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FUNCTIONAL SEPARATION OF INDUCTIVE LIMITS AND REPRESENTATION OF PRESHEAVES BY SECTIONS PART THREE: SOME SPECIAL CASES OF SEPARATION OF INDUCTIVE LIMITS OF PRESHEAVES

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To complete the investigations from Part One we discuss here some cases when the inductive limit of a presheaf of closure spaces has a completely regular, normal or metrizable topology coarser than that canonically defined in it. The last case enables us to prove a representation theorem in Part Four. Further, we get a sufficient condition for the canonical maps of the projective limit of a presheaf into its entries to be homeomorphisms. It is also shown that sometimes the means developed in the foregoing parts can be used for the verification of nonemptiness of projective limits of some presheaves. The second and the third section is, however, not used later and may be therefore skipped.

To establish the separation theorems in Part One we had to assume that the maps between the entries of the presheaf were $1-1$. Passing to factorpresheaves we can sometimes get rid of the condition and obtain some sufficient conditions for the inductive limit of a presheaf to be functionally separated. This in turn yields a representation theorem in Part Four.

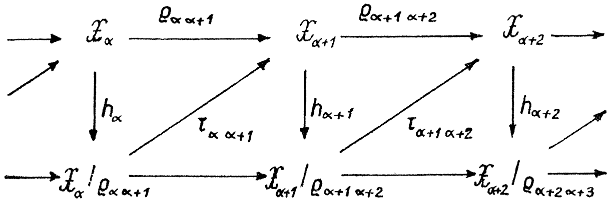
The basic notation and definitions were introduced at the beginning of Part One, which is very often reffered to. If we refer, say, to Theorem 1.1.7 or to 0.5, we mean Theorem 1.1.7 in the first section of Part One or Definition 0.5 at the beginning of Part One, respectively.

[1]. FACTORPRESHEAVES

Given a closured presheaf $\mathcal{S} = \{\mathcal{X}_\alpha = (X_\alpha, \tau_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ (see 0.12), $\langle A \leqq \rangle$ well ordered, then for each $\alpha \in A$ we can take the factorspace $\mathcal{X}_\alpha / \varrho_{\alpha\alpha+1}$ of $\mathcal{X}_\alpha$ by the equivalence "$a, b \in X_\alpha$, $a \sim b$ iff $\varrho_{\alpha\alpha+1}(a) = \varrho_{\alpha\alpha+1}(b)$", endowed with the closure t_α

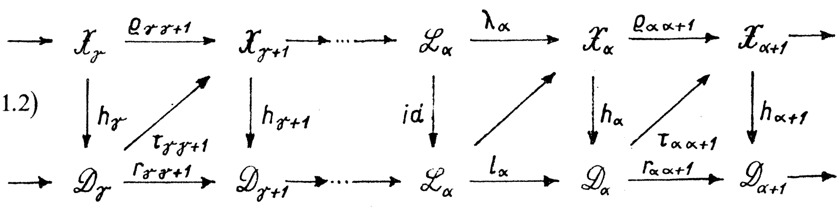
inductively defined by the canonical map $h_\alpha : \mathscr{X}_\alpha \to X_\alpha/\varrho_{\alpha\alpha+1}$. We get this diagram:

(3.1.1)



each triangle here is commutative, hence so is each square. We put $D_\alpha = X_\alpha/\varrho_{\alpha\alpha+1}$, $\mathscr{D}_\alpha = (D_\alpha, t_\alpha)$, $r_{\alpha\alpha} =$ identity, and for $\alpha, \beta \in A$, $\alpha \leqq \beta$ we define $r_{\alpha\beta} : \mathscr{D}_\alpha \to \mathscr{D}_\beta$ by $r_{\alpha\beta} = h_\beta \varrho_{\alpha+1\beta} \tau_{\alpha\alpha+1}$. If $\alpha \leqq \beta \leqq \gamma$ then the commutativity of 3.1.1 gives $r_{\alpha\gamma} = r_{\beta\gamma} r_{\alpha\beta}$, so $\mathscr{S}^\# = \{\mathscr{D}_\alpha | r_{\alpha\beta} | \langle A \leqq \rangle\}$ is a presheaf. If all the $\varrho_{\alpha\beta}$ are continuous (i.e. if $\mathscr{S}$ is from CLOS), then so is each h_α and $\tau_{\alpha\alpha+1}$, hence so is also each $r_{\alpha\beta} : \mathscr{D}_\alpha \to \mathscr{D}_\beta$. It can be easily seen that if for all $\alpha \in A$ with no $\alpha - 1$ we have $S_\alpha = \varinjlim \mathscr{S}_{A[\alpha]}$ (which means that $\varrho_{\beta\alpha} : S_\beta \to S_\alpha$ are the canonical maps for all $\beta \in A[\alpha]$), then $r_{\alpha\beta}$ is $1-1$ iff $\varrho_{\alpha+1\alpha+2}$ is $1-1$ on $\varrho_{\alpha\alpha+1}(X_\alpha)$. As $\tau_{\alpha\alpha+1}$ and h_α are continuous for any α and 3.1.1 is commutative, we get that $\mathscr{I} = \varinjlim \mathscr{S}$ is isomorphic with $\mathscr{K} = \varinjlim \mathscr{S}^\#$. Likewise, if $\alpha \in A$ and there is no $\alpha - 1$, then $\mathscr{L}_\alpha = \varinjlim \mathscr{S}_{A[\alpha]}$ and $\mathscr{L}^\#_\alpha = \varinjlim \mathscr{S}^\#_{A[\alpha]}$ are isomorphic; so we may write $\mathscr{L}^\#_\alpha = \mathscr{L}_\alpha$. Hence the diagram 3.1.1 in the "neighborhood" of the α looks like this:

(3.1.2)



The same construction can be done in terms of TOP, SEM, UNIF, PROX, ... (see 0.5). Thus we get

3.1.3. Proposition. *Given a presheaf* $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ *from* TOP *and a set* $B \subset A$ *such that* $\langle B \leqq \rangle$ *is well ordered and* $\mathscr{S}_B$ *is a compact presheaf* (see 2.1.2A), *suppose that*

(1) *either* B *is cofinal in* $\langle A \leqq \rangle$, *or* $\langle A \leqq \rangle$ *is ordered*, $\langle A - B \leqq \rangle$ *well ordered and* $A - B \subset \mathscr{L}$;

(2) (a) $\varrho_{\alpha+1\alpha+2}$ *is* $1-1$ *on* $\varrho_{\alpha\alpha+1}\mathscr{X}_\alpha$ *for all* $\alpha \in B$ ($\alpha + 1$ *is the follower of* α *in* $\langle B \leqq \rangle$),

(b) *the family* $\mathscr{E} = \{F_\alpha = C(\mathscr{X}_\alpha \to R) \mid \alpha \in B\}$ *has the following property*: *Given* $\alpha \in B$ *such that the predecessor* $\alpha - 1$ *of* α *in* $\langle B \leqq \rangle$ *does not exist*, $\beta \in B[\alpha]$ *and a thread* $\{f_\gamma \mid \gamma \in \langle \beta\alpha) \cap B\}$ *through* $\mathscr{E}$, *then there is* $f \in F_{\alpha+1}$ *with* $\varrho^*_{\gamma\alpha+1} f = f_\gamma$ *for all* $\gamma \in \langle \beta\alpha) \cap B$.

