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GENERALIZED PROJECTIVITY

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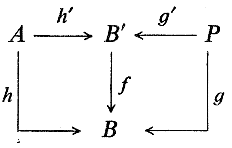
Throughout this paper, R stands for an associative ring with unity and R-mod denotes the category of all unitary left R-modules. A submodule A of B is *small* in B if $A + K = B$ implies $K = B$ for every submodule $K \subseteq B$. If f is an epimorphism from A to B (Ker f is small in A) then we shall say that (A, f) is *a coextension* (a cover) of B. This fact will be denoted by $A \leqq_f B$ ($A \leqq_f^s B$). (A, f) is said to be a *proper coextension of B* provided f is not an isomorphism. It is easy to see that $A \leqq_f^s B$ and $B \leqq_g^s C$ iff $A \leqq_{g \circ f}^s C$. A pair (P, φ) is a *projective cover* of the module M if (P, φ) is a cover of M and P a projective module. If M has a projective cover then all projective covers of M are canonically isomorphic and we can choose one of them and denote it by $(C(M), \varphi_M)$. A ring R is said to be *left perfect* if each of its left modules has a projective cover. A module C is said to be a *cogenerator* for the R-mod if for every $M \in R$-mod there is an index set A and a monomorphism $f : M \to \prod_{a \in A} C_a$, where $C_a = C$ for all $a \in A$.

1. GENERAL THEORY

In the class $\mathscr{M}$ of all $\langle A, h, B, g, P\rangle$ where $A, B, P \in R$-mod, $A \leqq_h B$ and $g \in \operatorname{Hom}_R(P, B)$, define the quasi-order $\leqq$ in the following way: $\langle A', h', B', g', P'\rangle \leqq \langle A, h, B, g, P\rangle$ if and only if $A = A'$, $P = P'$ and there exists an epimorphism $f : B' \to B$ such that the diagram



commutes.

In this paper $\mathscr{L}$ always denotes a subclass of $\mathscr{M}$. We shall say that $\mathscr{L}$ satisfies the condition

– (α) if $\langle A, h, B, g, P\rangle \in \mathscr{L}$, $\langle A, h', B', g', P\rangle \in \mathscr{M}$ and $\langle A, h', B', g', P\rangle \leqq$
$\leqq \langle A, h, B, g, P\rangle$ implies $\langle A, h', B', g', P\rangle \in \mathscr{L}$,

– (β) if $\langle A, h, B, g, P\rangle \in \mathscr{L}$ and $P' \leqq_f P$ implies $\langle A, h, B, g \circ f, P'\rangle \in \mathscr{L}$,

– (β′) if $\langle A, h, B, g, P\rangle \in \mathscr{L}$ and $P' \leqq_f^s P$ implies $\langle A, h, B, g \circ f, P'\rangle \in \mathscr{L}$,

– (γ) if $\langle A, h, B, g, P\rangle \in \mathscr{L}$ and $f : P' \to P$ an isomorphism implies $\langle A, h, B, g \circ f, P'\rangle \in \mathscr{L}$,

– (δ) if $\langle A, h, B, g, P\rangle \in \mathscr{L}$ and $f \in \mathrm{Hom}_R(P', P)$ implies $\langle A, h, B, g \circ f, P'\rangle \in \mathscr{L}$

For every $\langle A, h, B, g, P\rangle \in \mathscr{M}$ let us define $r_{\mathscr{L}}(A, h, B, g, P)$ $(s_{\mathscr{L}}(A, h, B, g, P))$ to be an intersection of all such $\mathrm{Ker}\, f$, $f \in \mathrm{Hom}_R(A, M)$ to which there exists a commutative (push-out) diagram

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & N \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \nu} \\ A & \xrightarrow[h]{} & B \\ & & \uparrow{\scriptstyle g} \\ & & P \end{array}$$

with $\langle M, \lambda, N, \nu \circ g, P\rangle \in \mathscr{L}$.

For every $\langle A, h, B, g, P\rangle \in \mathscr{M}$ we define

$$C_{\mathscr{L}}(A, h, B, g, P) = A/(r_{\mathscr{L}}(A, h, B, g, P) \cap \mathrm{Ker}\, h)\,,$$
$$D_{\mathscr{L}}(A, h, B, g, P) = A/(s_{\mathscr{L}}(A, h, B, g, P) \cap \mathrm{Ker}\, h)\,.$$

We also use the following abbreviation: $r_{\mathscr{L}}(A, h, B, 1_B, B) = r_{\mathscr{L}}(A, h, B)$. Similarly for $s_{\mathscr{L}}$, $C_{\mathscr{L}}$, $D_{\mathscr{L}}$.

Lemma 1.1. *Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ and let*

$$(*)\quad \begin{array}{ccccc} A & \xrightarrow{h} & B & \xleftarrow{g} & P \\ {\scriptstyle k_1}\uparrow & & {\scriptstyle k_2}\uparrow & & {\scriptstyle k_3}\uparrow \\ C & \xrightarrow[h_1]{} & D & \xleftarrow[g_1]{} & S \end{array} \qquad\qquad (**)\quad \begin{array}{ccc} M & \xrightarrow{\lambda} & N \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \nu} \\ A & \xrightarrow[h]{} & B \end{array}$$

be commutative diagrams, where h, h_1, λ are epimorphisms. If for every diagram (**) $\langle M, \lambda, N, \nu \circ g \circ k_3, S\rangle \in \mathscr{L}$ whenever $\langle M, \lambda, N, \nu \circ g, P\rangle \in \mathscr{L}$ then $k_1(r_{\mathscr{L}}(C, h_1, D, g_1, S)) \subseteq r_{\mathscr{L}}(A, h, B, g, P)$. In particular, $k_1(r_{\mathscr{L}}(C, h_1, D)) \subseteq$ $\subseteq r_{\mathscr{L}}(A, h, B)$.

Proof. Obvious.

Definition 1.2. A module P is said to be *$\mathscr{L}$-projective* if every diagram

$$(1)\qquad \begin{array}{ccc} & & P \\ & & \big\downarrow{\scriptstyle g} \\ N & \xrightarrow[h]{} & M \end{array}$$

with $\langle N, h, M, g, P\rangle \in \mathscr{L}$ can be completed to a commutative one.

Theorem 1.3. *Suppose P has a projective cover. Then the following conditions* (i), (ii), (iii) *are equivalent,* (iii) *implies* (iv) *and* (iv) *implies* (v). *Moreover, if $\mathscr{L}$ satisfies* (α) *then all the following conditions are equivalent.*

(i) *for every epimorphism $N \xrightarrow{h} P$, where N has a coextension (Q, q) with $r_{\mathscr{L}}(Q, h \circ q, P) \cap \operatorname{Ker} h \circ q \subseteq \operatorname{Ker} q$, there is a homomorphism $\psi : P \to N$ with $h \circ \psi = 1_P$;*

(ii) $\operatorname{Ker} \varphi_P \subseteq r_{\mathscr{L}}(C(P), \varphi_P, P)$;

(iii) *every diagram* (1), *where N has a coextension (Q, q) with $r_{\mathscr{L}}(Q, h \circ q, M, g, P) \cap$ $\cap \operatorname{Ker} h \circ q \subseteq \operatorname{Ker} q$ can be made commutative;*

(iv) *P is $\mathscr{L}$-projective;*

(v) $\operatorname{Ker} \varphi_P \subseteq s_{\mathscr{L}}(C(P), \varphi_P, P)$.

