $g \in G_0$ such that the set $s(g_0)$ is finite. Then G is an l-subgroup of G_0, thus G is linearly ordered. Obviously $A(G) = \{0\}$ and hence according to [8], we have $D_1(G) = G$. Let $g_0 \in G_0$ be such that $g_0(n) = 1$ for each $n \in N$. Put

$$X = \{g \in G : g < g_0\}\,, \quad Y = \{g \in G : g > g_0\}\,, \quad Z = \{y - x : y \in Y,\ x \in X\}\,.$$

Let $0 < u \in G$. There is $n_0 \in N$ such that $u(n_0) > 0$ and $u(n) = 0$ for each $n \in N$ with $b < n_0$. Let $x, y \in G_0$ be defined as follows:

$x(n) = 1$ for each $n < n_0$ and $x(n) = 0$ otherwise;

$y(n) = 1$ for each $n \leqq n_0$, $y(n_0 + 1) = 2$, and $y(n) = 0$ for $n > n_0 + 1$.

Then $x \in X$, $y \in Y$ and $y - x < u$. Thus $\inf Z = 0$. Hence by 1.3 there exists $h \in M(G)$ such that $\sup X = h = \inf Y$ holds in $M(G)$. Assume that h belongs to G. Then $\sup X = h$ is valid in G. But it is not difficult to verify that $\sup X$ does not exist in G, which is a contradiction. Hence $G \neq M(G)$.

3.7. Lemma. *Let $x_0 \in H$, $X \subset G$, $\sup X = x_0$, $0 < g \in G$. Then there is $x \in X$ with $x + g$ non $\leqq x_0$.*

Proof. Suppose that $x + g \leqq x_0$ for each $x \in X$. Put $X = \{x_i\}$. Then

$$x_0 < x_0 + g = \sup \{x_i\} + g = \sup \{x_i + g\} \leqq x_0 ,$$

a contradiction.

3.8. Proposition. *Let G be linearly ordered, $A(G) \neq \{0\}$. Then $M(G) = D_1(G)$.*

Proof. Let $0 < x_0 \in M(G)$. There exists $X \subset G$ such that $\sup X = x_0$ is valid in $M(G)$. Choose $0 < a \in A(G)$. According to 3.7 there exists $x \in X$ such that $x + a$ non $\leqq x_0$. Hence

$$x \leqq x_0 < x + a$$

and thus $x_0 \in G_1 = D_1(G)$. Therefore $(M(G))^+ \subseteq D_1(G)$ and this implies $M(G) \subseteq \subseteq D_1(G)$. Thus $M(G) = D_1(G)$.

3.9. Example. There exists a lattice ordered group G' with $A(G') \neq \{0\}$ such that $D_1(G') = G' \neq M(G')$.

Let G be as in 3.6 and let G_0 be the additive group of all reals with the natural linear order. Put $G' = G_0 \times G$. According to [9], we have

$$D_1(G') = D_1(G_0) \times D_1(G) = G_0 \times G = G' ,$$

since $A(G_0) = G_0$, $D_1(G_0) = D(G_0) = G_0$ and $A(G) = \{0\}$. On the other hand, from 2.7 we infer

$$M(G') = M(G_0) \times M(G) = G_0 \times M(G) \neq G' ,$$

since $M(G) \neq G$ (cf. 3.6).

A convex l-subgroup B of G is said to be a large l-subgroup of G if $B \cap K \neq \{0\}$ for each convex l-subgroup $K \neq \{0\}$ of G. In other wo..ds, B is large in G if for each $0 < g \in G$ there exists $0 < g_1 \in B$ with $g_1 \leqq g$.