Then $\mathscr{I} = \varinjlim \mathscr{S}$ and $\mathscr{J} = \varinjlim \mathscr{S}_B$ are f.s. If there is a countable cofinal set C in B then the condition (2b) *may be left out.*

Proof. If $\mathscr{S}_B^\# = \{\mathscr{D}_\alpha | r_{\alpha\beta}| \langle B \leqq \rangle\}$ is the factorpresheaf of $\mathscr{S}_B$ constructed above then we put $\mathscr{H} = \{H_\alpha = C(\mathscr{D}_\alpha \to R) \mid \alpha \in B\}$. By [1, Chap. 1, Sec. 10(6), Cor. 1 of Prop. 8, p. 97], $\mathscr{D}_\alpha$ are Hausdorff, hence compact, being continuous images of the compact spaces $\mathscr{X}_\alpha$. As $r_{\alpha\alpha+1}$ is $1-1$ for all $\alpha \in B$, 1.3.4A yields the left smoothness of $\mathscr{H}$. Let $\alpha \in B$ such that there is no $\alpha - 1$, $\beta \in B[\alpha]$ and a thread $\{g_\gamma \mid \gamma \in \langle \beta\alpha) \cap B\}$ through $\mathscr{H}$ be given. As 3.1.2 is commutative, we get that $\{f_\gamma = g_\gamma h_\gamma \mid \gamma \in \langle \beta\alpha) \cap B\}$ is a thread through $\mathscr{E}$, hence there is $f \in F_{\alpha+1}$ with $\varrho^*_{\gamma\alpha+1} f = f_\gamma$ for all $\gamma \in \langle \beta\alpha) \cap B$. Putting $g = f\sigma_{\alpha\alpha+1}$, we get from 3.1.2 that $r^*_{\gamma\alpha} g = g_\gamma$ for all $\gamma \in \langle \beta\alpha) \cap B$, so $\mathscr{H}$ is connected. By 1.1.7, $\mathscr{K} = \varinjlim \mathscr{S}_B^\#$ is f.s. Since $\mathscr{K}$ and $\mathscr{J} = \varinjlim \mathscr{S}_B$ are isomorphic, the statement follows. If there is a countable cofinal set C in B then 1.5.6B yields that $\mathscr{K}$ is f.s. without 2b, which finishes the proof.

2. SOME REMARKS ON FUNCTIONAL SEPARATION, COMPLETE REGULARITY AND NORMALITY OF INDUCTIVE LIMITS

We start with a generalization of Th. 1.1.7.

3.2.1. Remark. *Given a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ from an i.c. category $\mathfrak{L}$ which is endowed with a leftward smooth and connected family $\mathscr{E} = \{F_\alpha = C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$ (which need not be separating), $\langle A \leqq \rangle$ well ordered, suppose for any $p, q \in \mathscr{I} = \varinjlim \mathscr{S}$, $p \neq q$, there is $\alpha \in A$ and $f_\alpha \in F_\alpha$ such that $f = 1$ and $f = 0$ on the set $M_\alpha = \xi_\alpha^{-1}(p)$ and $N_\alpha = \xi_\alpha^{-1}(q)$, respectively ($\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{I}$ are the canonical maps). Then $\mathscr{I} = \varinjlim \mathscr{S}$ is f.s. by $D = \{f \in C(\mathscr{I} \to R|\mathfrak{L})|$ there is $\beta \in A$ with $\xi_\alpha^* f \in F_\alpha$ for all $\alpha \geqq \beta\}$.*

If, moreover, for every $\alpha \in A$ there is $f_\alpha \in F_\alpha$ such that $f_\alpha = 1$ on M_α and $f_\alpha = 0$ on N_α and if there is a countable cofinal set D in C then the condition of connectedness of $\mathscr{E}$ may be left out.

Proof. Let $p, q \in \mathscr{I}$, $p \neq q$. There is $\alpha \in A - \mathscr{L}$ and $f_\alpha \in F_\alpha$ such that $M_\alpha, N_\alpha \neq \emptyset$, $f_\alpha / M_\alpha = 1$, $f_\alpha / N_\alpha = 0$. We put $M_\gamma = \xi_\gamma^{-1}(p)$, $N_\gamma = \xi_\gamma^{-1}(q)$ for $\gamma \in A(\alpha) = \{\beta \in A \mid \beta \geqq \alpha\}$. As $\mathscr{E}$ is leftward smooth and connected, we can make a thread $\mathscr{G} = \{g_\gamma \mid \gamma \in A(\alpha)\}$ through $\mathscr{E}$ with $g_\alpha = f_\alpha$ by induction (see the proof of 1.1.7). If $\gamma \in A(\alpha)$, $a \in M_\gamma$, then there is $b \in M_\alpha$ and $\delta \geqq \gamma$ with $\varrho_{\gamma\delta}(a) = \varrho_{\alpha\delta}(b) = c$. Thus $g_\gamma(a) = g_\delta(c) = g_\alpha(b)$, hence $g_\gamma / M_\gamma = 1$ for all $\gamma \in A(\alpha)$. Likewise $g_\gamma / N_\gamma = 0$, thus for $f = \varinjlim \mathscr{G}$ we have $f(p) = 1$, $f(q) = 0$ as desired. For the proof of the rest take a countable cofinal set D in C of the ordinal type ω_0. Then $\mathscr{S}_D$, $\mathscr{E}_D$ fulfil the conditions of 3.2.1 and 1.4.2 completes the proof.

Since ξ in 3.2.1 need not be separating, the maps $\varrho_{\alpha\alpha+1}$ need not be $1-1$ (by 1.3.1c, 1.1.6, if $\mathscr{E}$ is leftward smooth, connected and separating, then $\varrho_{\alpha\beta}$ are $1-1$).

3.2.2. Remark. *Let* $\mathscr{S} = \{\mathscr{X}_\alpha = (X_\alpha, \tau_\alpha) |\varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ *be a presheaf from* CLOS, *let* $\varrho_{\alpha\beta}$ *be* $1-1$ *for all* $\alpha, \beta \in A$, $\alpha \leqq \beta$. *If* $\alpha \in A$, *we put* $D_\alpha = \{f \in C(\mathscr{X}_\alpha \to R) \mid f \circ \varrho_{\alpha\beta}^{-1}$ *is continuous on* $(\varrho_{\alpha\beta}(X_\alpha)$, ind $\tau_\beta)$ *for all* $\beta \geqq \alpha\}$. *Then for* $\alpha, \beta \in A$, $\alpha \leqq \beta$ *we have*:

(1) A. $\varrho_{\alpha\beta}^* D_\beta \subset D_\alpha$. B. $\varrho_{\alpha\beta}^* D_\beta = D_\alpha$ *if* $\varrho_{\alpha\beta}$ *is onto.*

(2) *If* $\mathscr{J} = \varinjlim \mathscr{S}$ *is f.s., then each* D_α *separates points of* X_α, *so a necessary condition for* $\mathscr{J}$ *to be f.s. is that each* D_α *separates points.*

(3) *If* $\mathscr{J}$ *is completely regular and every* $\varrho_{\alpha\beta}$ *carries any* τ_α*-closed set onto a* τ_β*-closed one, then* D_α *separates points and closed sets. This gives us a necessary condition for* $\mathscr{J}$ *to be completely regular.*

(4) *Let* $\langle A \leqq \rangle$ *be well ordered and let* $\varrho_{\alpha\beta}$ *be onto for all* $\alpha, \beta \in A$. *If every* D_α *separates points of* X_α, *then* $\mathscr{J}$ *is f.s. The statements* (1), (2), (4) *hold in terms of any i.c. category.*