Proof. (i) implies (ii). It suffices to put $N = C_{\mathscr{L}}(C(P), \varphi_P, P)$ and set $h = \bar{\varphi}_P$, where $\bar{\varphi}_P$ is induced by φ_P. Since (N, h) is a cover of P, h is an isomorphism by (i) and consequently $\operatorname{Ker} \varphi_P \subseteq r_{\mathscr{L}}(C(P), \varphi_P, P)$.

(ii) implies (iii). Consider the diagram (1). Then there is $f : C(P) \to Q$ with $h \circ q \circ f = g \circ \varphi_P$. By Lemma 1.1, $f(r_{\mathscr{L}}(C(P), \varphi_P, P)) \subseteq r_{\mathscr{L}}(Q, h \circ q, M, g, P)$ and the inclusions $f(\operatorname{Ker} \varphi_P) = f(r_{\mathscr{L}}(C(P), \varphi_P, P) \cap \operatorname{Ker} \varphi_P) \subseteq r_{\mathscr{L}}(Q, h \circ q, M, g, P) \cap$ $\cap \operatorname{Ker} h \circ q \subseteq \operatorname{Ker} q$ complete the proof.

(iii) implies (i). Obvious.

(ii) implies (iv). Easy.

(iv) implies (v). Consider the push-out diagram

$$\begin{array}{ccc} C(P) & \xrightarrow{\varphi_P} & P \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow[h]{} & B \end{array}$$

where $\langle A, h, B, g, P\rangle \in \mathscr{L}$. Then $g = h \circ f_1$ for some $f_1 : P \to A$ and $f = f_1 \circ \varphi_P$ since $\operatorname{Ker} \varphi_P + \operatorname{Ker} (f - f_1 \circ \varphi_P) = C(P)$ as is easily seen. Therefore $\operatorname{Ker} \varphi_P \subseteq$ $\subseteq \operatorname{Ker} f$.

Finally, if $\mathscr{L}$ satisfies (α) then $r_{\mathscr{L}} = s_{\mathscr{L}}$.

Corollary 1.4. *Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ satisfying* (β) *and* $P, Q \in R$-mod *with projective covers* $(C(P), \varphi_P)$, $(C(Q), \varphi_Q)$, *respectively. Consider the commutative diagram*

$$\begin{array}{ccc} C(Q) & \xrightarrow{C(f)} & C(P) \\ {\scriptstyle \varphi_Q}\downarrow & & \downarrow{\scriptstyle \varphi_P} \\ Q & \longrightarrow & P \end{array}$$

where f is an epimorphism. Then P is $\mathscr{L}$-projective provided

$$\operatorname{Ker} \varphi_P \subseteq C(f)\,(r_{\mathscr{L}}(C(Q), \varphi_Q, Q)).$$

Proof. We have $C(f)\,(r_{\mathscr{L}}(C(Q), \varphi_Q, Q)) \subseteq r_{\mathscr{L}}(C(P), \varphi_P, P)$ by Lemma 1.1 and P is $\mathscr{L}$-projective by Theorem 1.3.

Definition 1.5. For any ordinal α and any epimorphism $\varphi : A \to B$ let us define sequences $C^{\alpha}_{\mathscr{L}}(A, \varphi, B)$ of modules and epimorphisms $\varphi_\alpha : A \to C^{\alpha}_{\mathscr{L}}(A, \varphi, B)$ inductively as follows:

$$C^{0}_{\mathscr{L}}(A, \varphi, B) = B\,, \quad \varphi_0 = \varphi\,;$$

$$C^{\alpha+1}_{\mathscr{L}}(A, \varphi, B) = C_{\mathscr{L}}(A, \varphi_\alpha, C^{\alpha}_{\mathscr{L}}(A, \varphi, B)),\ \varphi_{\alpha+1} \text{ is a natural epimorphism and}$$

$$C^{\alpha}_{\mathscr{L}}(A, \varphi, B) = A/\bigcap_{\beta<\alpha} \operatorname{Ker} \varphi_\beta,\ \varphi_\alpha \text{ is a natural epimorphism, } \alpha\text{-limit.}$$

Further put $\bar{C}_{\mathscr{L}}(A, \varphi, B) = C^{\alpha}_{\mathscr{L}}(A, \varphi, B)$, $\bar{r}_{\mathscr{L}}(A, \varphi, B) = \operatorname{Ker} \varphi_\alpha$, where $\operatorname{Ker} \varphi_\alpha = \operatorname{Ker} \varphi_{\alpha+1}$, $\alpha \geqq 1$.

Corollary 1.6. *For every module P with a projective cover $(C(P), \varphi_P)$ the module $\bar{C}_{\mathscr{L}}(C(P), \varphi_P, P)$ is $\mathscr{L}$-projective.*

Proof. Denote $Q = \bar{C}_{\mathscr{L}}(C(P), \varphi_P, P) = C^{\alpha}_{\mathscr{L}}(C(P), \varphi_P, P)$. Then $\operatorname{Ker}(\varphi_P)_\alpha = \operatorname{Ker}(\varphi_P)_{\alpha+1} = r_{\mathscr{L}}(C(P), (\varphi_P)_\alpha, Q) \cap \operatorname{Ker}(\varphi_P)_\alpha \subseteq r_{\mathscr{L}}(C(P), (\varphi_P)_\alpha, Q)$ and apply Theorem 1.3.

Lemma 1.7. *Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ satisfying* (γ), $N, P \in R$-mod *and let $h : N \to P$ be an epimorphism. If $f : N_1 \to N_2$ is a canonical isomorphism of two projective covers (N_1, g_1), (N_2, g_2) of N then $f_{[\alpha]}(C^{\alpha}_{\mathscr{L}}(N_1, h \circ g_1, P)) = C^{\alpha}_{\mathscr{L}}(N_2, h \circ g_2, P)$, where $f_{[\alpha]}$ is an isomorphism induced by f for $\alpha \geqq 1$, $f_{[0]} = 1_P$.*

Proof. By transfinite induction and Lemma 1.1.

Lemma 1.8. *Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ satisfying* (β′) *and let P be a module having a projective cover $(C(P), \varphi_P)$. If $C(P) \leqq_g Q \leqq_f P$ where $f \circ g = \varphi_P$ then* $\operatorname{Ker} g_\alpha \subseteq \operatorname{Ker}(\varphi_P)_\alpha$ *for every ordinal α under the notation from* 1.5.

Proof. It follows easily by transfinite induction using Lemma 1.1.

Remark 1.9. If $\mathscr{L}$ satisfies (γ) then the class of all $\mathscr{L}$-projective modules is abstract.