3.10. Proposition. *Let G be a direct product of linearly ordered groups. Suppose that $A(G)$ is a large l-subgroup of G. Then $M(G) = D_1(G)$.*

Proof. Let $G = \prod_{i \in I}^0 G_i$, where all lattice ordered groups G_i are linearly ordered. Without loss of generality we can assume that $G_i \neq \{0\}$ for each $i \in I$. Since $A(G)$ is large in G, $A(G) \cap G_i \neq \{0\}$ for each $i \in I$. Clearly $A(G_i) = A(G) \cap G_i$ and hence according to 3.8 we have $M(G_i) = D_1(G_i)$ for each $i \in I$. Hence using 2.7 and [9], 2.17 we obtain

$$M(G) = \prod_{i \in I}^0 M(G_i) = \prod_{i \in I}^0 D_1(G_i) = D_1(G) .$$

3.10.1. Open question: Let $A(G)$ be a large l-subgroup of G. Is then $M(G) = D_1(G)$?

3.11. Proposition. *Let $A(G) \neq 0$. The following conditions for G are equivalent:*

(a) *If X, Y are nonempty subsets of G such that* (i) *$x < y$ for each $x \in X$ and each $y \in Y$, and* (ii) $\inf\{y - x : x \in X, y \in Y\} = 0$, *then there are elements $0 < a \in A(G)$ and $g \in G$ such that for each $x \in X$ and $y \in Y$ we have $x \vee g \leqq y$, $x \leqq y \wedge (g + a)$.*

(b) $M(G) = D_1(G)$.

Proof. Suppose that (a) holds and $x_0 \in M(G)$. According to 1.3 there are subsets X, Y of G such that (i) and (ii) are valid and $\sup X = x_0 = \inf Y$. From (a) we obtain $g \leqq x_0 < g + a$ and hence $x_0 \in G_1 = D_1(G)$. Thus $M(G) \subseteq D_1(G)$ and therefore $M(G) = D_1(G)$.

Conversely, suppose that $M(G) = D_1(G)$ and let X, Y be subsets of G fulfilling (i) and (ii). According to 1.3 there is $x_0 \in M(G)$ with $\sup X = x_0 = \inf Y$. By the assumptions we have $x_0 \in D_1(G)$ and hence there are elements $g \in G$ and $0 < a \in A(G)$ such that $g \leqq x_0 < g + a$. Hence for each $x \in X$ and each $y \in Y$ we have $g \vee x \leqq y$, $x \leqq y \wedge (g + a)$. Thus (a) is valid.

A lattice ordered group G is said to be generalized complete [9] if $D_1(G) = G$.

3.12. Proposition. *If G is M-complete, then it is genereali̇zed complete.*

Proof. Suppose that G is M-complete. From 1.4 it follows that each convex l-subgroup of G is M-complete. Hence $A(G)$ is M-complete and so, being archimedean, it is complete. Therefore, according to [8], G is generalized complete.

In [10] it was shown that in each lattice ordered group G (that need not be commutative), the greatest convex generalized complete l-subgroup $d_1(G)$ exists. From 3.12 we obtain immediately:

3.12.1. Corollary. *Let G be an abelian lattice ordered group. Then $m(G) \subseteq d_1(G)$.*

3.12.2. Example. There exists an abelian lattice ordered group G such that $m(G) \subset d_1(G)$. Let G be as in Example 3.6. Then $D_1(G) = G$, hence G is generalized complete and so $d_1(G) = G$. It is not hard to verify that $m(G) = \{0\}$.

4. SOME FURTHER PROPERTIES OF $M(G)$

Let G be an abelian lattice ordered group. In this section some relations between G and $H = M(G)$ will be investigated.

4.1. Proposition. *G is a dense l-subgroup of H.*

Proof. For each $0 < x_0 \in H$ there exists $\emptyset \neq X \subseteq G$ such that $\sup X = x_0$ holds in H, hence for each $0 \neq x \in X$ we have $0 < |x| \leqq x_0$, $|x| \in G$.

It is easy to verify that if $X = \{x_i\} \subset H$, $x_0 \in H$, $\sup \{x_i\} = x_0$, then for each positive integer n we have $\sup \{nx_i\} = nx_0$, and dually.

4.2. Proposition. *If G is divisible, then H is divisible.*

Proof. Suppose that G is divisible. It suffices to verify that for each positive integer n and for each $0 < x_0 \in H$ there exists $y_0 \in H$ with $ny_0 = x_0$.

