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NON-COMMUTATIVE INTERPOLATION OF SOBOLEV-BESOV AND LEBESGUE SPACES WITH WEIGHTS

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The paper contains results about interpolation of weighted Sobolev-Besov and Lebesgue spaces in domains. Further, we give examples and counter-example of non-commutative interpolation. These results arise from the problem of finding conditions ensuring

$$(*) \qquad F(\{A_0, A_1 \cap A_2\}) = F(\{A_0, A_1\}) \cap F(\{A_0, A_2\}),$$

where A_0, A_1 and A_2 are Banach spaces contained in a linear Hausdorff space and F is an interpolation functor. Results of the type $(*)$ are well-known if the so-called commutative interpolation is considered. The first result may be found in Lions [8]. Later, Peetre [12] showed that $(*)$ is true using the theory of quasilinearisable interpolation couples; he supposed that the operators involved commute with each other. The interpolation of domains of infinitesimal generators of commuting semi-groups of bounded operators or the interpolation of domains of positive operators with commuting resolvents are special cases of this result. All these assertions are based on hypotheses of commutativity and were mainly obtained by the authors mentioned above and by T. Muramatu [9], H. Komatsu [7] and by P. Grisvard [3]. These statements are contrasted not only by counter-examples (see H. Triebel [15]), but also by results of the type $(*)$ which were obtained independently of the assertions mentioned above; in particular, the commutativity assumptions are not fulfilled. For details we refer to H. Triebel [13, 14, 15, 16]. So it is interesting to extend $(*)$ to the non-commutative case. J. Peetre [10] and P. Grisvard [4] obtained general results for the real interpolation functor.

In our work we will only make use of the very applicable result by P. Grisvard. $(*)$ is proved by him in the case that A_0 contains A_2 and A_1 is the domain of a positive operator A acting in A_0, with a positive restriction on A_2 (see Section 1). On the one hand, our paper applies the theory of non-commutative interpolation to function spaces of Sobolev-Besov or Lebesgue types, respectively, with and without weights; on the other hand, we want to give an idea of how far these methods can be used.

In our considerations, the Banach spaces A_0 and A_2 will be weighted or unweighted Sobolev-Besov or Lebesgue spaces, for instance: $A_0 = L_p(\Omega)$, $A_2 = B^s_{pp}(\Omega)$ (the well-known Besov space; if $s \neq$ integer then the Slobodeckij spaces arise). The positive operator A, acting in A_0, is the multiplication operator: $(Au)(x) = \varphi(x) u(x)$ where $\varphi(x)$ is a measurable positive function. Thus, the domain $D(A)$, of definition of A is a weighted Sobolev-Besov space and for several classes of functions $\varphi(x)$ (described by certain growth conditions) we get results of the type

$$(L_p(\Omega), L_p(\Omega, \varphi) \cap B^s_{pp}(\Omega))_{\theta,q} =$$

$$= (L_p(\Omega), L_p(\Omega, \varphi))_{\theta,q} \cap (L_p(\Omega), B^s_{pp}(\Omega))_{\theta,q} .$$

With the aid of non-commutative interpolation we get, for a class of weight-functions $\varphi_0(x)$ and $\varphi_1(x)$,

$$(W^m_p(\Omega, \varphi_0), W^m_p(\Omega, \varphi_1) \cap W^l_p(\Omega, \varphi_0))_{\theta,p} =$$

$$= (W^m_p(\Omega, \varphi_0), W^m_p(\Omega, \varphi_1))_{\theta,p} \cap (W^m_p(\Omega, \varphi_0), W^l_p(\Omega, \varphi_0))_{\theta,p} .$$

Simultaneously we extend earlier results by A. Favini [1] about the interpolation $(W^m_p(\Omega, \varphi_0), W^m_p(\Omega, \varphi_1))_{\theta,p}$.

Finally, we present a series of examples and counter-examples to $(*)$.

1. MULTIPLICATION OPERATOR IN UNWEIGHTED SOBOLEV SPACES

First we describe the result by P. Grisvard [4]. We use the following statement which is important for the whole work:

Proposition. *Let A_0 and A_2 be Banach spaces, $A_2 \subset A_0$. Let A be a closed unbounded operator acting in A_0 with $D(A) = A_1$ (note that A_1 is a Banach space equipped with the norm $\|u\|_{A_1} = \|u\|_{A_0} + \|Au\|_{A_0}$, $u \in D(A)$). We assume that the operator A satisfies the following conditions:*

1. *For $0 < t < \infty$ the operator $tA + E$ is invertible in A_0 and*

(1) $$\|(tA + E)^{-1} u\|_{A_0} \leqq c_1 \|u\|_{A_0} , \quad u \in A_0 ;$$

2. *for $0 < t < \infty$ the operator $tA + E$ is invertible in A_2 and*

(2) $$\|(tA + E)^{-1} u\|_{A_2} \leqq c_2 \|u\|_{A_2} , \quad u \in A_2$$

with constants c_1 and c_2 independent of t.

I take the opportunity to thank H. Triebel for pushing me on.