Definition 1.10. Let A, B be modules and let $\varphi : A \to B$ be an epimorphism. A pair (A, φ) is said to be an *$\mathscr{L}$-projective cover* of the module B if A is $\mathscr{L}$-projective, $A \leqq_f C \leqq_g B$ with $g \circ f = \varphi$ and C $\mathscr{L}$-projective implies f is an isomorphism.

Theorem 1.11. *Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ satisfying (α), (β') and let P be a module with a projective cover $(C(P), \varphi_P)$. Then $(\bar{C}_{\mathscr{L}}(C(P), \varphi_P, P), \bar{\varphi}_P)$, where $\bar{\varphi}_P$ is an epimorphism induced by φ_P, is an $\mathscr{L}$-projective cover of P.*

Proof. The module $N = \bar{C}_{\mathscr{L}}(C(P), \varphi_P, P) = C^{\alpha}_{\mathscr{L}}(C(P), \varphi_P, P)$ is $\mathscr{L}$-projective by Corollary 1.6.Let a module Q, $N \leqq_h Q \leqq_g P$, $g \circ h = \bar{\varphi}_P$, be $\mathscr{L}$-projective. From Theorem 1.3 we get $\operatorname{Ker}(h \circ (\varphi_P)_\alpha)_1 = \operatorname{Ker}(h \circ (\varphi_P)_\alpha)_\alpha$ and Lemma 1.8 yields $\operatorname{Ker}(h \circ (\varphi_P)_\alpha)_\alpha \subseteq \operatorname{Ker}(\varphi_P)_\alpha$. Therefore $r_{\mathscr{L}}(C(P), h \circ (\varphi_P)_\alpha, Q) \cap \operatorname{Ker}(h \circ (\varphi_P)_\alpha) \subseteq$ $\subseteq \operatorname{Ker}(\varphi_P)_\alpha$ and h is an isomorphism by Theorem 1.3 (i).

Theorem 1.12. *Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ satisfying (α), (β) and let P be a module with a projective cover $(C(P), \varphi_P)$. Then all $\mathscr{L}$-projective covers of the module P are canonically isomorphic provided at least one of the following two conditions holds:*

(i) *Every $\mathscr{L}$-projective module has a projective cover.*

(ii) *There is a natural number $n \geqq 1$ such that $C^n_{\mathscr{L}}(C(P), \varphi_P, P) = C^{n+1}_{\mathscr{L}}(C(P), \varphi_P, P)$.*

Proof. The existence of an $\mathscr{L}$-projective cover of P follows from Theorem 1.11. Let $(\tilde{P}, g)$ be an arbitrary $\mathscr{L}$-projective cover of P.

(i) There is an epimorphism $\tilde{g} : C(\tilde{P}) \to C(P)$ with $g \circ \varphi_{\tilde{P}} = \varphi_P \circ \tilde{g}$. For every ordinal number $\alpha \geqq 1$, $\tilde{g}$ induces an epimorphism $\tilde{g}_{[\alpha]} : C^{\alpha}_{\mathscr{L}}(C(\tilde{P}), \varphi_{\tilde{P}}, \tilde{P}) \to C^{\alpha}_{\mathscr{L}}(C(P), \varphi_P, P)$. By Theorem 1.3, $\bar{C}_{\mathscr{L}}(C(\tilde{P}), \varphi_{\tilde{P}}, \tilde{P}) = C_{\mathscr{L}}(C(\tilde{P}), \varphi_{\tilde{P}}, \tilde{P}) \cong \tilde{P}$. Hence $\bar{\varphi}_P \circ h = g$ for an epimorphism $h : \tilde{P} \to \bar{C}_{\mathscr{L}}(C(P), \varphi_P, P)$ and h is an isomorphism since $(\tilde{P}, g)$ is an $\mathscr{L}$-projective cover of P.

(ii) Put $P_k = C^k_{\mathscr{L}}(C(P), \varphi_P, P)$, $g_{[0]} = g$ and suppose that $g_{[i]} : \tilde{P} \to P_i$ is an epimorphism. From $\operatorname{Ker}(\varphi_P)_{i+1} = r_{\mathscr{L}}(C(P), (\varphi_P)_i, P_i) \cap \operatorname{Ker}(\varphi_P)_i$ we obtain $r_{\mathscr{L}}(C(P), (\varphi_P)_i, P_i, g_{[i]}, \tilde{P}) \cap \operatorname{Ker}(\varphi_P)_i \subseteq \operatorname{Ker}(\varphi_P)_{i+1}$ by Lemma 1.1 and $g_{[i]}$ factors through a homomorphism $g_{[i+1]} : \tilde{P} \to P_{i+1}$. As is easy to see, $g_{[i+1]}$ is an epimorphism. The proof is completed similarly as in (i).

Definition 1.13. A coextension (N, h) of a module P is said to be *$\mathscr{L}$-codense* if there are $Q \in R$-mod and an epimorphism $q : Q \to N$ such that $r_{\mathscr{L}}(Q, h \circ q, P) \cap$ $\cap \operatorname{Ker}(h \circ q) \subseteq \operatorname{Ker} q$.

Suppose now that N has a projective cover $(C(N), \varphi_N)$. As is easy to see, (N, h) is an $\mathscr{L}$-codense coextension of P if and only if $r_{\mathscr{L}}(C(N), h \circ \varphi_N, P) \cap \operatorname{Ker} h \circ \varphi_N \subseteq$ $\subseteq \operatorname{Ker} \varphi_N$.

An $\mathscr{L}$-codense coextension which is a cover is said to be an *$\mathscr{L}$-codense cover.*

Theorem 1.14. *Let $\mathscr{L}$ satisfy (α) and let P be a module having a projective cover. Then P is $\mathscr{L}$-projective iff P has no proper $\mathscr{L}$-codense cover.*

Proof. The "if" part of the proof follows from Theorem 1.3 since every $\mathscr{L}$-codense coextension of an $\mathscr{L}$-projective module splits. For the "only if" part $(C_{\mathscr{L}}(C(P), \varphi_P, P), \bar{\varphi}_P)$, where $\bar{\varphi}_P$ is an epimorphism induced by φ_P, is an $\mathscr{L}$-codense cover of P so that P is $\mathscr{L}$-projective by Theorem 1.3.

Definition 1.15. Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ satisfying (γ) and let N be a module having a projective cover. A coextension (N, h) of a module P is said to be *weakly $\mathscr{L}$-codense* if $\bar{r}_{\mathscr{L}}(C(N), h \circ \varphi_N, P) \subseteq \operatorname{Ker} \varphi_N$. A weakly $\mathscr{L}$-codense coextension which is a cover is said to be a *weakly $\mathscr{L}$-codense cover.*

Theorem 1.16. *Let $\mathscr{L}$ satisfy (α), (γ) and let P be a module having a projective cover. Then P is $\mathscr{L}$-projective iff it has no proper weakly $\mathscr{L}$-codense cover.*

Proof. It follows easily from Theorem 1.14.