Let n be positive integer and let $0 < x_0 \in H$. There exist subsets X, Y of G^+ such that $\sup X = x_0 = \inf Y$ holds in H and $\inf Z = 0$, where $Z = \{y - x : y \in Y, x \in X\}$. Let $X = \{x_i\}$, $Y = \{y_j\}$. Denote $X_1 = \{(1/n)\, x_i\}$, $Y_1 = \{(1/n)\, y_j\}$. Then $x_1 < y_1$ for each $x_1 \in X_1$ and each $y_1 \in Y_1$. Put $Z_1 = \{y_1 - x_1 : x_1 \in X_1, y_1 \in Y_1\}$. For each $z_1 \in Z_1$ we have $0 < z_1$ and there is $z \in Z$ with $z_1 = (1/n)\, z$; from this and from $\inf Z = 0$ we infer that $\inf Z_1 = 0$. Thus according to 1.3 there exists $y_0 \in H$ such that

$$\sup X_1 = y_0 = \inf Y_1 .$$

4.3. Proposition. *Let G be a vector lattice. Then H is a vector lattice.*

Proof. Let $0 < x_0 \in H$ and let r be a positive real. Let X, Y and Z be as in 4.2. Put $X_1 = \{rx_i\}$, $Y_1 = \{ry_j\}$, $Z_1 = \{y_1 - x_1 : x_1 \in X_1, y_1 \in Y_1\}$. We have $y_1 > x_1$ for each $x_1 \in X_1$ and each $y_1 \in Y_1$. Suppose that $\inf Z_1 = 0$ fails to hold in G. Then there is $0 < u \in G$ such that $u \leqq z_1$ for each $z_1 \in Z_1$. Hence $0 < (1/r)\, u \leqq (1/r)\, z_1$ for each $z_1 \in Z_1$ and thus $(1/r)\, u \leqq z$ for each $z \in Z$, which is a contradiction. Thus $\inf Z_1 = 0$. Hence there is $y_0 \in H$ with

$$\sup X_1 = y_0 = \inf Y_1 .$$

Now we define rx_0 by putting $rx_0 = y_0$. If t_0 is any element of H, then we put $rt_0 = rt_0^- - rt_0^+$ and $(-r)\, t_0 = -(rt_0)$. It is a routine to verify that under this definition of multiplication of elements of H by reals, H turns out to be a vector lattice such that G is a vector sublattice of H.

Let K be a convex l-subgroup of G with the property that $g > k$ for each $k \in K$ provided $0 < g \in G \setminus K$. Under this assumption G is called a lexicographic extension of K and we write $G = \langle K \rangle$. If, moreover, $G \neq K$, then G is said to be a proper lexicographic extension of K. From $G = \langle K \rangle$ it follows that the factor l-group G/K is linearly ordered and that, for $g_1, g_2 \in G$ with $g_1 + K \neq g_2 + K$, the relation $g_1 + K < g_2 + K$ is valid in G/K if and only if $g_1 < g_2$ holds.

4.4. Proposition. *Let K be a convex l-subgroup of G, $\{0\} \neq K \neq G$, such that $G = \langle K \rangle$. Then $H = \langle c(K) \rangle$ and $c(K) \neq H$.*

Proof. Since $G \neq K$, there exists $g \in G^+ \setminus K$. Then $g > k$ for each $k \in K$, thus g does not belong to $c(K)$ and hence $c(K) \neq H$. Let $x_0 \in H \setminus c(K)$. Put

$$X = \{x \in G : x \leqq x_0\}, \quad Y = \{y \in G : y \geqq x_0\}.$$

If $X \cap K \neq \emptyset \neq Y \cap K$, then $x_0 \in c(K)$, a contradiction. Assume that $Y \cap K = \emptyset$. Then $y > k$ for each $y \in Y$ and each $k \in K$. Hence $K \subseteq X$. Now we distinguish two cases.

(a) Suppose that there is $x \in X$ such that $x > k$ for each $k \in K$. Since $x_0 = \sup X$, we have $x_0 > k$ for each $k \in K$ and thus $x_0 > x_1$ for each $x_1 \in c(K)$.

(b) Suppose that no $x \in X$ exceeds the whole set K. Then for each $x \in X$, either $x \in K$ or $x < k$ for each $k \in K$. From this and from $K \subseteq X$ we obtain

$$x_0 = \sup K.$$

There exists $0 < k \in K$. Put $K = \{k_i\}$. Then in H we have $k + K = K$ and

$$x_0 < k + x_0 = k + \bigvee k_i = \bigvee (k + k_i) = x_0$$

which is a contradiction.