Then for $0 < \theta < 1$, $1 < q < \infty$,

$$(3) \qquad (A_0, A_1 \cap A_2)_{\theta,q} = (A_0, A_1)_{\theta,q} \cap (A_0, A_2)_{\theta,q}$$

is valid.

Here $(\cdot, \cdot)_{\theta,q}$ denotes the real interpolation functor (for instance the K-functor in the sense of J. Peetre [11]). We denote the conditions (1) and (2) in the proposition by "A_0-condition" and "A_2-condition", respectively, and state:

If, for the closed operator $tA + E$; $0 < t < \infty$, there exists a suitable operator $I_t : A_0 \to A_0$; $0 < t < \infty$, such that

(4) for $0 < t < \infty$, $tA + E$ is an operator from $D(A)$ onto A_0;

(5) the operator $tA + E$; $0 < t < \infty$, is a bounded operator from A_0 into A_0;

(6) in A_0, $I_t(tA + E) = E$ holds,

then the operators $tA + E$; $0 < t < \infty$, are invertible in A_0: $(tA + E)^{-1} = I_t$. If, additionally, (1) is valid, then the A_0-condition is fulfilled. The same can be said about the A_2-condition. Replacing the operators I_t by I'_t, and the space A_0 by A_2 in (4), (5) and (6) we get the properties (4′), (5′) and (6′), respectively.

Basic notations. Let Ω be an arbitrary domain in R_n with a boundary $\partial\Omega$ and a closure $\bar{\Omega} : \partial\Omega = \bar{\Omega} - \Omega$. $L^\infty(\Omega)$ denotes the set of all functions, Lebesgue-measurable in Ω with bounded essential suprema. By $W_p^m(\Omega)$; $m = 1, 2, \ldots$; $1 < p < \infty$, we denote the usual Sobolev spaces equipped with the norms

$$(7) \qquad \|u\|_{W_p{}^m(\Omega)} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L_p(\Omega)} \Big)^{1/p}, \quad u \in W_p^m(\Omega).$$

Let $C_0^\infty(\Omega)$ be the set of all infinitely differentiable finite functions; we denote the closure of this set in the Banach space $W_p^m(\Omega)$ by $\mathring{W}_p^m(\Omega)$. Denote by $L_p(\Omega, \varphi(x))$ the usual weighted space with a non-negative measurable function $\varphi(x)$. The norm is given by

$$(8) \qquad \|u\|_{L_p(\Omega,\, \varphi(x))} = \int_\Omega |\varphi(x)\, u(x)|^p \, \mathrm{d}x^{1/p}, \quad u \in L_p(\Omega, \varphi(x)).$$

The spaces arising by the real interpolation of Sobolev spaces $W_p^m(\Omega)$ are the Besov spaces $B^s_{p,q}(\Omega)$; by the complex interpolation we have the Lebesgue (Liouville or Bessel-potential) spaces $H_p^s(\Omega)$. We introduce a smoothness-property for bounded domains in R_n:

Definition. A bounded domain Ω belongs to a class C^m; $m = 1, 2, \ldots$ or $m = \infty$, if the following conditions are satisfied:

(9) There exist balls K_j; $j = 1, \ldots, N$, with

$$\bigcup_{j=1}^{N} K_j \supset \partial\Omega \quad \text{and} \quad K_j \cap \partial\Omega \neq \emptyset .$$

(10) There exist vector-functions $f^{(j)}(x) = (f_1^{(j)}(x), \ldots, f_n^{(j)}(x))$ defined on $\bar{K}_j$ and satisfying the following conditions:

(i) The functions $f^{(j)}(x)$ are continuously differentiable up to the m-th order.

(ii) $y = f^{(j)}(x)$ is a one-to-one map from K_j onto a bounded domain in R_n^+ . f^j maps $\partial\Omega \cap K_j$ onto a part of the domain in R_n $\{y : y \in R_n;\ y_n = 0\}$ and $\Omega \cap K_j$ onto a simple-connected domain in the half-space R_n^+.

(iii)
$$\frac{\partial(f_1^{(j)}, \ldots, f_n^{(j)})}{\partial(x_1, \ldots, x_n)} \neq 0 \quad \text{for} \quad x \in \bar{K}_j\,;$$

$\bar{f}^{(j)}$ denotes the transformation inverse to $f^{(j)}$.

Remark. If Ω is a domain belonging to C^m, then there exists a compact domain ω, $\bar{\omega} \subset \Omega$, with

(11)
$$\Omega \subset \bigcup_{j=1}^{N} K_j \cup \omega\,.$$

Domains Ω belonging to a class C^m are called *domains with a C^m-boundary.*

1.1. THE CLASSES $K_1^{(m)}$ OF WEIGHT-FUNCTIONS

A weight-function is a function $\varphi(x) : \Omega - R_1$ which is measurable, positive and infinitely differentiable.