Proof. 1. Let $\alpha, \beta, \gamma \in A$, $\alpha \leqq \beta$, $\alpha \leqq \gamma$, $f \in D_\beta$. We put $g_\gamma = f \circ \varrho_{\alpha\beta} \circ \varrho_{\alpha\gamma}^{-1}$. If $\gamma \geqq \beta$ then $g_\gamma = f \circ \varrho_{\beta\gamma}^{-1} / \varrho_{\alpha\gamma}(X_\alpha)$. However, $\varrho_{\alpha\gamma}(X_\alpha) \subset M = \varrho_{\beta\gamma}(X_\beta)$ and $f \circ \varrho_{\beta\gamma}^{-1}$ is continuous on M. If $\alpha \leqq \gamma < \beta$ then $g_\gamma = f \circ \varrho_{\gamma\beta} / \varrho_{\alpha\gamma}(X_\alpha)$ which is continuous, hence (1A) is proved. To prove (1B), we take $\alpha \in A$, $f \in D_\alpha$ and put $g = f \circ \varrho_{\alpha\beta}^{-1}$. If $\gamma \geqq \beta$ then $g \circ \varrho_{\beta\gamma}^{-1} = f \circ \varrho_{\alpha\gamma}^{-1}$ which is continuous, hence $g \in D_\beta$ and clearly $\varrho_{\alpha\beta}^* g = f$ as desired. Let $\mathscr{J}$ be f.s., $\alpha \in A$, $a, b \in X_\alpha$, $a \neq b$. If $\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{J}$ is the canonical map, then $\xi_\alpha(a) = p \neq q = \xi_\alpha(b)$. There is $f \in C(\mathscr{J} \to R)$ with $f(a) = 0$, $f(b) = 1$. Then $g = f \circ \xi_\alpha \in D_\alpha$ and $g(a) = 0$, $g(b) = 1$ which proves (2). Given $\alpha \in A$, $a \in X_\alpha$, $Q \subset X_\alpha$ closed, $a \notin Q$, then $S = \xi_\alpha(Q) \subset \mathscr{J}$ is closed for $\xi^{-1}_\beta(S) = \varrho_{\alpha\beta}(Q)$ is τ_β-closed for all $\beta \geqq \alpha$ (we have $\mathscr{J} = \varinjlim \mathscr{S}_M$, where $M = A(\alpha) = \{\beta \in A \mid \beta \geqq \alpha\}$, for M is confinal in $\langle A \leqq \rangle$). Further, the closure in $\mathscr{J}$ is inductively defined by $\xi_\gamma : \mathscr{X}_\gamma \to |\mathscr{J}|$ (see 0.19), $\xi_\alpha(a) \notin S$, thus there is $f \in C(\mathscr{J} \to R)$ with $f = 0$ on S, $f \circ \xi_\alpha(a) = 1$. Then $g = f \circ \xi_\alpha$ separates Q and a as desired. (4): The family $\mathscr{D} = \{D_\alpha \mid \alpha \in A\}$ is smooth by (1B). Since $\varrho_{\alpha\beta}$ are onto, the maps $\varrho_{\alpha\beta}^*$ are $1-1$. By 1.2.4, $\mathscr{D}$ is connected, hence 1.1.7 works. Likewise we can proceed in terms of i.c. categories. The proof is complete.

Part (4) of the foregoing remark can be modified for the case when either $\mathscr{S}$ is not from CLOS (i.e. $\varrho_{\alpha\beta}$ need not be continuous) or $\langle A \leqq \rangle$ is not well ordered, in the following way:

3.2.3. Remark. *Let* $\mathscr{S} = \{\mathscr{X}_\alpha |\varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ *be a closured inductive family* (*see* 0.12) *and let every* $\varrho_{\alpha\beta}$ *be* $1-1$ *and onto. Suppose there is* $\gamma \in A$ *such that the set* $T_\gamma = D_\gamma \cap \{f \in C(\mathscr{X}_\gamma \to R) \mid f \circ \varrho_{\beta\gamma} \in C(\mathscr{X}_\beta \to R)$ *if* $\beta \leqq \gamma\}$ *separates points of* $\mathscr{X}_\gamma$ (D_γ *are from* 3.2.2). *If* $\mathscr{J} = \varinjlim \mathscr{S}$ *exists then it is* **f.s.** ($\mathscr{J}$ *is defined by* 0.12). *The same holds in terms of* TOP, SEM, ... – *see* 0.5.

Proof. Under our conditions the canonical maps $\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{J}$ are onto. We have $H = \{f \circ \xi_\gamma^{-1} \mid f \in T_\gamma\} \subset C(\mathscr{J} \to R)$. Indeed, if $h = f \circ \xi_\gamma^{-1} \in H$, $\alpha \in A$ then $h \circ \xi_\alpha = f \circ \varrho_{\alpha\gamma}$ for $\alpha \leqq \gamma$, and $h \circ \xi_\alpha = f \circ \varrho_{\gamma\alpha}^{-1}$ for $\gamma < \alpha$. In both the cases $h \circ \xi_\alpha$ is continuous (see the definition of D_α in 3.2.2) hence so is h as desired. The proof in the other categories is the same.

3.2.4. Remark. *Let a presheaf* $\mathcal{S} = \{\mathcal{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ *from an i.c. category* $\mathfrak{L}$ *be endowed with a family* $\mathcal{E} = \{F_\alpha \subset C(\mathcal{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$.

(A) *If for every* $\alpha \in A$ *the set* $V_\alpha = \bigcap\{\varrho^*_{\alpha\beta} F_\beta \mid \beta \geqq \alpha\}$ *separates points of* $\mathcal{X}_\alpha$, $\varrho^*_{\alpha\beta}$ *sends* F_β *in to* F_α *and is* $1-1$ *in* F_β *for all* $\alpha, \beta \in A$, $\alpha \leqq \beta$, *then* $\mathcal{J} = \underleftarrow{\lim}\, \mathcal{S}$ *is f.s.* ($\langle A \leqq \rangle$ *need not be well ordered*).

(B) *Let* $A(\alpha) = \{\beta \in A \mid \beta \geqq \alpha\}$ *be countable for any* $\alpha \in A$. *Suppose that* $\varrho^*_{\alpha\beta}$ *carries* F_β *onto a subset of* F_α *which is of the Baire type* G_δ *and dense in the usual sup-norm for all* $\alpha, \beta \in A$, $\alpha \leqq \beta$. *If every* F_α *is complete in the norm and separates points of* $\mathcal{X}_\alpha$ *then so does* V_α.