The case $X \cap K \neq \emptyset$ is analogous. We have verified that $H = \langle c(K) \rangle$ is valid.

4.5. Proposition. *Let K be a convex l-subgroup of G, $\{0\} \neq K \neq G$, $G = \langle K \rangle$. Then the linearly ordered groups G/K and $H/c(K)$ are isomorphic.*

Proof. According to 4.4 we have $H = \langle c(K) \rangle$, hence $H/c(K)$ is a linearly ordered group. For $g \in G$ we denote

$$\varphi(g + K) = g + c(K).$$

Let $g_1, g_2 \in G$. If $g_1 + K = g_2 + K$, then $g_1 + c(K) = g_2 + c(K)$, hence φ is a mapping of the set G/K into $H/c(K)$. Suppose that $g_1 + c(K) = g_2 + c(K)$; hence $g_1 - g_2 \in c(K)$ and thus $g_1 - g_2 \in K$. Therefore φ is a monomorphism. Obviously φ is regular with respect to the operation $+$. If $g_1 + K < g_2 + K$ in G/K, then $g_1 < g_2$ and hence $g_1 + c(K) < g_2 + c(K)$; conversely, if $g_1 + c(K) < g_2 + c(K)$, then $g_1 + K < g_2 + K$. Hence φ is an isomorphism of the linearly ordered group G/K into $H/c(K)$.

Let x_0, X and Y be as in the proof of 4.4. Put

$$\bar{X} = \{g + K : g \in G, (g + K) \cap X \neq \emptyset\}$$

and let $\bar{Y}$ be defined analogously. There exists $0 < k \in K$. If X is a join of some classes $g + K$, then $x + k \in X$ for each $x \in X$ and thus $x + k \leqq x_0$ for each $x \in X$, which contradicts 3.7. Hence there is $g \in G$ such that

$$(g + K) \cap X \neq \emptyset, \quad g + K \not\subset X.$$

Thus there is $g_1 \in g + K$ such that $g_1 \in X$. If $g' \in G$, $g' + K < g + K$, then $g' + K \subset X$. If $g'' \in G$, $g'' + K > g + K$, then g'' cannot belong to X, since $g'' \in X$

would imply $g + K \subset X$, which is a contradiction. Hence

$$X = [(g + K) \cap X] \cup X',$$

where X' is the join of all $g' + K$ with $g' + K < g + K$. Thus $g'' + K \subset Y$ for each $g'' + K > g + K$. Similarly as we did for X we can now verify that Y cannot be a join of some classes $g'' + K$ with $g'' \in G$. From this we infer that $(g + K) \cap Y \neq \emptyset$. Thus there is $g_2 \in g + K$ with $g_2 \in Y$. Then we have

$$g_1 - g \leqq x_0 - g \leqq g_2 - g$$

and $g_1 - g$, $g_2 - g \in K$, thus $x_0 - g \in c(K)$ and so $x_0 \in g + c(K)$, $x_0 + c(K) = g + c(K)$. Hence φ is an epimorphism. This completes the proof.

4.6. Lemma. *Let $\{P_i\}$ $(i \in I)$ be an upper-directed system of convex l-subgroups of G. Then $\bigcup_{i \in I} c(P_i) = c(\bigcup_{i \in I} P_i)$.*

The proof is a routine and so it will be omitted. From 4.6 and from 1.11 (ii) we obtain

4.6.1. Corollary. *Let $\{P_i\}$ $(i \in I)$ be an upper-directed system of convex l-subgroups of G. Then $\bigcup_{i \in I} M(P_i) = M(\bigcup_{i \in I} P_i)$.*

A subset $B = \{g_i\}_{i \in I}$ of G is said to be a basis for G if B is a maximal disjoint subset of G and the interval $[0, g_i]$ of G is a chain for each $i \in I$.