Definition. A weight-function $\varphi(x)$ belongs to a class $K_1^{(m)}$; $m \in \{1, 2, \ldots\}$, if:

(1)
$$\frac{1}{\varphi(x)} \in L^\infty(\Omega)\,,$$

(2)
$$\frac{D^\alpha \varphi(x)}{\varphi(x)} \in L^\infty(\Omega)\,, \quad 0 < |\alpha| \leqq m\,.$$

The classes $K_1^{(m)}$ are closed with respect to addition, multiplication and involution. Now we specify the Banach spaces A_0, A_1 and A_2 is Grisvard's proposition:

We assume that Ω is an arbitrary domain in R_n, m is an integer and $\varphi(x)$ is a weight-function. We set:

(3)
$$A_0 = L_p(\Omega)\,, \quad A_2 = W_p^m(\Omega)$$

and

$$(4) \qquad A_0 = L_p(\Omega)\,, \quad A_2 = \mathring{W}_p^m(\Omega)\,,$$

respectively. In both cases we put:

$$(Au)(x) = \varphi(x)\,u(x)\,, \quad D(A) = \{u : u \in L_p(\Omega),\ \varphi u \in L_p(\Omega)\}\,,$$

$$A_1 = [D(A), \|\cdot\|_A] = L_p(\Omega, \varphi(x))\,.$$

Theorem. *Let Ω be an arbitrary domain in R_n. Let the Banach spaces A_0, A_1 and A_2 be given by* (3).

For weight-functions $\varphi(x)$ belonging to $K_1^{(m)}$ the conditions of Grivard's proposition are satisfied and for $0 < \theta < 1$, $1 < p < \infty$ we have

$$(5) \qquad (L_p(\Omega), W_p^m(\Omega) \cap L_p(\Omega, \varphi(x)))_{\theta,p} = W_p^m(\Omega) \cap L_p(\Omega, \varphi(x)^\theta)\,.$$

Proof. We prove the A_0-condition. First we show 1. (1), (4), (5), (6). For $0 < t < \infty$ we put

$$(I_t u)(x) = \frac{1}{t\,\varphi(x) + 1}\,u(x)\,, \quad D(I_t) = L_p(\Omega)\,.$$

1. (1), (4), (5), (6) are obvious, since

$$(6) \qquad \int_\Omega \left|\frac{u}{t\varphi + 1}\right|^p \mathrm{d}x \leqq c \int_\Omega |u|^p\,\mathrm{d}x\,, \quad c \text{ independent of } t\,,$$

and

$$(7) \qquad \int_\Omega \left(\frac{\varphi}{t\varphi + 1}\right)^p |u|^p\,\mathrm{d}x \leqq c(t) \int_\Omega |u|^p\,\mathrm{d}x$$

are valid.

Thus, the A_0-condition is satisfied with $(tA + E)^{-1} = I_t$.

Secondly we prove the A_2-condition. We show 1. (2), (4'), (5'), (6'). For $0 < t < \infty$ we put

$$(8) \qquad (I_t' u)(x) = \frac{1}{t\,\varphi(x) + 1}\,u(x)\,, \quad D(I_t') = W_p^m(\Omega)\,.$$

Using the formula

$$(9) \qquad \mathrm{D}^\alpha\left(\frac{1}{t\varphi + 1}\right) =$$

$$= \sum_{K_1 a_1 + \ldots + K_n a_n = |\alpha|} \frac{t^{K_1 + \ldots + K_n}}{(t\varphi + 1)^{K_1 + \ldots + K_n + 1}} \left(\frac{\partial^{a_1}\varphi}{\partial x_1^{a_1}}\right)^{K_1} \cdots \left(\frac{\partial^{a_n}\varphi}{\partial x_n^{a_n}}\right)^{K_n}$$

$(K_1, \ldots, K_n;\ a_1, \ldots, a_n$ non-negative integers) we get

$$(10)\qquad \int_\Omega \left|\frac{u}{t\varphi + 1}\right|^p \mathrm{d}x + \sum_{0<|\alpha|\leqq m} \int_\Omega \left|\mathrm{D}^\alpha \left(\frac{u}{t\varphi + 1}\right)\right|^p \mathrm{d}x \leqq$$

$$\leqq c\left[\int_\Omega |u|^p \,\mathrm{d}x + \sum_{0<|\alpha|\leqq m} \sum_{\beta_1+\beta_2=\alpha} \int_\Omega \left|\mathrm{D}^{\beta_1}\left(\frac{1}{t\varphi+1}\right)\mathrm{D}^{\beta_2} u\right|^p \mathrm{d}x\right] \leqq c' \|u\|^p_{W_p{}^m(\Omega)}$$

for $u \in W_p^m(\Omega)$, where c is a positive constant independent of t. Similarly,

$$(11)\quad \int_\Omega \left|\frac{\varphi}{t\varphi + 1}\, u(x)\right|^p \mathrm{d}x + \sum_{0<|\alpha|\leqq m} \int_\Omega \left|\mathrm{D}^\alpha\left(\frac{\varphi}{t\varphi+1}\, u(x)\right)\right|^p \mathrm{d}x \leqq c(t)\, \|u\|^p_{W_p{}^m(\Omega)}$$

follows. Thus, the A_2-condition is satisfied with $(tA + E)^{-1} = I'_t$. Using

$$(12)\qquad (L_p(\Omega), L_p(\Omega, \varphi(x)))_{\theta,p} = L_p(\Omega, \varphi(x)^\theta)$$

and

$$(13)\qquad (L_p(\Omega), W_p^m(\Omega))_{\theta,p} = W_p^{m\theta}(\Omega)$$

(cf. H. Triebel [16] Ch. 1 and Ch. 4, resp.) we get (5). If l is an integer, $1 \leqq l < m$, then under the same conditions as in the Theorem, the identity

$$(L_p(\Omega), W_p^l(\Omega) \cap L_p(\Omega, \varphi(x)))_{\theta,p} = W_p^{l\theta}(\Omega) \cap L_p(\Omega, \varphi^\theta)$$

holds.