Lemma 1.17. *Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ satisfying (β'), let A be a module having a projective cover, $B, C \in R$-*mod *and $A \leqq_f B \leqq_g C$. Then*

(i) *if $(A, g \circ f)$ is a weakly $\mathscr{L}$-codense cover of C then (A, f) is a weakly $\mathscr{L}$-codense cover of B,*

(ii) *if (A, f) is a weakly $\mathscr{L}$-codense cover of B and (B, g) a weakly $\mathscr{L}$-codense cover of C then $(A, g \circ f)$ is a weakly $\mathscr{L}$-codense cover of C.*

Proof. (i) Obvious, since Lemma 1.8 yields $\bar{r}_{\mathscr{L}}(C(A), f \circ \varphi_A, B) \subseteq \bar{r}_{\mathscr{L}}(C(A), g \circ f \circ \varphi_A, C)$.

(ii) By assumption $\bar{r}_{\mathscr{L}}(C(A), f \circ \varphi_A, B) \subseteq \operatorname{Ker} \varphi_A$ and $\bar{r}_{\mathscr{L}}(C(A), g \circ f \circ \varphi_A, C) \subseteq \subseteq \operatorname{Ker} f \circ \varphi_A$. Hence $C(A) \leqq_\pi \bar{C}_{\mathscr{L}}(C(A), g \circ f \circ \varphi_A, C) \leqq_{\overline{f \circ \varphi_A}} B$, $\overline{f \circ \varphi_A} \circ \pi = f \circ \circ \varphi_A$, π is a natural epimorphism and Lemma 1.8 gives $\bar{r}_{\mathscr{L}}(C(A), g \circ f \circ \varphi_A, C) = = \bar{r}_{\mathscr{L}}(C(A), \pi, \bar{C}_{\mathscr{L}}(C(A), g \circ f \circ \varphi_A, C)) \subseteq \bar{r}_{\mathscr{L}}(C(A), f \circ \varphi_A, B) \subseteq \operatorname{Ker} \varphi_A$.

Definition 1.18. A cover (N, h) of P is a said to be a *minimal weakly $\mathscr{L}$-codense cover* of P if it holds: (N, h) is a weakly $\mathscr{L}$-codense cover of P and $M \leqq_g N \leqq_h P$, where $(M, h \circ g)$ is a weakly $\mathscr{L}$-codense cover of P, implies g is an isomorphism.

Theorem 1.19. *Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ satisfying (α), (β) and let P be a module with a projective cover such that at least one of the conditions which are given in Theorem* 1.12 *holds. Then the following conditions are equivalent:*

(i) *(N, h) is a minimal weakly $\mathscr{L}$-codense cover of P,*

(ii) *(N, h) is an $\mathscr{L}$-projective cover of P,*

(iii) *N is $\mathscr{L}$-projective and (N, h) is a weakly $\mathscr{L}$-codense cover of P.*

Proof. (i) implies (ii). The $\mathscr{L}$-projectivity of N follows from Theorem 1.16 and Lemma 1.17. If $N \leqq_g M \leqq_f P$, $f \circ g = h$ and M is $\mathscr{L}$-projective then Theorem 1.16 and Lemma 1.17 imply that g is an isomorphism.

(ii) implies (iii). It follows from Theorem 1.11 and Theorem 1.12 that there is an isomorphism $f : N \to \bar{C}_{\mathscr{L}}(C(P), \varphi_P, P)$ such that $\bar{\varphi}_P \circ f = h$ and it suffices to use Lemma 1.17 (ii), since $(\bar{C}_{\mathscr{L}}(C(P), \varphi_P, P), \bar{\varphi}_P)$ is a weakly $\mathscr{L}$-codense cover of P.

(iii) implies (i). By Theorem 1.16 and Lemma 1.17 (i).

Proposition 1.20. *Let $\mathscr{L}$ be a subclass of $\mathscr{M}$ satisfying* (α), (δ) *and let P be a module with a projective cover. Then P is $\mathscr{L}$-projective if and only if every diagram* (1) *with (N, h) being $\mathscr{L}$-codense coextension of M can be made commutative.*

Proof. If the condition is satisfied then P is $\mathscr{L}$-projective by Theorem 1.3. Conversely, $r_{\mathscr{L}}(Q, h \circ q, M) \cap \operatorname{Ker} h \circ q \subseteq \operatorname{Ker} q$ for some $Q \in R$-mod and an epimorphism $q : Q \to N$ and it suffices to use Theorem 1.3 (iii), since $r_{\mathscr{L}}(Q, h \circ q, M, g, P) \subseteq r_{\mathscr{L}}(Q, h \circ q, M)$ by Lemma 1.1.

We can use the following notation. For every $C \in R$-mod let us denote by $\mathscr{L}_C$ the class of all $\langle N, h, M, g, P \rangle \in \mathscr{L}$, where $N = C$.

Definition 1.21. A module C is said to be a *test module for $\mathscr{L}$-projectivity* if every $\mathscr{L}_C$-projective module is $\mathscr{L}$-projective.

Theorem 1.22. (Test criterion for $\mathscr{L}$-projectivity.) *Let R be a left perfect ring and $\mathscr{L}$ a subclass of $\mathscr{M}$ satisfying the following condition* (ε):

(ε) *If*

$$
(2) \qquad \begin{array}{ccc}
 & & P \\
 & & \downarrow{\scriptstyle \pi} \\
C(P)/r_{\mathscr{L}}(C(P), \varphi_P, P) & \xrightarrow{\bar{\varphi}_P} & P/\varphi_P(r_{\mathscr{L}}(C(P), \varphi_P, P)) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
C & \xrightarrow{\quad h \quad} & D
\end{array}
$$

is a push-out diagram then $\langle C, h, D, g \circ \pi, P \rangle \in \mathscr{L}$. ($\pi$ *is a natural epimorphism.*)

Then every cogenerator for the R-*mod is a test module for* $\mathscr{L}$-*projectivity.*

Proof. Suppose that C is a cogenerator for the R-mod and P is $\mathscr{L}_C$-projective. With respect to Theorem 1.3, it suffices to prove that $K = \operatorname{Ker} \bar{\varphi}_P = 0$, where $\bar{\varphi}_P : C(P)/r_{\mathscr{L}}(C(P), \varphi_P, P) \to P/\varphi_P(r_{\mathscr{L}}(C(P), \varphi_P, P))$ is an epimorhism induced by φ_P. Suppose on the contrary that $K \neq 0$. Then there are $k \in K$ and a homomorphism $f : C(P)/r_{\mathscr{L}}(C(P), \varphi_P, P) \to C$ with $f(k) \neq 0$, since C is a cogenerator. Consider the push-out diagram (2). By the assumption, there is $p : P \to C$ with $h \circ p = g \circ \pi$.

Since

$$\begin{array}{ccc} C(P) & \xrightarrow{\varphi_P} & P \\ {\scriptstyle f \circ \pi_1}\downarrow & & \downarrow{\scriptstyle g \circ \pi} \\ C & \xrightarrow{h} & D \end{array}$$

(π_1, π are natural epimorphisms) is a push-out diagram and $\operatorname{Ker} \varphi_P$ is small in $C(P)$, we have $f \circ \pi_1 = p \circ \varphi_P$. Further, $k = \pi_1(k')$ for some $k' \in \operatorname{Ker} \varphi_P$. Therefore $f(k) = (f \circ \pi_1)(k') = (p \circ \varphi_P)(k') = 0$, a contradiction.