Proof. Given $p, q \in \mathcal{J}$, $p \neq q$, there is $\alpha \in A$ such that there are representatives a_α of p and b_α of q in $\mathcal{X}_\alpha$. If $\alpha, \beta \in A$, $\alpha \leqq \beta$ then $V_\alpha = \varrho^*_{\alpha\beta} V_\beta$. Indeed, if $f \in V_\alpha$ then for every $\gamma \geqq \beta$ there is $f_\gamma \in F_\gamma$ with $\varrho^*_{\alpha\gamma} f_\gamma = f = \varrho^*_{\alpha\beta}\varrho^*_{\beta\gamma} f_\gamma$. As $f_\beta, \varrho^*_{\beta\gamma} f_\gamma \in F_\beta$ for all $\gamma \geqq \beta$ and $\varrho^*_{\alpha\beta}$ is $1-1$ in F_β so $\varrho^*_{\beta\gamma} f_\gamma = f_\beta$ for all $\gamma \geqq \beta$, hence $f_\beta \in V_\beta$. As $\varrho^*_{\alpha\beta} f_\beta = f$, we get $V_\alpha \subset \varrho^*_{\alpha\beta} V_\beta$. Clearly $\varrho^*_{\alpha\beta} V_\beta \subset V_\alpha$ which proves the equality. As V_α separates points of $\mathcal{X}_\alpha$, we get from 1.3.1c that $\varrho_{\alpha\beta}$ is $1-1$. By 0.9, the canonical maps $\xi_\alpha : \mathcal{X}_\alpha \to \mathcal{J}$ are $1-1$ for all $\alpha \in A$, so a_α, b_α are unique. We can find $f_\alpha \in V_\alpha$ with $f_\alpha(a_\alpha) = 0$, $f_\alpha(b_\alpha) = 1$. Setting $a_\gamma = \varrho_{\alpha\gamma}(a_\alpha)$, $b_\gamma = \varrho_{\alpha\gamma}(b_\alpha)$ for $\gamma \geqq \alpha$, we can find to every $\gamma \in A(\alpha)$ a unique $f_\gamma \in F_\gamma$ with $\varrho^*_{\alpha\gamma} f_\gamma = f_\alpha (\varrho^*_{\alpha\gamma}$ are $1-1)$. Thus we get a family $\mathcal{F} = \{f_\gamma \mid \gamma \in A(\alpha)\}$ with $f_\gamma(a_\gamma) = 0$, $f_\gamma(b_\gamma) = 1$ for all $\gamma \in A(\alpha)$. As every $\varrho^*_{\alpha\beta}$ is $1-1$ in F_β, we have $f_\gamma = \varrho^*_{\gamma\beta} f_\beta$ for all $\gamma, \beta \in A(\alpha)$, $\gamma \leqq \beta$, hence $\mathcal{F}$ is a thread through $\mathcal{E}$. Further, $\mathcal{J} = \underleftarrow{\lim}\, \mathcal{S}_{A(\alpha)}$ for $A(\alpha)$ is confinal in $\langle A \leqq \rangle$. Thus $f = \underleftarrow{\lim}\, \mathcal{F} \in C(\mathcal{J} \to R)$, which proves (A) for $f(p) = 0$, $f(q) = 1$.

(B) By the Baire category theorem [5, Chap. XIV, sec. 4, Th. 4.1, p. 299], every V_α is norm-dense in F_α, hence V_α separates points.

We can easily prove 3.2.4 if $\langle A \leqq \rangle$ is well ordered. The proof used here allowed to leave out that condition.

3.2.5. Remark. *Given a presheaf* $\mathcal{S} = \{\mathcal{X}_\alpha = (X_\alpha, \tau_\alpha) | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ *from* TOP *such that every* $\mathcal{X}_\alpha$ *is Hausdorff, let* $\varrho_{\alpha\beta}$ *be a homeomorphism of* $\mathcal{X}_\alpha$ *onto a* τ_β*-open set* $\varrho_{\alpha\beta}(X_\alpha)$ *for all* $\beta \geqq \alpha$. *Then* $\mathcal{J} = \underleftarrow{\lim}\, \mathcal{S}$ *is Hausdorff.*

Proof. Let $p, q \in \mathcal{J}$, $p \neq q$. We take $\alpha \in A$ such that there are $a, b \in X_\alpha$ with $\xi_\alpha(a) = p$, $\xi_\alpha(b) = q$ ($\xi_\alpha : \mathcal{X}_\alpha \to \mathcal{J}$ are the canonical maps). As $\mathcal{X}_\alpha$ is Hausdorff, there are τ_α-open sets $G, H \subset X_\alpha$ with $a \in G$, $b \in H$, $G \cap H = \emptyset$. Then $G_\beta = \varrho_{\alpha\beta}(G)$, $H_\beta = \varrho_{\alpha\beta}(H)$ are τ_β-open and disjoint for any $\beta \in A(\alpha) = \{\gamma \in A \mid \gamma \geqq \alpha\}$. As $A(\alpha)$ is confinal in $\langle A \leqq \rangle$, we have $\mathcal{J} = \underleftarrow{\lim}\, \mathcal{S}_{A(\alpha)}$. Clearly $\xi_\beta^{-1}\, \xi_\beta(G) = G_\beta$, $\xi_\beta^{-1}\, \xi_\beta(H) = H_\beta$ are τ_β-open for any $\beta \in A(\alpha)$, hence $M = \xi_\alpha(G)$, $N = \xi_\alpha(H)$ are open and disjoint, $p \in M$, $q \in N$ as desired.

The above remark can be generalised as follows: Instead of $\varrho_{\alpha\beta}(X_\alpha)$ to be τ_β-open we may assume that for any $\alpha \in A$, $a, b \in X_\alpha$ there is $\beta \geqq \alpha$ and τ_β-neighborhoods U, V of $\varrho_{\alpha\beta}(a), \varrho_{\alpha\beta}(b)$ such that $P = \varrho_{\alpha\gamma}(U)$, $Q = \varrho_{\beta\gamma}(V)$ are open τ_γ-neighborhoods of $\varrho_{\alpha\gamma}(a)$, $\varrho_{\alpha\gamma}(b)$, respectively, for all $\gamma \geqq \beta$. The proof is the same. (If P, Q are not always τ_γ-open then only CLOS $\underleftarrow{\lim}\, \mathcal{S}$ is Hausdorff.)

3.2.6. Remark. *Let* $\mathscr{S} = \{(X_\alpha, \tau_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ *be a presheaf from* CLOS, $\langle A \leqq \rangle$ *well ordered and containing a countable confinal set C. Then* $(I, t_I^*) = \varinjlim \mathscr{S}$ *is a Hausdorff closure space if every* (X_α, τ_α) *is Hausdorff and* $\varrho_{\alpha\beta}$ *is a homeomorphism into* (X_β, τ_β) *for all* $\alpha, \beta \in A$.

Proof. There is a confinal set $D \subset C$ such that $\langle D \leqq \rangle$ is of the ordinal type ω_0; $\varinjlim \mathscr{S}_D$ is isomorphic to $\varinjlim \mathscr{S}$, so we may assume that $\langle A \leqq \rangle$ is of the ordinal type ω_0.

It is known that there is an order preserving map of A either on the set N of natural numbers or on its segment $N[k]$. Hence $\mathscr{S}$ is either over $N[k]$ (but then $(I, t_I^*) = (X_{k-1}, \tau_{k-1})$ and the statement holds) or over N. Let $A = N$, $p, q \in I$, $p \neq q$. We set $a_k = \xi_k^{-1}(p)$, $b_k = \xi_k^{-1}(q)$. Let n be the smallest number for which there are a_n, b_n. We may assume that $n = 1$ as $N(n) = \{k \in N \mid k \geqq n\}$ is confinal in N and thus $(I, t_I^*) = \varinjlim \mathscr{S}_{N(n)}$. The maps ϱ_{nn+1} are homeomorphisms into (X_{n+1}, τ_{n+1}). Thus we can construct by induction a sequence $\{U_k\}$, $\{V_k\}$ of τ_k-neighborhoods of a_k, b_k such that $U_k \cap V_k = \emptyset$ and that $\varrho_{jk}^{-1}(U_k) \subset U_j$, $\varrho_{jh}^{-1}(V_k) \subset V_j$ for all $k \in N$ and all $j \leqq k$. Then $U = \bigcup\{\xi_k(U_k) \mid k \in N\}$ and $V = \bigcup\{_k(V_k) \mid k \in N\}$ are t_I^*-neighborhoods of p and q, respectively (see 0·19), and $U \cap V = \emptyset$. Indeed, if $r \in U \cap V$, then there are $k, l \in N$ and $c \in U_k$, $d \in V_l$ with $\xi_k(c) = \xi_l(d) = r$. We may assume $k \leqq l$. Then $c = \varrho_{kl}^{-1}(d) \in V_k$ and $c \in U_k$, so $U_k \cap V_k \neq \emptyset$ – a contradiction, which proves the Remark.