Remark 1. An analogous theorem holds, if the Banach spaces A_0, A_1 and A_2 are specified in the sense of (4). For the spaces $(L_p(\Omega), \mathring{W}_p^l(\Omega))_{\theta,p}$, $l \in \{1, 2, \ldots, m\}$, we refer to P. Grisvard [5] or to Section 4.

Remark 2. For $\Omega = R_n$, the functions $\varphi(x) = (1 + |x|^2)^\eta$; $\eta > 0$, are examples of weight-functions belonging to all classes $K_1^{(m)}$; $m = 1, 2, 3, \ldots$.

1.2. THE CLASSES $K_2^{(m)}$ OF WEIGHT-FUNCTIONS

Let Ω be an arbitrary domain in R_n.

Definition. A weight-function $\varphi(x)$ belongs to a class $K_2^{(m)}$, $m \in \{1, 2, \ldots\}$, if:

$$(1)\qquad \frac{1}{\varphi(x)} \in L^\infty(\Omega),$$

(2) for all $0 < |\alpha| \leqq m$ we have $\mathrm{D}^\alpha\varphi/\varphi \in L_{p_\alpha}$ with $p_\alpha = n/|\alpha|$.

The classes $K_2^{(m)}$ are closed with respect to addition, multiplication and involution.

Theorem. *Let Ω be an arbitrary domain in R_n. The Banach spaces A_0, A_1 and A_2*

are specified in the sense of 1.1 (3). *The weight-functions* $\varphi(x)$ *belonging to* $K_2^{(m)}$ *satisfy the conditions of Grisvard's proposition and for* $0 < \theta < 1$, $1 < p < \infty$, 1.1 (5) *holds.*

The proof is obtained by Hölder's inequality and by the fact that for suitable real numbers p, l, q, k the spaces $W_p^l(\Omega)$ are embedded in the spaces $W_q^k(\Omega)$. More exactly, let Ω be an arbitrary domain in R_n. Then for $0 \leqq t \leqq s < \infty$ and $\infty > q \geqq \geqq p > 1$,

$$W_p^s(\Omega) \subseteq W_q^t(\Omega)\,, \quad s - \frac{n}{p} \geqq t - \frac{n}{q}\,, \tag{3}$$

is valid.

Remark 1. Using the statement

$$\text{for } \frac{n}{|\alpha|} < q < \infty\,, \quad L_q(\Omega) \subset L_{n/|\alpha|}(\Omega) + L^\infty(\Omega) \quad \text{holds}\,, \tag{4}$$

we put Theorems 1.1 and 1.2 together as follows:

Corollary. *Let* Ω *be an arbitrary domain in* R_n. *The spaces* A_0, A_1 *and* A_2 *are given by* 1.1(3). *Let* $\varphi(x)$ *be a weight-function satisfying*

$$\frac{1}{\varphi(x)} \in L^\infty(\Omega)\,, \tag{5}$$

(6) *for every* α, $1 \leqq |\alpha| \leqq m$, *there exists a number* q_α,

$$\frac{n}{|\alpha|} < q_\alpha \leqq \infty\,, \quad \textit{such that } \frac{D^\alpha \varphi}{\varphi} \in L_{q_\alpha}(\Omega)\,.$$

Then Grisvard's-conditions are fulfilled and 1.1 (5) *is valid.*

Remark 2. Remark 1.1.1 again applies.

Remark 3. We give an example of a weight-function $\varphi(x)$ belonging to all classes $K_2^{(m)}$, $m = 1, 2, \ldots$.

We set $\Omega = \{(x, y) : 1 < x < \infty;\ 0 < y < e^{-x}\}$ and $\varphi(x) = e^{x^2}$. The function $\varphi(x)$ is a weight-function. For multi-indices $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 \neq 0$ we have

$$D^\alpha \varphi(x) = e^{x^2} P_{|\alpha|}(x) \quad \text{with } P_{|\alpha|} \text{ a polynomial in } x \text{ of degree } |\alpha|\,.$$

Thus, for $1 < q < \infty$,

$$\int_\Omega \left| \frac{1}{\varphi(x)} D^\alpha \varphi \right|^q dx = \int_1^\infty \int_0^{e^{-x}} |P_{|\alpha|}(x)|^q \, dy \, dx = \int_1^\infty |P_{|\alpha|}(x)|^q \, e^{-x} \, dx < \infty$$

is valid, so that $\varphi(x)$ belongs to all classes $K_2^{(m)}$, $m = 1, 2, \ldots$. It is clear that $\varphi(x)$ does not belong to any class $K_1^{(m)}$, $m \in \{1, 2, \ldots\}$.