2. APPLICATIONS

Let $\mathscr{P}$ be a subclass of the class $\mathscr{R}$ of all couples (M, N), $M \subseteq N$. We say that $\mathscr{P}$ satisfies the condition (a) if $(M, N) \in \mathscr{P}$, $M' \subseteq M \subseteq N$ implies $(M', N) \in \mathscr{P}$.

Remark 2.1. Let $\mathscr{K}$, $\mathscr{P}$ be subclasses of $\mathscr{R}$ and $\mathscr{L}$ a class of all $\langle A, h, B, g, P \rangle \in \mathscr{M}$, where $(\operatorname{Ker} h, A) \in \mathscr{P}$ and $(h^{-1}(\operatorname{Im} g), A) \in \mathscr{K}$. Obviously $\mathscr{L}$ satisfies (β). Moreover, if both $\mathscr{K}$ and $\mathscr{P}$ satisfy (a) then $\mathscr{L}$ satisfies (α) and (δ).

We start with some basic definitions from the theory of preradicals (for details see [4] and [5]).

A *preradical* s for R-mod is any subfunctor of the identity functor, i.e., s assigns to each module M its submodule $s(M)$ in such a way that every homomorphism of M into N induces a homomorphism of $s(M)$ into $s(N)$ by restriction. A preradical s is said to be

- *idempotent* if $s(s(M)) = s(M)$ for every module M,
- *a radical* if $s(M/s(M)) = 0$ for every module M,
- *cohereditary* if $s(M/N) = (s(M) + N)/N$ for every submodule N of a module M,
- *hereditary* if $s(N) = N \cap s(M)$ for every submodule N of a module M.

A module M is an *s-torsion* if $s(M) = M$ and *s-torsionfree* if $s(M) = 0$. If r and s are preradicals then we write $r \leqq s$ if $r(M) \subseteq s(M)$ for all $M \in R$-mod. The zero functor is denoted by zer and the identity functor by id.

For every $M \in R$-mod we define $r^{\{M\}}(N) = \bigcap \operatorname{Ker} f$, f ranging over all $f \in \operatorname{Hom}_R(N, M)$. It is easy to see that $r^{\{M\}}$ is a radical and, in fact, the largest preradical for which M is a torsionfree module. The *idempotent core* $\bar{s}$ of a preradical s is defined by $\bar{s}(M) = \sum K$, where K runs through all s-torsion submodules K of M, and the radical closure $\tilde{s}$ is defined by $\tilde{s}(M) = \bigcap L$, where L runs through all submodules L with M/L s-torsionfree. Further, the cohereditary core $ch(s)$ is defined by $ch(s)(M) = s(R)M$.

For a preradical s and modules $N \subseteq M$ let us define $C_s(N : M)$ by $C_s(N : M)/N = s(M/N)$.

Let s be a preradical for R-mod. A coextension (A, h) of a module B is said to be:

- $(s, 1)$-*codense* if there exist $C \in R$-mod and $g : C \to A$ an epimorphism with $s(g^{-1}(\operatorname{Ker} h)) \subseteq \operatorname{Ker} g$,
- $(s, 2)$-*codense* if $s(\operatorname{Ker} h) = 0$,
- $(s, 3)$-*codense* if $\operatorname{Ker} h \cap s(A) = 0$.

Further if $N \subseteq M$ is a submodule and (M, π) is an $(s, 1)$-codense coextension of M/N where π is a natural epimorphism, then we shall write $N \subseteq^{(s,1)} M$. Similarly $N \subseteq^{(s,2)} M$ $(N \subseteq^{(s,3)} M)$.

A preradical s is said to be *cobalanced* if $A \subseteq B$, $C \subseteq D$, $A \cong C$ and $A \subseteq^{(s,1)} B$ implies $C \subseteq^{(s,1)} D$.

It is easy to prove the following assertions for a preradical s:

(i) if $P \leqq_g A$, P projective then (A, h) is an $(s, 1)$-codense coextension of B iff $s(g^{-1}(\operatorname{Ker} h)) \subseteq \operatorname{Ker} g$;

(ii) if $N \subseteq M$ are modules then $N \subseteq^{(s,3)} M$ implies $N \subseteq^{(s,2)} M$ and $N \subseteq^{(s,2)} M$ implies $N \subseteq^{(s,1)} M$;

(iii) if s is a cohereditary preradical and $N \subseteq M$ are modules then $N \subseteq^{(s,1)} M$ implies $N \subseteq^{(s,2)} M$;

(iv) if s is a hereditary preradical and $N \subseteq M$ are modules then $N \subseteq^{(s,2)} M$ implies $N \subseteq^{(s,3)} M$;

(v) $M \subseteq^{(s,1)} M$ iff M is $ch(s)$-torsionfree, $M \subseteq^{(s,2)} M$ iff $M \subseteq^{(s,3)} M$ iff M is s-torsionfree;

(vi) if $K \subseteq N \subseteq M$ are modules then $N \subseteq^{(s,i)} M$ implies $K \subseteq^{(s,i)} M$ for all $i \in \{1, 2, 3\}$;

(vii) every cohereditary preradical is cobalanced;

(viii) if R is left hereditary then every preradical is cobalanced.

Definition 2.2. Let r, s be two preradicals, $i, j \in \{1, 2, 3\}$ and $M \in R$-mod. Let $\mathscr{P}_i$ be the class of all couples (N, M) of modules with $N \subseteq^{(r,i)} M$ and let $\mathscr{K}_j$ be the class of all couples (N, M) with $N \subseteq^{(s,j)} M$. Now let $\mathscr{L}_{i,j}$ be the class of all $\langle M, h, B, g, P\rangle \in \mathscr{M}$ such that $(\operatorname{Ker} h, M) \in \mathscr{P}_i$ and $(h^{-1}(\operatorname{Im} g), M) \in \mathscr{K}_j$. We say that a module P is (r, i, s, j, M)-projective, if it is $\mathscr{L}_{i,j}$-projective. A module P is said to be (r, i, s, j)-*projective*, if it is (r, i, s, j, N)-projective for all $N \in R$-mod.

A module P is said to be (r, i, M)-*projective* ((r, i)-*projective*), if it is $(r, i, \text{zer}, 1, M)$-projective ($(r, i, \text{zer}, 1)$-projective). A module P is said to be (i, r, M)-*projective* ((i, r)-*projective*), if it is $(\text{zer}, 1, r, i, M)$-projective ($(\text{zer}, 1, r, i)$-projective).

Proposition 2.3. *Every module P having a projective cover has an (r, i, s, j, M)-projective cover ((r, i, s, j)-projective cover), $i, j \in \{1, 2, 3\}$. If at least one of the* *conditions which are given in Theorem* 1.12 *holds, then all the* (r, i, s, j, M)*-projective covers* ((r, i, s, j)*-projective covers*) *of* P *are canonically isomorphic,* $i, j \in \in \{1, 2, 3\}$.

Proof. Apply Theorems 1.11, 1.12 and Remark 2.1.