Under the conditions of the foregoing remark we get that (I, t_I^*) is Hausdorff, but we do not know whether there is a Hausdorff topology in I, coarser than t_I^*, as t_I^* is only a closure. We do not know it, either, when $\mathscr{S} \subset$ TOP. Now we shall shortly deal with the question when $\varinjlim \mathscr{S}$ is completely regular or normal.

3.2.7. Remark. *Let* $\mathscr{S} = \{(\mathscr{X}_\alpha, \tau_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ be a normal *presheaf from* TOP, $\langle A \leqq \rangle$ *well ordered. Suppose*

(a) $\varrho_{\alpha\alpha+1} : \mathscr{X}_\alpha \to \mathscr{X}_{\alpha+1}$ *is a homeomorphism onto a* $\tau_{\alpha+1}$*-closed set* $\varrho_{\alpha\alpha+1}(X_\alpha)$ *for all* $\alpha \in A$.

(b) $\{\alpha \in A \mid \alpha - 1$ *does not exist*$\} \subset \mathscr{L}$.

Then $\mathscr{I} =$ TOP $\varinjlim \mathscr{S}$ *is normal. The condition* (b) *may be left out if there is a countable cofinal set in A and if* (a) *holds for any pair* $\alpha, \beta \in A$, $\alpha \leqq \beta$.

Proof. Let $P, Q \subset \mathscr{I}$ be closed, $P \cap Q = \emptyset$, $M_\alpha = \xi_\alpha^{-1}(P)$, $N_\alpha = \xi_\alpha^{-1}(Q)$ ($\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{I}$ are the canonical maps). There is $\alpha \in A$ such that $M_\alpha \neq \emptyset$, $N_\alpha \neq \emptyset$. As $M_\alpha \cap N_\alpha = \emptyset$, there is $f_\alpha \in F_\alpha$ with $f_\alpha/M_\alpha = 1$, $f_\alpha/N_\alpha = 0$. Since $A(\alpha) = \{\beta \in A \mid \beta \geqq \alpha\}$ is confinal in $\langle A \leqq \rangle$, we may assume $\alpha = 1$ (α is the smallest element of $\langle A \leqq \rangle$). Let $\beta \in A$ and let us have a thread $\{f_\gamma \mid \gamma \in A[\beta]\}$ through $\mathscr{E}$ with $f_\gamma \mid M_\gamma = 1$, $f_\gamma \mid N_\gamma = 0$ for all $\gamma \in A[\beta]$. If there is no $\beta - 1$, then $\beta \in \mathscr{L}$ and there is $f_\beta \in F_\beta$ with $\varrho_{\gamma\beta}^* f_\beta = f_\gamma$ for all $\gamma \in A[\beta]$. Clearly $\{f_\gamma \mid \gamma \in \langle 1, \beta\rangle\}$ is a thread through $\mathscr{E}_{\langle 1, \beta\rangle}$ with $f_\gamma \mid M_\gamma = 1$, $f_\gamma \mid N_\gamma = 0$ for all $\gamma \in \langle 1, \beta\rangle$. If there is $\beta - 1$ then $g = f_{\beta-1} \circ \varrho_{\beta-1\beta}^{-1}$ is τ_β-continuous on $O_\beta = \varrho_{\beta-1\beta}(X_{\beta-1})$, so we can define $\tilde{g}$ on $R_\beta = O_\beta \cup$

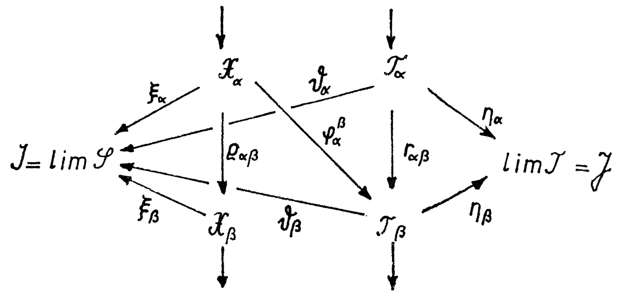
$\cup M_\beta \cup N_\beta$ as follows: $\tilde{g} = g$ on O_β, $\tilde{g} = 1$ on M_β, $\tilde{g} = 0$ on N_β. Then $\tilde{g}$ is τ_β-continuous on R_β, hence by 1.3.4A, there is a continuous extension f_β of $\tilde{g}$ to the whole $\mathscr{X}_\beta$. Clearly $\{f_\gamma \mid \gamma \in \langle 1\beta \rangle\}$ is a thread through $\mathscr{E}$ with $f_\gamma \mid M_\gamma = 1$, $f_\gamma \mid N_\gamma = 0$ for all $\gamma \in \langle 1, \beta \rangle$. By induction, there is $f \in C(\mathscr{I} \to R)$ with $f/P = 1$, $f/Q = 0$ which completes the proof. The last statement readily follows from 1.2.6.

3.2.8. Remark. *Let* $\mathscr{S} = \{\mathscr{X}_\alpha = (X_\alpha, n_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ *be a presheaf from* UNIF. *We put* $\mathrm{cl}\,\mathscr{S} = \{(X_\alpha, \mathrm{cl}\, n_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ ($\mathrm{cl}\, n_\alpha$ *is the closure generated in* X_α *by* n_α; *it is always a topology hence* $\mathrm{cl}\,\mathscr{S}$ *is from* TOP – *see* 0.9), $(I, n) =$ UNIF $\varinjlim \mathscr{S}$, $(I, t^*) =$ TOP $\varinjlim \mathrm{cl}\,\mathscr{S}$ (*see* 0.7). *Then the topology t, induced in I by n is completely regular and coarser than t** (*notice that* (I, n) *need not be separated*). *If* (I, n) *is f.s. then so are* (I, t) *and* (I, t_I^*).

Proof. By [3, Chap. V, sec 29, Th. 28A5(f), p. 504], t is completely regular. If (I, n) is f.s. (by $U((I, n) \to R)$), then so is (I, t) (by $C((I, t) \to R)$). The canonical maps $\xi_\alpha : (X_\alpha, \mathrm{cl}\, n_\alpha) \to (I, t)$ are continuous thus so is the identity $e : (I, t^*) \to$ $\to (I, t)$ as desired.