Throughout this section Ω is a bounded domain in R_n with a C^∞-boundary. The properties of such domains gave rise to classes $K_3^{(m)}$; $m \in \{1, 2, \ldots\}$.

Definition. Let Ω be a bounded domain in R_n with a C^∞-boundary; $d(x)$ denotes the distance from the point $x \in \Omega$ to the boundary $\partial\Omega$. The class $K_3^{(m)}$, $m \in \{1, 2, \ldots\}$, consists of all weight-functions $\varphi(x)$ satisfying

$$\frac{1}{\varphi(x)} \in L^\infty(\Omega)\,, \tag{1}$$

$$\text{for all} \quad 1 \leqq |\alpha| \leqq m\,, \quad \frac{D^\alpha \varphi}{\varphi}\, d(x)^{|\alpha|} \in L^\infty(\Omega) \quad \text{holds}\,. \tag{2}$$

The classes $K_3^{(m)}$ are closed with respect to addition, multiplication and involution. Next we give a statement without proof which we use throughout this section.

Lemma (Hardy's inequality applied to domains). *Let Ω be a bounded domain in R_n with a C^∞-boundary. We assume that a natural number m and real numbers α and p, $1 < p < \infty$, satisfy $\alpha - mp + kp \neq -1$, $k = 0, \ldots, m - 1$. Then there exists a positive constant c and a domain ω, $\bar{\omega} \subset \Omega$, such that*

$$\int_\Omega d(x)^{\alpha - mp}\, |u(x)|^p\, dx \leqq c \int_\Omega d(x)^\alpha \sum_{|\gamma| = m} |D^\gamma u^p|\, dx + c \int_\omega |u|^p\, dx \tag{3}$$

holds for all functions $u(x) \in \bar{C}^\infty(\Omega)$, for which the left integral in (3) converges. ($\bar{C}^\infty(\Omega)$ is defined as usual.)

Remark 1. We state a special case of this lemma:

Functions $u(x) \in \mathring{W}_p^m(\Omega)$ satisfy the estimate

$$\left\| \frac{u}{d(x)^m} \right\|_{L_p(\Omega)} \leqq c \|u\|_{W_p{}^m(\Omega)}\,. \tag{4}$$

Theorem. *Let Ω be a bounded domain in R_n with a C^∞-boundary. The Banach spaces A_0, A_1 and A_2 are specified by 1.1 (4). For weight-functions $\varphi(x)$ belonging to $K_3^{(m)}$ the conditions of Grisvard's proposition are satisfied and*

$$(L_p(\Omega), \mathring{W}_p^m(\Omega) \cap L_p(\Omega, \varphi))_{\theta,p} = (L_p(\Omega), \mathring{W}_p^m(\Omega))_{\theta,p} \cap L_p(\Omega, \varphi^\theta) \tag{7}$$

holds for $0 < \theta < 1$, $1 < p < \infty$.

Proof. The proof of the A_0-condition is similar to that in Section 1.1. To prove the A_2-condition we use Remark 1 and formula 1.1 (9), thus obtaining the following estimate, for functions $u(x) \in \mathring{W}_p^m(\Omega)$:

$$\sum_{|\alpha| \leqq m} \int_\Omega \left| D^\alpha \left(\frac{u}{t\varphi + 1} \right) \right|^p dx \leqq \tag{8}$$

$$\leqq c \left(\int_\Omega |u|^p \, dx + \sum_{0 < |\alpha| \leqq m} \sum_{\beta_1 + \beta_2 = \alpha} \int_\Omega \left| D^{\beta_1} \left(\frac{1}{t\varphi + 1} \right) D^{\beta_2} u \right|^p dx \right) \leqq c' \|u\|_{W_p^m(\Omega)},$$

where c' is a positive constant independent of t.

Analogously we estimate for the same u:

$$\sum_{|\alpha| \leqq m} \left\| D^\alpha \left(\frac{\varphi}{t\varphi + 1} u \right) \right\|_{L_p(\Omega)}^p \leqq c(t) \, \|u\|_{W_p^m(\Omega)}^p < \infty, \tag{9}$$

where the constant c depends on the parameter t.

Thus, the functions $u(x)/(t\,\varphi(x)+1)$ and $(\varphi(x)/(t\,\varphi(x)+1))\,u(x)$ belong to $\mathring{W}_p^m(\Omega)$ so that Grisvard's-conditions are satisfied. By means of 1.1 (12) we get (7). To determine the spaces $(L_p(\Omega), \mathring{W}_p^m(\Omega))_{\theta,p}$ we refer to P. Grisvard [5] or to Section 4 of our paper. There we state Grisvard's result.

Remark 2 (Examples of weight-functions belonging to $K_3^{(m)}$). Let Ω be a bounded domain in R_n with a C^∞-boundary.

1. The functions $\varphi(x) = d(x)^\alpha$, α real, $\alpha \leqq 0$, belong to all classes $K_3^{(m)}$, $m = 1, 2, \ldots$; they belong neither to any class $K_1^{(m)}$, $m = 1, 2, \ldots$, nor to any class $K_2^{(m)}$, $m = 1, 2, \ldots$.