Lemma 2.4. *Let* r s *be preradicals for* R-mod $i, j \in \{1, 2, 3\}$ *and let* (A, h) *be a coextension of a module* B. *Consider a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{\psi} & Y & \longrightarrow & 0 \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle v} & & \\ A & \xrightarrow[h]{} & B & & \end{array}$$

with an exact row, where $\operatorname{Ker} \psi \subseteq^{(r,i)} X$ *and* $\psi^{-1}(\operatorname{Im} v) \subseteq^{(s,j)} X$. *Then the following implications hold*:

(i) *if* $i = 1, j = 1$, A *projective then* $r(\operatorname{Ker} h) + ch(s)(A) = C_{ch(s)}(r(\operatorname{Ker} h) : A) \subseteq \subseteq \operatorname{Ker} f$;

(ii) *if* $i = 1, j = 2$, A *projective then* $C_{\tilde{s}}(r(\operatorname{Ker} h) : A) \subseteq \operatorname{Ker} f$;

(iii) *if* $i = 2, j = 1$ *then* $\tilde{r}(\operatorname{Ker} h) + \mathrm{ch}(s)(A) = C_{ch(s)}(\tilde{r}(\operatorname{Ker} h) : A) \subseteq \operatorname{Ker} f$;

(iv) *if* $i = 2, j = 2$ *then* $C_{\tilde{s}}(\tilde{r}(\operatorname{Ker} h) : A) \subseteq \operatorname{Ker} f$;

(v) *if* $i = 3, j = 1$ *then* $ch(s)(A) + C_r(ch(s)(A) : A) \cap \operatorname{Ker} h = C_{ch(s)}(C_r(ch(s)(A) : : A) \cap \operatorname{Ker} h : A) \subseteq \operatorname{Ker} f$;

(vi) *if* $i = 3.\ j = 2$ *then* $C_{\tilde{s}}(C_r(\tilde{s}(A) : A) \cap \operatorname{Ker} h : A) \subseteq \operatorname{Ker} f$.

Proof. Easy.

Lemma 2.5. *Let* r, s *be preradicals for* R*-mod,* $i, j \in \{1, 2, 3\}$ *and let* (A, h) *be a coextension of a module* B. *Let* $\mathcal{L}_{i,j}$ *be the class of all* $\langle M, f, N, g, P\rangle \in \mathcal{M}$ *such that* $\operatorname{Ker} f \subseteq^{(r,i)} M$ *and* $f^{-1}(\operatorname{Im} g) \subseteq^{(s,j)} M$. *Then the following implications hold*:

(i) *if* $i = 1, j = 1, s = \mathrm{zer}$ *then* $r_{\mathcal{L}_{i,j}}(A, h, B) \subseteq r(\operatorname{Ker} h)$;

(ii) *if* $i = 1, j = 1$, r *is cobalanced then* $r_{\mathcal{L}_{i,j}}(A, h, B) \subseteq r(\operatorname{Ker} h) + ch(s)(A)$;

(iii) *if* $i = 2, j = 1, s = \mathrm{zer}$ *then* $r_{\mathcal{L}_{i,j}}(A, h, B) \subseteq \tilde{r}(\operatorname{Ker} h)$;

(iv) *if* $i = 2, j = 1$ *and* $\operatorname{Ker} h/(\tilde{r}(\operatorname{Ker} h) + ch(s)(A) \cap \operatorname{Ker} h)$ *is* $\tilde{r}$*-torsionfree then* $r_{\mathcal{L}_{i,j}}(A, h, B) \subseteq \tilde{r}(\operatorname{Ker} h) + ch(s)(A)$;

(v) *if* $i = 1, j = 2$, r *is cobalanced then* $r_{\mathcal{L}_{i,j}}(A, h, B) \subseteq C_{\tilde{s}}(r(\operatorname{Ker} h) : A)$;

(vi) *if* $i = 2, j = 2$ *and* $\operatorname{Ker} h/(\operatorname{Ker} h \cap C_{\tilde{s}}(\tilde{r}(\operatorname{Ker} h) : A))$ *is* $\tilde{r}$*-torsionfree then* $r_{\mathcal{L}_{i,j}}(A, h, B) \subseteq C_{\tilde{s}}(\tilde{r}(\operatorname{Ker} h) : A)$;

(vii) *if* $i = 1$, $j = 1$ *and* B *is* $ch(s)$*-torsionfree then* $r_{\mathscr{L}_{i,j}}(A, h, B) \subseteq r(\mathrm{Ker}\, h) + ch(s)(A)$;

(viii) *if* $i = 1$, $j = 2$ *and* B *is* s*-torsionfree then* $r_{\mathscr{L}_{i,j}}(A, h, B) \subseteq C_{\tilde{s}}(r(\mathrm{Ker}\, h) : A)$;

(ix) *if* $i = 3$, $j = 1$ *and* r *is a radical then* $r_{\mathscr{L}_{i,j}}(A, h, B) \subseteq C_r(ch(s)(A) : A)$;

(x) *if* $i = 3$, $j = 2$, r *is a radical and* $A/C_r(\tilde{s}(A) : A)$ *is* $\tilde{s}$*-torsionfree then* $r_{\mathscr{L}_{i,j}}(A, h, B) \subseteq C_r(\tilde{s}(A) : A)$.

Proof. (i) By (vii).

(ii) and (vii) Consider the commutative diagram

$$\begin{array}{ccc} A/(r(\mathrm{Ker}\, h) + ch(s)(A)) & \xrightarrow{\bar{h}} & B/h(ch(s)(A)) \\ {\scriptstyle \pi_1}\uparrow & & \uparrow{\scriptstyle \pi_2} \\ A & \xrightarrow[h]{} & B \end{array}$$

where π_1, π_2 are natural epimorphisms. It is easy to see that $\mathrm{Ker}\, \bar{h} \subseteq^{(r,1)} A/(r(\mathrm{Ker}\, h) + ch(s)(A))$ and $(\bar{h})^{-1}(\mathrm{Im}\, \pi_2) \subseteq^{(s,1)} A/(r(\mathrm{Ker}\, h) + ch(s)(A))$.

(iii) By (iv).

(iv), (v), (vi), (viii). Similarly.

(ix) Consider the natural epimorphism $A \to A/C_r(ch(s)(A) : A)$.

(x) Similarly.