4.7. Proposition. *Let $B = \{g_i\}_{i \in I}$ be a basis for G. Then B is a basis for H.*

Proof. Let $i \in I$ and let A_i be the interval in H with the endpoints 0, g_i. Suppose that A_i fails to be a chain. The there are elements $0 < x_0$ and $0 < y_0$ in A_i such that $x_0 \wedge y_0 = 0$. Hence there are elements x_1, y_1 in G such that $0 < x_1 \leqq x_0$, $0 < y_1 \leqq y_0$ Then $x_1 \wedge y_1 = 0$ and both x_1, y_1 belong to the interval A_i' in G with the endpoints 0 and g_i; since A_i' is a chain, we have a contradiction. Hence A_i is a chain for each $i \in I$. Let $0 < z_0 \in H$. There is $z_1 \in G$ with $0 < z_1 \leqq z_0$. Further (since B is maximal disjoint in G), there exists $i \in I$ with $0 < g_i \wedge z_1$. Hence $0 < g_i \wedge z_0$. Thus B is a basis for H.

Let us consider the following condition for G (cf. [3]):

(F) Each disjoint subset of G that is upper bounded in G is finite.

4.8. Proposition. *Suppose that G fulfils* (F). *Then H fulfils* (F).

Proof. Let $0 < x_0 \in H$. Assume that there exists an infinite disjoint system $\{x_i\}$ $(i \in I)$ in H such that x_0 exceeds all x_i. There is $y \in G$ with $x_0 \leqq y$. For each $i \in I$ there is $g_i \in G$ with $0 < g_i \leqq x_i$. Hence $\{g_i\}$ $(i \in I)$ is an infinite disjoint system in G and y exceeds each g_i; this is a contradiction.

4.9. Definition. (Cf. [3].) A lattice ordered group G is said to be a *lexicographic sum* of the system $\Gamma^0 = \{A_i^0\}$ $(i \in I_0)$ of its convex l-subgroups if there exists an ordinal β and convex l-subgroups A^α of G for each α with $0 \leqq \alpha < \beta$ such that the following conditions are fulfilled:

(i) $A^0 = \sum^0 A_i^0$ $(i \in I_0)$, $A^{\alpha_1} \subseteq A^{\alpha_2}$ whenever $0 \leqq \alpha_i < \beta$ $(i = 1, 2)$ and $\alpha_1 < \alpha_2$;
(ii) $\bigcup_{\alpha<\beta} A^\alpha = G$;
(iii) for each ordinal α with $0 < \alpha < \beta$ there exists a system $\Gamma^\alpha = \{A_i^\alpha\}$ $(i \in I_\alpha)$ of convex l-subgroups of G with $A^\alpha = \sum^0 A_i^\alpha$ $(i \in I)$ such that

(a) if α is non-limit, $\alpha = \gamma + 1$, and if $A_i^\alpha \in \Gamma^\alpha$, then either A_i^α equals to some l-subgroup belonging to Γ^γ, or there exists a subset I of I_γ with card $I > 0$ and a convex l-subgroup A of G such that

$$A = \sum^0 A_i^\gamma \ (i \in I)$$

and A_i^α is a proper lexicographic extension of A;
(b) if α is a limit ordinal and $A_i^\alpha \in \Gamma^\alpha$, then there exists a system $\{A_{i(\gamma)}^\gamma\}$ $(\gamma < \alpha)$ such that $A_{i(\gamma)}^\gamma$ belongs to Γ^γ for each $\gamma < \alpha$, $A_{i(\gamma_1)}^{\gamma_1} \subseteq A_{i(\gamma_2)}^{\gamma_2}$ whenever $\gamma_1 < \gamma_2$ and $A_i^\alpha = \bigcup_{\gamma<\alpha} A_{i(\gamma)}^\gamma$.

Now let us again suppose that G fulfils (F). Let $G \neq \{0\}$. It is easy to verify that then G possesses a basis $\{b_i\}$ $(i \in I_0)$ and for each $i \in I_0$ there exists a largest convex linearly ordered subgroup A_i^0 of G containing b_i. The following theorem has been proved by Conrad [3]:

4.10. Theorem. *Under the above notation, G is a lexicographic sum of the system $\{A_i^0\}$ $(i \in I_0)$.*

From 4.8, 4.10 and 4.7 we obtain:

4.11. Theorem. *Let $G \neq \{0\}$. Suppose that G fulfils (F) and let $\{b_i\}$ $(i \in I_0)$ be a basis for G. For each $i \in I_0$ let B_i^0 be the largest convex linearly ordered subgroup of H containing b_i. Then H is a lexicographic sum of the system B_i^0 $(i \in I_0)$.*

A more detailed description of the representation of H as a lexicographic sum of linearly ordered groups is contained in the following theorem. Let G be as in 4.10. Hence there is an ordinal β and there are systems A^α $(\alpha < \beta)$ of convex l-subgroups of G such that the conditions (i)–(iii) from 4.9 are fulfilled. Then the following assertion is valid:

4.12. Theorem. *For each ordinal α with $0 < \alpha < \beta$, $c(A^\alpha)$ is a convex l-subgroup of H such that the following conditions are fulfilled:*

(i_1) *$c(A^0) = \sum^0 c(A_i^0)$ $(i \in I)$ and $c(A^{\alpha_1}) \subseteq c(A^{\alpha_2})$ whenever $0 \leqq \alpha_i < \beta$ $(i = 1, 2)$ and $\alpha_1 < \alpha_2$;*
(ii_1) *$\bigcup_{\alpha<\beta} c(A^\alpha) = H$;*
(iii_1) *for each ordinal α with $0 \leqq \alpha < \beta$ we have $c(A^\alpha) = \sum^0 c(A_i^\alpha)$ $(i \in I)$ and*

(a) *if* α *is non-limit,* $\alpha = \gamma + 1$, *and if* $A_i^\alpha \in \Gamma^\alpha$, *then either* $c(A_i^\alpha)$ *equals to some* $c(A_j^\gamma)$ *with* $j \in I_\gamma$, *or there is a subset* I *of* I_γ *with* $\operatorname{card} I > 0$ *and a convex l-subgroup* A_1 *of* H *such that*

$$A_1 = \sum^0 c(A_i^\gamma) \; (i \in I)$$

and $c(A_i^\alpha)$ *is a proper lexicographic extension of* A_1;

(b) *if* α *is a limit ordinal and* $i \in I_\alpha$, *then there exists a system* $\{c(A_{\gamma(i)}^\gamma\}$ $(\gamma < \alpha)$ *with* $i(\gamma) \in I_\gamma$ *for each* $\gamma < \alpha$, $c(A_{i(\gamma_1)}^{\gamma_1}) \subseteq c(A_{i(\gamma_2)}^{\gamma_2})$ *whenever* $\gamma_1 < \gamma_2$ *and* $c(A_i^\alpha) = \bigcup_{\gamma < \alpha} c(A_{i(\gamma)}^\gamma)$.

In particular, H *is a lexicographic sum of the system* $c(A_i^0)$ $(i \in I)$ *and all* $c(A_i^0)$ *are linearly ordered. Moreover, if* A *and* A_1 *are as in* (iii) *of* 4.9 *or in* (iii_1), *respectively, then* $c(A_i^\alpha)/A_1$ *is isomorphic with* A_i^α/A.

Proof. Obviously all $c(A^\alpha)$ and all $c(A_i^\alpha)$ are convex l-subgroups of H. According to 1.11 we have $c(A^\alpha) = M(A^\alpha)$, $c(A_i^\alpha) = M(A_i^\alpha)$. Now (i_1) follows from (i) and 2.8. The assertion (ii_1) is a consequence of (ii) and 4.6 (because $H = c(G)$). From (iii), 2.8, 4.4 and 4.6 we obtain that (iii_1) is valid. Hence according to 4.9, H is a lexicographic sum of the system $c(A_i^0)$ $(i \in I_0)$. If A and A_1 are as in 4.9 (iii) or in (iii_1), respectively, then by 4.5 the linearly ordered groups $c(A_i^\alpha)/A_1$ and A_i^α/A are isomorphic. Since a maximal Dedekind completion of a linearly ordered group is linearly ordered, all $c(A_i^0) = M(A_i^0)$ are linearly ordered.

Hence H is constructed from the system $\{c(A_i^0)\}$ $(\text{i} \in I_0)$ by the same steps (using the operations of the direct sum and the lexicographic extension) as G is constructed from the system $\{A_i^0\}$ $(i \in I_0)$.

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