We assume that $\sup_{x \in \Omega} d(x) < 1$. Then the functions $\varphi(x) = \ln(d(x)^\alpha)$, $\alpha < 0$, belong to all classes $K_3^{(m)}$ but not to any class $K_1^{(m)}$ or $K_2^{(m)}$, $m = 1, 2, \ldots$.

Remark 3. The space $\mathring{W}_p^m(\Omega)$ can be represented by the following intersection:

$$\mathring{W}_p^m(\Omega) = W_p^m(\Omega) \cap L_p(\Omega, d(x)^{-m}), \quad m = 1, 2, \ldots, \quad 1 < p < \infty. \tag{10}$$

Setting $A_0 = L_p(\Omega)$, $A_2 = W_p^m(\Omega)$ and $(Au)(x) = d(x)^{-m} u(x)$, $D(A) = L_p(\Omega, d(x)^{-m})$, the A_0-condition is proved immediately. The weight-function $d(x)^{-m}$ belongs to the class $K_3^{(m)}$, so that we would conjecture

$$(L_p(\Omega), \mathring{W}_p^m(\Omega))_{\theta,p} = W_p^{m\theta}(\Omega) \cap L_p(\Omega, d(x)^{-m\theta}) = \mathring{W}_p^{m\theta}(\Omega). \tag{11}$$

However, this is not true. The next section contains statements about the spaces $(L_p(\Omega), \mathring{W}_p^m(\Omega))_{\theta,p}$; it turns out that the interpolation satisfies (11) for certain values of θ, but not for all.

In this section we want to suggest in which way the above results can be generalized.

Remark 1. Theorems 1.1, 1.2, 1.3 are also true if the functor $(\cdot, \cdot)_{\theta,p}$ is replaced by a functor $(\cdot, \cdot)_{\theta,q}$, $1 < q < \infty$. The spaces $(L_p(\Omega), L_p(\Omega, \varphi))_{\theta,q}$ are the so-called Beurling spaces (see J. E. Gilbert [2]). The spaces $(L_p(\Omega), W_p^m(\Omega))_{\theta,q}$ are the usual Besov spaces (see H. Triebel [16] Sect. 4.3.1.).

This remark can be applied to all the following results. Pay attention to the question whether the spaces on the right hand side in $(*)$ are known.

Remark 2. Let Ω be an arbitrary domain in R_n. We put in 1.1 and in 1.2:

(i) $A_0 = L_p(\Omega)$,

$$A_2 = W_q^t(\Omega), \quad 1 < q, \quad p < \infty; \quad 0 \leqq t < \infty; \quad t - n/q \geqq -n/p.$$

Let Ω be a bounded domain in R_n with a C^∞-boundary. We put in 1.1, 1.2, 1.3:

(ii) $A_0 = L_p(\Omega)$,

$$A_2 = \mathring{W}_q^t(\Omega), \quad 1 < q, \quad p < \infty; \quad 0 \leqq t < \infty; \quad t - n/q \geqq -n/q.$$

In (i) and in (ii) the operator A is the same as in Sections 1.1, 1.2.

In these cases Theorems analogous to Theorems 1.1, 1.2 and 1.3, respectively, are true; the proof is based on the embedding assertions

$$W_q^t(\Omega) \subset L_p(\Omega), \quad \mathring{W}_q^t(\Omega) \subset L_p(\Omega),$$

which hold due to the assumptions on the domain Ω and the parameters p, q and t mentioned above.

For the real K-functor $(\cdot, \cdot)_{\theta,r}$ with

$$\frac{1}{r} = \frac{1 - \theta}{p} + \frac{\theta}{q}$$

the spaces on the right hand side of the equation mentioned in Remark 1 are known. We refer to Remark 1 and to H. Triebel [16], Section 4.3.

Remark 3. We obtain an essential generalisation of the above results to other spaces by using the well-known interpolation property (see H. Triebel [16] Ch. 1). The proofs of Theorems 1.1, 1.2 and 1.3 show that the operators $(tA + E)^{-1}$, $0 < t < \infty$, acting in A_0 belong to $\mathscr{L}(\{A_0, A_2\}, \{A_0, A_2\})$. (This set consists of all linear maps from $A_0 + A_2$ into $A_0 + A_2$, whose restrictions to A_0 and A_2 are continuous maps in A_0 and A_2, respectively.) For operators belonging to this set the interpolation property implies that their restrictions to $F(\{A_0, A_2\})$ are bounded operators. Thus provided the A_0- and A_2-conditions are satisfied, we obtain an $F(\{A_0, A_2\})$-condition and

$$(A_0, F(\{A_0, A_2\}) \cap A_2)_{\theta,q} = (A_0, F(\{A_0, A_2\}))_{\theta,q} \cap (A_0, A_2)_{\theta,q}$$

holds.