Corollary 2.6. *Let* r, s *be preradicals for* R-mod *and let* P *be a module possessing a projective cover* $(C(P), \varphi_P)$. *Then the following implications hold*:

(i) *if* r *is cobalanced then* P *is* $(r, 1, s, 1)$*-projective iff* $\mathrm{Ker}\, \varphi_P \subseteq s(C(P)) + r(\mathrm{Ker}\, \varphi_P)$;

(ii) *if* r *is cobalanced then* P *is* $(r, 1, s, 2)$*-projective iff* $\mathrm{Ker}\, \varphi_P \subseteq C_{\tilde{s}}(r(\mathrm{Ker}\, \varphi_P) : C(P))$;

(iii) *if* P *is* s*-torsionfree then* P *is* $(r, 1, s, 2)$*-projective iff* $\mathrm{Ker}\, \varphi_P \subseteq C_{\tilde{s}}(r(\mathrm{Ker}\, \varphi_P) : C(P))$;

(iv) *if* $\mathrm{Ker}\, \varphi_P/(\tilde{r}(\mathrm{Ker}\, \varphi_P) + s(C(P)) \cap \mathrm{Ker}\, \varphi_P)$ *is* $\tilde{r}$*-torsionfree then* P *is* $(r, 2, s, 1)$*-projective iff* $\mathrm{Ker}\, \varphi_P \subseteq \tilde{r}(\mathrm{Ker}\, \varphi_P) + s(C(P))$;

(v) *if* $\mathrm{Ker}\, \varphi_P/(\mathrm{Ker}\, \varphi_P \cap C_{\tilde{s}}(\tilde{r}(\mathrm{Ker}\, \varphi_P) : C(P)))$ *is* $\tilde{r}$*-torsionfree then* P *is* $(r, 2, s, 2)$*-projective iff* $\mathrm{Ker}\, \varphi_P \subseteq C_{\tilde{s}}(\tilde{r}(\mathrm{Ker}\, \varphi_P) : C(P))$;

(vi) *if* r *is a radical then* P *is* $(r, 3, s, 1)$*-projective iff* $\mathrm{Ker}\, \varphi_P \subseteq C_r(s(C(P)) : C(P))$;

(vii) *if r is a radical and $C(P)/C_r(\bar{s}(C(P)) : C(P))$ is $\bar{s}$-torsionfree then P is $(r, 3, s, 2)$-projective iff* $\operatorname{Ker} \varphi_P \subseteq C_r(\bar{s}(C(P)) : C(P))$.

Proof. It follows from 1.3, 2.4 and 2.5.

Proposition 2.7. *Let r, s be preradicals for R-mod, $i \in \{1, 2, 3\}$ and $P \in R$-mod. Then*

(i) *if $K \subseteq ch(s)(P)$ then P is $(r, i, s, 1)$-projective iff P/K is so,*

(ii) *if P is $(r, i, s, 2)$-projective and $K \subseteq \bar{s}(P)$ then P/K is $(r, i, s, 2)$-projective.*

Proof. Use the fact that P is (r, i, s, j)-projective iff it is (r, i, M)-projective for all $M \in R$-mod with $M \subseteq^{(s,j)} M$, $i, j \in \{1, 2, 3\}$.

Corollary 2.8. *Let r, s be two preradicals and P a module such that $\bar{P} = P/ch(s)(P)$ has a projective cover $(C(\bar{P}), \varphi_{\bar{P}})$. Then P is $(r, 1, s, 1)$-projective iff* $\operatorname{Ker} \varphi_{\bar{P}} \subseteq s(C(\bar{P})) + r(\operatorname{Ker} \varphi_{\bar{P}})$.

Proof. By 2.5 (vii), 2.4, 1.3 and 2.7(i).

Proposition 2.9. *Let r, s be preradicals, $i \in \{1, 2, 3\}$ and $P \in R$-mod. Then the following assertions hold:*

(i) *P is $(r, i, s, 2)$-projective iff it is $(r, i, s, 3)$-projective;*

(ii) *P is $(r, 2, s, i)$-projective iff it is $(\bar{r}, 2, s, i)$-projective;*

(iii) *P is $(r, i, s, 2)$-projective iff it is $(r, i, \bar{s}, 2)$-projective;*

(iv) *P is $(r, i, s, 1)$-projective iff it is $(r, i, ch(s), 1)$-projective;*

(v) *if P has a projective cover $(C(P), \varphi_P)$ and* $\operatorname{Ker} \varphi_P/(\bar{r}(\operatorname{Ker} \varphi_P) + s(C(P)) \cap \operatorname{Ker} \varphi_P)$ *is $\bar{r}$-torsionfree then P is $(r, 2, s, 1)$-projective iff it is $(\bar{\bar{r}}, 2, s, 1)$-projective;*

(vi) *if P has a projective cover $(C(P), \varphi_P)$ then P is $(r, 1)$-projective iff it is $(\bar{r}, 1)$-projective;*

(vii) *if r is cobalanced then P is $(r, 1, s, i)$-projective iff it is $(ch(r), 1, s, i)$-projective;*

(viii) *if r cobalanced and P has a projective cover $(C(P), \varphi_P)$ then P is $(r, 1, s, 1)$-projective iff it is $(\overline{ch}(r), 2, s, 1)$-projective.*

Proof. (i), (ii), (iii), (iv), (vii) are obvious.

(vi) follows from 2.8.

(v) With respect to (ii) we can assume that r is a radical. Since $\bar{r} \subseteq r$, the $(\bar{r}, 2, s, 1)$-projectivity implies the $(r, 2, s, 1)$-projectivity. Conversely, consider a diagram (1) with $\operatorname{Ker} h \subseteq^{(\bar{r},2)} N$ and $N \subseteq^{(s,1)} N$ and suppose that P is $(r, 2, s, 1)$-projective.

Then $h \circ \bar{g} = g \circ \varphi_P$ for a homomorphism $\bar{g} : C(P) \to N$. Further, let $K_0 = \text{Ker } h$, $K_{\alpha+1} = r(K_\alpha)$ for every ordinal α and $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$ if α is limit. Then $0 = \bar{r}(\text{Ker } h) = K_\gamma$ for an ordinal γ. By (2.6) (iv) $\text{Ker } \varphi_P \subseteq r(\text{Ker } \varphi_P) + s(C(P))$, and therefore $\bar{g}(\text{Ker } \varphi_P) \subseteq K_\alpha$ for every ordinal α. Hence $\bar{g}(\text{Ker } \varphi_P) = 0$ and so P is $(\bar{r}, 2, s, 1)$-projective.

(viii) By (vii) and (v).

Proposition 2.10. *Let r, s be two preradicals and P a module with a projective cover $(C(P), \varphi_P)$. Then the following assertions hold:*

(i) *if r is a cobalanced idempotent preradical then $(C(P)/(r(\text{Ker } \varphi_P) + s(C(P)) \cap \text{Ker } \varphi_P), \bar{\varphi}_P)$ is an $(r, 1, s, 1)$-projective cover of P;*

(ii) *if r is a cobalanced idempotent preradical then $(C(P)/(C_{\bar{s}}(r(\text{Ker } \varphi_P) : C(P)) \cap \text{Ker } \varphi_P), \bar{\varphi}_P)$ is an $(r, 1, s, 2)$-projective cover of P;*

(iii) *$(C(P)/\bar{r}(\text{Ker } \varphi_P), \bar{\varphi}_P)$ is an $(r, 1)$ projective cover of P;*

(iv) *if R is left perfect and r cobalanced then $(C(P)/(\overline{ch}(r) (\text{Ker } \varphi_P) + s(C(P)) \cap \text{Ker } \varphi_P), \bar{\varphi}_P)$ is an $(r, 1, s, 1)$-projective cover of P;*

(v) *if R is a left perfect left hereditary ring then $(C(P)/(\bar{r}(\text{Ker } \varphi_P) + s(C(P)) \cap \text{Ker } \varphi_P), \bar{\varphi}_P)$ is an $(r, 1, s, 1)$-projective cover of P;*