3.2.9. Remark. *Given an i.c. category* $\mathfrak{L}$ *and an inductive family* $\mathscr{S} \doteq$ $= \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ *of* $\mathfrak{L}$-*objects* (*see* 0.12), *suppose that* $A[\alpha]$ *is right directed for every* $\alpha \in A$. *If* $\alpha \in A$, *we put* $\mathscr{T}_\alpha = \varinjlim \mathscr{S}_{A[\alpha]}$ *in the sense of* 0.12 (*see* 1.1.3, 0.2). *Let* $r_{\alpha\beta} : \mathscr{T}_\alpha \to \mathscr{T}_\beta$ *be the canonical* $\mathfrak{L}$-*morphisms. Then* $\mathscr{T} = \{\mathscr{T}_\alpha \mid r_{\alpha\beta} | \langle A \leqq \rangle\}$ *is from* $\mathfrak{L}$ (*see* 0.2) *and* $\mathscr{J} = \varinjlim \mathscr{T}$ is $\mathfrak{L}$-*isomorphic with* $\mathscr{I} = \varinjlim \mathscr{S}$ *if* $\mathscr{I}$ exists- ($\varinjlim \mathscr{S}$ *is meant in the sense of* 0.12).

Proof. Clearly $\mathscr{T}$ is from $\mathfrak{L}$. Look at this commutative diagram:



Here ϑ_α, φ_α^β, ξ_α, η_α are canonical maps. Every arrow except $\varrho_{\alpha\beta}$ is an $\mathfrak{L}$-morphism. Thus there are $\mathfrak{L}$-morphisms $i : \mathscr{I} \to \mathscr{J}$ and $j : \mathscr{J} \to \mathscr{I}$ with $i \circ \xi_\alpha = \eta_\beta \circ \varphi_\alpha^\beta$ for all $\beta > \alpha$, and $j \circ \eta_\alpha = \vartheta_\alpha$ for all $\alpha \in A$. Clearly $i \circ j$ and $j \circ i$ are identical as desired.

The foregoing remark can be useful if $\mathscr{S}$ is not from $\mathfrak{L}$. We can form $\mathscr{T}$ and deal with the functional separatedness of $\mathscr{T}$.

Given an i.c. category $\mathfrak{L}$ and an inductive family $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ of $\mathfrak{L}$-objects, which is endowed with a family $\mathscr{E} = \{F_\alpha \subset C_\alpha = C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$, then if $\cdot\leqq$ is the inverse order in A (i.e. $\alpha \cdot\leqq \beta$ iff $\beta \leqq \alpha$), then $\mathscr{E}^* = \{F_\alpha | \varrho^*_{\alpha\beta} | \langle A \cdot\leqq \rangle\}$ is an inductive family from SET (see 0.5). We can construct the projective limit $P = \varprojlim \mathscr{E}^*$ (see 0.6). Further, we assume that there is $\mathscr{J} = \varinjlim \mathscr{S}$ (see 0.12).

3.3.1. Remark. *$P = \varprojlim \mathscr{E}^*$ is isomorphic with the set $S_{\mathscr{E}} = \{f \in C(\mathscr{J} \to R \mid \mathfrak{L}) \mid f \circ \xi_\alpha \in F_\alpha$ for all $\alpha \in A\}$ ($\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{J}$ are the canonical maps). Thus if $\mathscr{C} = \{C_\alpha = C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$, then $\varprojlim \mathscr{C}^* \neq \emptyset$.*

Proof. Each element of P is a thread $\mathscr{F} = \{f_\alpha \mid \alpha \in A\}$ through $\mathscr{E}$ (see 0.6). Such a thread generates a unique map $f : C(\mathscr{J} \to R)$ with $f_\alpha = f \circ \xi_\alpha$ for all $\alpha \in A$. Putting $G(\mathscr{F}) = f$, we have $f \in S_{\mathscr{E}}$. On the other hand, if $f \in S_{\mathscr{E}}$, then $\mathscr{F} = \{f_\alpha = f \circ \xi_\alpha \mid \alpha \in A\}$ is from P as $f_\alpha = \varrho^*_{\alpha\beta} f_\beta$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$. Setting $A(f) = \mathscr{F}$ we get $A \circ G(\mathscr{F}) = \mathscr{F}$, $G \circ A(f) = f$, so G and A are isomorphisms. As $S_{\mathscr{C}} = C(\mathscr{J} \to R \mid \mathfrak{L}) \neq \emptyset$, we get $\varprojlim \mathscr{C}^* \neq \emptyset$.

3.3.2. Definition. *We say that $P = \varprojlim \mathscr{E}^*$ separates points of $\mathscr{J} = \varinjlim \mathscr{S}$ if the set $S_{\mathscr{E}}$ from* 3.3.1 *separates points of $\mathscr{J}$.*

3.3.3. Remark. *$\mathscr{J} = \varinjlim \mathscr{S}$ is f.s. iff there is a family $\mathscr{E}$ for $\mathscr{S}$ such that $\varprojlim \mathscr{E}^*$ separates points of $\mathscr{J}$. Thus a necessary and sufficient condition for $\mathscr{J}$ to be f.s. is that $\varprojlim \mathscr{C}^*$ separates points of $\mathscr{J}$, where $\mathscr{C} = \{C_\alpha = C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$.*

Proof. If $\mathscr{J}$ is f.s., then there is $S \subset C = C(\mathscr{J} \to R \mid \mathfrak{L})$ which is separating (and then so is also C itself). Setting $F_\alpha = \{f \circ \xi_\alpha \mid f \in S\}$ for $\alpha \in A$, $\mathscr{E} = \{F_\alpha \mid \alpha \in A\}$ then $\varprojlim \mathscr{E}^*$ is separating for $\mathscr{J}$. The converse is clear. Since $\varprojlim \mathscr{E}^* \subset \varprojlim \mathscr{C}^*$ for any $\mathscr{E}$, the last statement follows.

3.3.4. Remark. *Let an inductive family $\mathscr{D} = \{D_\alpha | \tilde{\varrho}_{\alpha\beta} | \langle A \leqq \rangle\}$ from* SET *be given. Denoting by $\cdot\leqq$ the inverse order for $\leqq$, we assume that $\langle A \cdot\leqq \rangle$ is right directed and that there is an i.c. category $\mathfrak{L}$ and a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \cdot\leqq \rangle\}$ of $\mathfrak{L}$-objects such that $\mathscr{J} = \varinjlim \mathscr{S}$ exists and that at least one of the following conditions holds:*

(1) *$D_\alpha \subset C_\alpha = C(\mathscr{X}_\alpha \to R \mid \mathfrak{L})$, $\tilde{\varrho}_{\alpha\beta} = \varrho^*_{\beta\alpha}$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$, and $S = \{f \in C(\mathscr{J} \to R \mid \mathfrak{L}) \mid f \circ \xi_\alpha \in D_\alpha,\ \alpha \in A\}$ ($\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{J}$ are the canonical maps).*
(2) *$C_\alpha \subset D_\alpha$, $\tilde{\varrho}_{\alpha\beta} \mid C_\alpha = \varrho^*_{\beta\alpha}$ for all $\alpha, \beta \in A$, $\beta \cdot\leqq \alpha$.*
Then $P = \varprojlim \mathscr{D} \neq \emptyset$.

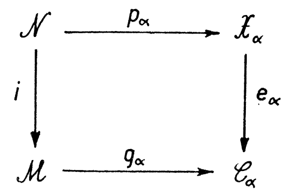
Proof. If (1) holds, then P is isomorphic with S. If (2) holds, then by 3.1.1, $P' = \varprojlim \mathscr{C}^* \neq \emptyset$, where $\mathscr{C} = \{C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$. As $\varprojlim \mathscr{C}^* \subset \varprojlim \mathscr{D}$, we get $P \neq \emptyset$.