We give some examples: Let Ω be a bounded domain in R_n with a C^∞-boundary or $\Omega = R_n$. We denote by $H_p^s(\Omega)$, $1 < p < \infty$, $0 \leqq s < \infty$, the usual Lebesgue (Bessel potential or Liouville) spaces and by $B_{p,q}^s(\Omega)$ the usual Besov spaces. We refer to H. Triebel [16] for the definition, the basic properties and the following assertions:

(1) $$1 < p < \infty\,; \quad s = 0, 1, 2, \ldots\,; \quad H_p^s(\Omega) = W_p^s(\Omega)\,;$$

(2) $$[H_{p_0}^{s_0}(\Omega), H_{p_1}^{s_1}(\Omega)]_\theta = H_p^s(\Omega) \begin{cases} 0 < \theta < 1\,, \quad \dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}\,, \\ 1 < p < \infty\,, \\ 0 < s_0, s_1 < \infty\,, \quad s = (1-\theta)\, s_0 + \theta s_1\,; \end{cases}$$

(3) $$(H_p^{s_0}(\Omega), H_p^{s_1}(\Omega))_{\theta,p} = B_{p,p}^s(\Omega) \quad 0 < \theta < 1\,, \quad 1 < p < \infty\,; \quad s = (1-\theta)\, s_1 + \theta s_2\,;$$

(4) $$(W_p^{s_0}(\Omega), W_p^{s_1}(\Omega))_{\theta,q} = B_{p,q}^s(\Omega) \begin{cases} 0 < \theta < 1\,, \\ 0 \leqq s_0\,, \quad s_1 < \infty\,, \quad s_0 \neq s_1\,, \\ 1 < p\,,\ q < \infty\,, \\ s = (1-\theta)\, s_0 + \theta s_1\,. \end{cases}$$

In (2) $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor with parameter θ. If, in addition, the parameters s_0, s_1 and s_2 fulfil $s_0 - (1/p) \neq$ integer, $s_1 - (1/p) \neq$ $\neq$ integer, $s - (1/p) \neq$ integer, then, replacing $H_p^s(\Omega)$ by $\mathring{H}_p^s(\Omega)$, $W_p^s(\Omega)$ by $\mathring{W}_p^s(\Omega)$ and $B_{p,q}^s(\Omega)$ by $\mathring{B}_{p,q}^s(\Omega)$, similar formulas are true. Thus, we obtain analogous results as in the sections above. We only have to replace $W_p^m(\Omega)$ by $H_p^s(\Omega) = [L_p(\Omega), W_p^m(\Omega)]_{s/m}$ and by $B_{p,q}^s(\Omega) = (L_p(\Omega), W_p^m(\Omega))_{s/m,q}$, respectively. Under the mentioned conditions on the parameters the same holds for $A_2 = \mathring{H}_p^s(\Omega)$ or $A_2 = \mathring{B}_{p,q}^s(\Omega)$.

2. MULTIPLICATION OPERATORS IN WEIGHTED SPACES

In this section we get results of the type (*) in the case that the Banach spaces A_0 and A_2 as well as the space A_1 are weighted Sobolev spaces. We consider two types of weighted spaces; the spaces $W_p^m(\Omega, \sigma)$ (the ewight-function is the same for all the orders of differentiation) and the spaces $W_p^m(\Omega; \varrho^\mu; \varrho^\nu)$ (the weight-function depends on the order of differentiation). In both cases we get results of the type (*) for some classes of weight-functions (described by growth-conditions as in Sections 1.1, 1.2

and 1.3). We state the theorems without proofs, because these can be obtained by obvious modifications of the proofs of Theorems 1.1, 1.2 and 1.3. Nevertheless, we give the necessary definitions.

2.1. MULTIPLICATION OPERATORS IN THE SPACES $W_p^m(\Omega, \sigma)$

Let Ω be a domain in R_n. Let $1 < p < \infty$ and let m be an integer, $m \geqq 0$. Then we denote by $W_p^m(\Omega, \sigma)$ the space consisting of all measurable complexvalued functions $u(x)$, which have measurable derivatives up to the m-th order belonging to $L_p(\Omega, \sigma)$. The norm is given by

$$\|u\|_{W_p{}^m(\Omega,\sigma)} = \Big(\sum_{0 \leqq |\alpha| \leqq m} \|D^\alpha u\|^p_{L_p(\Omega,\sigma)}\Big)^{1/p}; \tag{1}$$

$\mathring{W}_p^m(\Omega, \sigma)$ denotes the closure of $C_0^\infty(\Omega)$ in $W_p^m(\Omega, \sigma)$.