(vi) *if r is an idempotent preradical and $\bar{P} = P/ch(s)(P)$ has a projective cover $(C(\bar{P}), \varphi_{\bar{P}})$ then $(C(P)/(\text{Ker } \varphi_P \cap \pi^{-1}(r(\text{Ker } \varphi_{\bar{P}}) + s(C(\bar{P})))), \bar{\varphi}_P)$, where $\pi : C(P) \to C(\bar{P})$ is an epimorphism with $\varphi_{\bar{P}} \circ \pi = \nu \circ \varphi_P$, $\nu : P \to \bar{P}$ natural, is an $(r, 1, s, 1)$-projective cover of P;*

(vii) *$(C(P)/\bar{\bar{r}}(\text{Ker } \varphi_P), \bar{\varphi}_P)$ is an $(r, 2)$-projective cover of P;*

(viii) *if r is an idempotent preradical and $\text{Ker } \varphi_P/(\tilde{r}(\text{Ker } \varphi_P) + s(C(P)) \cap \text{Ker } \varphi_P)$ is $\tilde{r}$-torsionfree then $(C(P)/(\tilde{r}(\text{Ker } \varphi_P) + s(C(P)) \cap \text{Ker } \varphi_P), \bar{\varphi}_P)$ is an $(r, 2, s, 1)$-projective cover of P;*

(ix) *if r is an idempotent preradical and $\text{Ker } \varphi_P/(\text{Ker } \varphi_P \cap C_{\bar{s}}(\tilde{r}(\text{Ker } \varphi_P) : C(P)))$ is $\tilde{r}$-torsionfree then $(C(P)/(\text{Ker } \varphi_P \cap C_{\bar{s}}(\tilde{r}(\text{Ker } \varphi_P) : C(P))), \bar{\varphi}_P)$ is an $(r, 2, s, 2)$-projective cover of P;*

(x) *if r is a radical then $(C(P)/(C_r(s(C(P)) : C(P)) \cap \text{Ker } \varphi_P), \bar{\varphi}_P)$ is an $(r, 3, s, 1)$-projective cover of P;*

(xi) *if r is a radical and $C(P)/C_r(\tilde{s}(C(P)) : C(P))$ is $\tilde{s}$-torsionfree then $(C(P)/(C_r(\tilde{s}(C(P)) : C(P)) \cap \text{Ker } \varphi_P), \bar{\varphi}_P)$ is an $(r, 3, s, 2)$-projective cover of P.*

Proof. (i), (ii), (iii), (vii), (viii), (ix), (x), (xi) Apply 2.4, 2.5 and 1.11.

(iv), By (i) and 2.9 (viii).

(v) It follows from (iv).

(vi) By 2.4 (i), 2.5 (vii) and 1.11, $(C(\bar{P})/(r(\text{Ker } \varphi_{\bar{P}}) + s(C(\bar{P}))), \bar{\varphi}_{\bar{P}})$ is an $(r, 1, s, 1)$-projective cover of $\bar{P}$. Put $Q = C(P)/(\text{Ker } \varphi_P \cap \pi^{-1}(K))$, where $K = r(\text{Ker } \varphi_{\bar{P}}) + s(C(\bar{P}))$. Obviously $Q/ch(s)(Q) \cong C(\bar{P})/K$ and Q is $(r, 1, s, 1)$-projective by 2.7 (i). Now, let $\text{Ker } \varphi_P \cap \pi^{-1}(K) \subseteq L \subseteq \text{Ker } \varphi_P$ and let $C(P)/L$ be $(r, 1, s, 1)$-projective. Then $(C(P)/L)/ch(s)(C(P)/L)$ is $(r, 1, s, 1)$-projective by 2.7 (i) and so $L = \text{Ker } \varphi_P \cap \pi^{-1}(K)$, since $(C(\bar{P})/K, \bar{\varphi}_{\bar{P}})$ is an $(r, 1, s, 1)$-projective cover of $\bar{P}$.

Lemma 2.11. *Let r be a cobalanced preradical and s a preradical for R-mod. If (A, h) is a coextension of B, $M \in R$-mod, $f \in \text{Hom}_R(A/(r(\text{Ker } h) + ch(s)(A)), M)$ then there is a pushout diagram*

$$\begin{array}{ccc} M & \xrightarrow{\sigma} & D \\ {\scriptstyle f\circ\pi}\uparrow & & \uparrow{\scriptstyle g} \\ A & \xrightarrow[h]{} & B \end{array}$$

with $\text{Ker } \sigma \subseteq^{(r,1)} M$ *and* $\sigma^{-1}(\text{Im } g) \subseteq^{(ch(s),1)} M$ *(π is a natural epimorphism).*

Proof. If

$$\begin{array}{ccc} M & \xrightarrow{\quad\quad\sigma\quad\quad} & D \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \xi} \\ A/(r(\text{Ker } h) + ch(s)(A)) & \xrightarrow{\bar{h}} & B/h(ch(s)(A)) \end{array}$$

is a push-out diagram then the diagram

$$\begin{array}{ccc} M & \xrightarrow{\sigma} & D \\ {\scriptstyle f\circ\pi}\uparrow & & \uparrow{\scriptstyle \xi\circ\pi_1} \\ A & \xrightarrow[h]{} & B \end{array}$$

is a push-out diagram with $\text{Ker } \sigma \subseteq^{(r,1)} M$ and $\sigma^{-1}(\text{Im}(\xi \circ \pi_1)) \subseteq^{(ch(s),1)} M$ (π_1 is a natural epimorphism).

Theorem 2.12. *Let R be a left perfect ring and C a cogenerator for the R-mod. If r is a cobalanced preradical, s a preradical then a module P is $(r, 1, s, 1)$-projective iff it is $(r, 1, ch(s), 1, C)$-projective. In particular, if both r and s are cobalanced then every cogenerator for the R-mod is a test module for the $(r, 1, s, 1)$-projectivity.*

Proof. See 2.9 (iv), 1.22 and the proof of Lemma 2.11.

Corollary 2.13. *Let r, s be two cobalanced preradicals, P a module possessing a projective cover $(C(P), \varphi_P)$ and $M \in R$-mod. Then P is $(r, 1, s, 1, M)$-projective iff*

$$\operatorname{Ker} \varphi_P \subseteq C_{r(M)}((r(\operatorname{Ker} \varphi_P) + s(C(P))) : C(P)) .$$

Proof. By 1.3, 2.4 (i) and 2.11.

Corollary 2.14. *Let r be a cobalanced idempotent preradical, s a cobalanced preradical, P a module with a projective cover $(C(P), \varphi_P)$ and $M \in R$-mod. Then $(C(P)/(C_{r(M)}((r(\operatorname{Ker} \varphi_P) + s(C(P))) : C(P)) \cap \operatorname{Ker} \varphi_P), \bar{\varphi}_P)$ is an $(r, 1, s, 1, M)$-projective cover of P.*

Proof. Apply 2.4 (i), 2.11 and 1.11.

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