3.3.5. Corollary. *Let each D_α from* 3.3.4 *be a topological locally convex linear reflexive space, so that $\tilde{\varrho}_{\alpha\beta}$ are linear maps. Then $\varprojlim \mathscr{D} \neq \emptyset$.*

Proof. If $\mathscr{X}_\alpha$ is the dual space of D_α, then $D_\alpha = C(\mathscr{X}_\alpha \to R \mid \mathfrak{L})$, where $\mathfrak{L} = \text{LIN}$ (see 0.5). Setting $\varrho_{\alpha\beta} = \tilde{\varrho}^*_{\beta\alpha}$ we have $\varrho^*_{\alpha\beta} = \tilde{\varrho}^{**}_{\beta\alpha} = \tilde{\varrho}_{\beta\alpha}$ and $\mathscr{S} = \{\mathscr{X}_\alpha | \tilde{\varrho}_{\alpha\beta} | \langle A \cdot \leqq \rangle\}$ is a presheaf of LIN – objects and LIN $\varinjlim \mathscr{S}$ exists. Now we use 3.3.4 (2).

3.3.6. Proposition. *Let $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be a topologised inductive family* (*see* 0·12) *for which there is a compact hull $\mathscr{C} = \{\mathscr{C}_\alpha | r_{\alpha\beta} | \langle A \leqq \rangle\}$* (*see* 2.1.2B) *such that all the $r_{\alpha\beta}$ are* $1-1$. *If $\langle A \leqq \rangle$ is right or left directed then all the canonical maps $\{p_\alpha : \mathscr{N} = \varprojlim \mathscr{S} \to \mathscr{X}_\alpha \mid \alpha \in A\}$* (*see* 0.6) *are homeomorphisms.*

Proof. There is $\mathscr{N} = \varprojlim \mathscr{S}$ and $\mathscr{M} = \varprojlim \mathscr{C}$. Denoting by $e_\alpha : \mathscr{X}_\alpha \to \mathscr{C}_\alpha$ ($g_\alpha : \mathscr{M} \to \mathscr{C}_\alpha$) the embeddings of $\mathscr{X}_\alpha$ into $\mathscr{C}_\alpha$ (the cannonical maps of $\mathscr{M}$ into $\mathscr{C}_\alpha$), we have $e_\alpha \circ p_\alpha : \mathscr{N} \to \mathscr{C}_\alpha$ for all $\alpha \in A$. By 0.6, there is a unique $1-1$ continuous map $i : \mathscr{N} \to \mathscr{M}$ such that $g_\alpha \circ i = e_\alpha \circ p_\alpha$ for all $\alpha \in A$. We get the commutative diagram



By [5, Appendix 2, Th. 2.5, p. 430, Th. 2.8, p. 432], i is a homeomorphism into $\mathscr{M}$. By [5, Appendix 2, 2.4, p. 429], the topological space $\mathscr{M}$ is compact. The maps $g_\alpha : \mathscr{M} \to \mathscr{C}_\alpha$ are $1-1$. Indeed, the set $|\mathscr{M}|$ consists of all threads $a = \{a_\alpha \in |\mathscr{C}_\alpha| \mid \alpha \in A\}$ – see 1.2.1, [5, Appendix 2, Def. 2.2, p. 427] and $g_\alpha(a) = a_\alpha$ for all $\alpha \in A$ [5, App. 2, p. 428]. Given $\alpha \in A$, $a = \{a_\alpha \mid \alpha \in A\}$, $b = \{b_\alpha \mid \alpha \in A\} \in \mathscr{M}$ with $g_\alpha(a) = g_\alpha(b)$, then $a_\alpha = b_\alpha$. If $a_\delta = b_\delta$ for some $\delta \in A$ then clearly $a_\gamma = b_\gamma$ for all $\gamma \in A$, $\gamma \geqq \delta$. Let $\beta \in A$. If $\langle A \leqq \rangle$ is right (left) directed, there is $\gamma \in A$ with $\gamma \geqq \alpha$, $\gamma \geqq \beta$ ($\gamma \leqq \alpha$, $\gamma \leqq \beta$). Then $r_{\beta\gamma}(a_\beta) = a_\gamma = b_\gamma = r_{\beta\gamma}(b_\beta)$, hence $a_\beta = b_\beta$ ($r_{\gamma\alpha}(a_\gamma) = a_\alpha = b_\alpha = r_{\gamma\alpha}(b_\gamma)$, hence $a_\gamma = b_\gamma$ so $a_\beta = b_\beta$) as all the $r_{\mu\nu}$ are $1-1$. This shows g_α to be $1-1$. Hence all the g_α are homeomorphisms into $\mathscr{C}_\alpha$ (see 0.15), thus so are also $g_\alpha^{-1} \circ e_\alpha = i \circ p_\alpha^{-1}$, which gives the continuity of p_α^{-1} as i is a homeomorphism.

3.3.7. Remark. In all the theorems, where we have a compact hull $\mathscr{C}$ of $\mathscr{S}$ over a right or left directed set such that all the $r_{\alpha\beta}$ are $1-1$, all the canonical maps $p_\alpha : \varprojlim \mathscr{S} \to \mathscr{X}_\alpha$ are homeomorphisms into $\mathscr{X}_\alpha$. This occurs namely in 2.1.7, 2.2.9 if $\mathscr{E}$ is strongly separating, in 2.2.7, 2.2.8 if $\mathscr{E}'$ is strongly separating, and in 2.3.6, 2.3.8, 3.1.3 (it can be easily seen by 0.6 that in 3.1.3 $\varprojlim \mathscr{S}$ and $\varprojlim \mathscr{S}^\#$ are isomorphic). Especially, this holds for $\mathscr{S}_B$ from 1.5.2, 1.5.5 and for $\mathscr{S}$ from the statement in 1.5.6B.

Now we will deal with the presheaves over countable sets.

If $\langle A \leqq \rangle$ is a well ordered and countable set whose ordinal number is at most ω_0 then there is an order preserving map of A either on to the set N of natural numbers or onto its segment $N[k]$. Thus if $\mathscr{S} = \{X_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ is a presheaf from a category $\mathfrak{K}$ then $\mathscr{S}$ is either over N or over $N[k] = \{1, \ldots, k-1\}$. If A is finite then $\varinjlim \mathscr{S}$ is $\mathfrak{K}$-isomorphic with X_{k-1}, where $A = N[k]$. The only case which is worth dealing with is $A = N$. Then we write $\mathscr{S} = \{X_m | \varrho_{mn} | N\}$ and say that $\mathscr{S}$ is over N.

3.4.1. Proposition. *Let $\mathscr{S} = \{(X_n, \tau_n) | \varrho_{mn} | N\}$ be a presheaf from* CLOS *over N such that all ϱ_{mn} are homeomorphisms of (X_m, τ_m) into (X_n, τ_n). Suppose that every (X_n, τ_n) is metrisable with the metric d_n (thus $\mathscr{S} \subset$* TOP *and $\mathscr{S}$ is a presheaf of* UNIF-*objects — see* 0·12, *but it need not be $\mathscr{S} \subset$* UNIF *as ϱ_{mn} need not be uniformly continuous). If $a, b \in (I, t_I^*) =$* CLOS $\varinjlim \mathscr{S}$ *(see* 0·7*) then there is the smallest $n \in N$ such that a, b have representatives $a_n, b_n \in X_n$. We put $a_m = \varrho_{nm}(a_n)$, $b_m = \varrho_{nm}(b_n)$ for $m \geqq n$.*

A. *Let us set $D(a, b) = \sum_{k=n}^{\infty} 2^{-k} d_k(a_k, b_k)(1 + d_k(a_k, b_k))^{-1}$. Then the function D is a metric in I such that all canonical maps $\xi_k : (X_k, d_k) \to (I, D)$ are homeomorphisms into (I, D). We have $D < 1$ and D yields a topology in I, coarser than t_I^*.*

B. *Given $p \in N$, we set $q = \max(n, p)$ and $D_p(a, b) = \sum_{k=q}^{\infty} 2^{-k} d_k(a_k, b_k)(1 + d_k(a_k, b_k))^{-1}$. Then D and D_p are equivalent metrics in I.*

Proof. A: It is an easy matter to check that D is a metric. We prove that all the $\xi_n : (X_n, d_n) \to (I, D)$ are continuous. Given $n \in N$, $a \in X_n$ and $\varepsilon > 0$, then there is $m \geqq n$ such that $\sum_{j=m} 2^{-j} < 2^{-1} \cdot \varepsilon$. The function $f(x) = x(1 + x)^{-1}$ is increasing, continuous and $f(0) = 0$, $f(x) < 1$ for all $x \geqq 0$, thus there is $\delta > 0$ such that $\delta(1 + \delta)^{-1} < 2^{-1} \cdot \varepsilon$. Moreover, $\eta > 0$ can be found such that for all $b \in X_n$ with $d_n(a, b) < \eta$ we have $d_k(\varrho_{nk}(a), \varrho_{nk}(b)) < \delta$ whenever $n \leqq k \leqq m$ (for all ϱ_{ij} are continuous) and that $d_k(\varrho_{kn}^{-1}(a), \varrho_{kn}^{-1}(b)) < \delta$ if $k < n$ and if $\varrho_{kn}^{-1}(a), \varrho_{kn}^{-1}(b)$ exist (for all $\varrho_{ij}^{-1} : (\varrho_{ij}(X_i), d_j) \to (X_i, d_i)$ are homeomorphisms — see 0.15). Given $b \in X_n$ with $d_n(a, b) < \eta$, let $l \leqq n$ be the smallest number for which there are $a_l, b_l \in X_l$ with $\varrho_{ln}(a_l) = a$, $\varrho_{ln}(b_l) = b$. Setting $a_j = \varrho_{lj}(a_l)$, $b_j = \varrho_{lj}(b_l)$ for $j \geqq l$, we have $D(\xi_n(a), \xi_n(b)) = \sum_{j=l}^{\infty} 2^{-j} d_j(a_j, b_j)(1 + d_j(a_j, b_j))^{-1} = \sum_{j=l}^{m} \ldots + \sum_{j=m+1}^{\infty} \ldots < \sum_{j=l}^{m} + 2^{-1}\varepsilon < \sum_{j=l}^{m} \delta(1 + \delta)^{-1} \cdot 2^{-j} + 2^{-1}\varepsilon < \delta(1 + \delta)^{-1} \sum_{j=1}^{\infty} 2^{-j} + 2^{-1}\varepsilon < 2^{-1}\varepsilon + 2^{-1}\varepsilon = \varepsilon$, which proves the continuity of $\xi_n : (X_n, d_n) \to (I, D)$. By 0.4, the identity $(I, t_I^*) \to (I, D)$ is continuous, hence D generates a coarser closure than t_I^*.

If $n \in N$, $a, b \in X_n$, then $2^{-n} d_n(a, b) (1 + d_n(a, b))^{-1} < D(\xi_n(a), \xi_n(b))$ which gives $d_n(a, b) < 2^n D(\xi_n(a), \xi_n(b)) (1 - 2^n D(\xi_n(a), \xi(b))^{-1}$ whenever $2^n D(\xi_n(a), \xi_n(b)) < 1$. As the function $g(x) = Cx(1 - Cx)^{-1}$ is continuous on $\langle 0, C^{-1})$, $g(0) = 0$, so for any $\varepsilon > 0$ there is δ with $0 < \delta < 1$ such that $g(x) < \varepsilon$ whenever $0 < x < \delta$. This proves the continuity of $\xi_n^{-1} : (\xi_n(X_n), D) \to (X_n, d_n)$. Indeed, given $p, q \in \xi_n(X_n)$, $\varepsilon > 0$, then we can find $\delta > 0$ with $0 < \delta < 1$ such that $|g(x)| < \varepsilon$ if $0 < x < \delta$. If $D(p, q) < \delta$ then $d_n(\xi_n^{-1}(p), \xi_n^{-1}(q)) < g(D(p, q)) < \varepsilon$ as desired.

B: As $A(p) = \{n \in N \mid n \geqq p\}$ is confinal in N, we have $(I, t_I^*) = \varinjlim \mathscr{S}_{A(p)}$. By A, D_p is a metric in I. Clearly $D_p \leqq D$, so the topology generated by D is finer than that generated by D_p. Conversely, given $x \in I$, $\varepsilon > 0$ and a D-nbd $M_\varepsilon = \{y \in I \mid D(x, y) < \varepsilon\}$ of x, we have to find a $\mu > 0$ such that $N_\mu = \{y \in I \mid D_p(x, y) < \mu\} \subset M_\varepsilon$. Let n be the smallest number such that there is a representative $a \in X_n$ of x. If $n \geqq p$ then it follows from the definition of D and D_p that $D_p(x, y) = D(x, y)$ for all $y \in I$, so $N_\varepsilon = M_\varepsilon$. It remains to deal with the case $n < p$. There is m with $m \geqq p$ so that $\sum_{k=m}^{\infty} 2^{-k} < \varepsilon$. As $\xi_m : (X_m, d_m) \to (I, D)$ is continuous, there is $\delta > 0$ such that for all $b \in X_m$ with $d_m(\varrho_{nm}(a), b) < \delta$, we have $D(x, \xi_m(b)) < \varepsilon$ (we have $\xi_m \circ \varrho_{nm}(a) = x$). Using A to $\mathscr{S}_{A(p)}$ and D_p, we get the continuity of $\xi_m^{-1} : (\xi_m(X_m), D_p) \to (X_m, d_m)$. Thus there is $\mu > 0$ so that $d_m(\xi_m^{-1}(x), \xi_m^{-1}(y)) < \delta$ for all $y \in N_\mu$, for which $\xi_m^{-1}(y)$ exists (we have $\xi_m^{-1}(x) = \varrho_{nm}(a)$). We have $N_\mu \subset M_\varepsilon$. Indeed, if $y \in N_\mu$ is such that $b_m = \xi_m^{-1}(y)$ does not exist, then $D(x, y) < \sum_{k=m}^{\infty} 2^{-k} < \varepsilon$, so $y \in M_\varepsilon$. If there is b_m, then $d_m(\xi_m^{-1}(x), b_m) < \delta$, hence $D(x, y) < \varepsilon$ as desired. Thus the topologies generated by D and D_p are equivalent.

3.4.2. Remark. In the light of Lemma 1.4.2, all remarks of the second section and also Proposition 3.4.1 remain valid in the following form: Let $\mathfrak{L}$ be the same category which we have in the remark or proposition, let $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\} \subset \mathfrak{L}$ be a presheaf. Let us have $B \subset A$ such that

(a) Either B is confinal in $\langle A \leqq \rangle$ or $\langle A \leqq \rangle$ is ordered, $\langle A - B \leqq \rangle$ well ordered and $A - B \subset \mathscr{L}$,

(b) $\mathscr{S}_B$ fulfils the conditions of the remark or proposition. Then the assertion of the remark (proposition) holds.

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