(2) We will consider only such spaces $W_q^m(\Omega, \sigma)$ and $\mathring{W}_p^m(\Omega, \sigma)$,

for which the interpolations $(L_p(\Omega, \sigma), W_p^m(\Omega, \sigma)_{\theta,q}$ and $(L_p(\Omega, \sigma), \mathring{W}_p^m(\Omega, \sigma))_{\theta,q}$, respectively, are well-known. This is true, for instance, for the weighted spaces considered by H. Triebel in [16], Ch. 3. In what follows we assume that the domains considered satisfy the same conditions as those introduced in the reference just mentioned. Now we specify the spaces A_0, A_1 and A_2 of Grisvard's proposition. Let $1 < p < \infty$ and let m be an integer, $m \geqq 0$. Let $\varphi(x)$ be a weight-function.

$$\begin{aligned} &A_0 = L_p(\Omega, \sigma), \quad A_2 = W_p^m(\Omega, \sigma), \\ &(Au)(x) = \varphi(x)\, u(x), \quad D(A) = \{u : u \in L_p(\Omega, \sigma),\ \varphi u \in L_p(\Omega, \sigma)\}, \\ &A_1 = [D(A), \|\cdot\|_A] = L_p(\Omega, \sigma\varphi). \end{aligned} \tag{3}$$

(3') $A_0 = L_p(\Omega, \sigma)$, $A_2 = \mathring{W}_p^m(\Omega, \sigma)$; the operator A is the same as in (3).

Theorem 1. *Let Ω be a domain described above. The Banach spaces A_0, A_1 and A_2 are given by (3) or (3'); we assume (2). For weight-functions $\varphi(x)$ belonging to the class $K_1^{(m)}$ (cf. Sect. 1.1) Grisvard's conditions are fulfilled. Thus,*

$$\begin{aligned} &(L_p(\Omega, \sigma), L_p(\Omega, \sigma\varphi) \cap W_p^m(\Omega, \sigma))_{\theta,p} = \\ &= (L_p(\Omega, \sigma \,.\, \varphi^\theta) \cap (L_p(\Omega, \sigma), W_p^m(\Omega, \sigma))_{\theta,p} \end{aligned} \tag{4}$$

or

$$\begin{aligned} &(L_p(\Omega, \sigma),\ L_p(\Omega, \sigma\varphi) \cap \mathring{W}_p^m(\Omega, \sigma))_{\theta,p} = \\ &= L_p(\Omega, \sigma \,.\, \varphi^\theta) \cap (L_p(\Omega, \sigma),\ \mathring{W}_p^m(\Omega, \sigma))_{\theta,p}, \end{aligned} \tag{5}$$

respectively.

We assume that $B^s_{pp}(\Omega, \sigma) = (L_p(\Omega, \sigma), W^m_p(\Omega, \sigma))_{s/m,p}$ with $s = (1 - \theta) m$ and that the spaces $B^s_{pp}(\Omega, \sigma)$ satisfy the embedding assertion

(6) for $0 \leqq t \leqq s$ and $\infty > q \geqq p > 1$, $B^s_{pp}(\Omega, \sigma) \subset B^t_{qq}(\Omega, \sigma)$;

$$s - \frac{n}{p} \geqq t - \frac{n}{q}.$$

Examples of such weighted spaces are given in the next section. (6) is true, for instance, if $\sigma(x)$ belongs to the class $K_1^{(m)}$, $s \leqq m$.

Theorem 2. *Let Ω be a domain in R_n described above. The Banach spaces A_0, A_1 and A_2 are given by* (3); *we assume* (2) *and* (6). *For weight-functions $\varphi(x)$ belonging to $K_2^{(m)}$ (cf. Sect.* 1.2), (4) *holds.*

2.2. MULTIPLICATION OPERATORS IN THE SPACES $W^m_p(\Omega; \varrho^\mu; \varrho^\nu)$

Let Ω be a domain in R_n. If $1 \leqq p < \infty$, then by $L^{\text{loc}}_p(\Omega)$ we mean the spaces, consisting of all complex-valued locally p-integrable functions $u(x)$. Outside Ω these functions are continued by zero. $C^\infty(\Omega)$ denotes the set of all complex-valued infinitely defferentiable functions in Ω. By $\varrho(x)$ we denote a weight-function mapping Ω into R_1, which satisfies

(1) $$|\nabla\varrho(x)| \leqq c\, \varrho^2(x)$$

and

(2) for each positive number K there exist numbers $\varepsilon_K > 0$ and $r_K > 0$ such that $\varrho(x) > K$ provided $d(x) \leqq \varepsilon_K$ or $|x| \geqq r_K$, $x \in \Omega$.

We introduce the space $W^s_p(\Omega; \varrho^\mu; \varrho^\nu)$ and refer to H. Triebel [16], Ch. 3 for details.

Definition. Let Ω be an arbitrary domain in R_n and $\varrho(x)$ a weight-function in the sense of (1) and (2). Let p, s, μ and ν be real numbers with $1 \leqq p < \infty$, $s \geqq 0$ and $\nu \geqq \mu + s$. Then we put

(3) $$W^s_p(\Omega; \varrho^\mu; \varrho^\nu) = \{u : u \in L^{\text{loc}}_p(\Omega);\ \|u\|_{W_p{}^s(\Omega;\varrho^\mu;\varrho^\nu)} =$$

$$= \left[\int_\Omega \Big(\sum_{|\alpha|=m} \varrho^{\mu p}(x)\, |D^\alpha u(x)|^p + \varrho^{\nu p}(x)\, |u(x)|^p\Big)\, dx\right]^{1/p} < \infty\}$$

for $s = 0, 1, 2, \ldots$; for $s = 0$ let $\mu = \nu$; $W^0_p(\Omega; \varrho^\mu; \varrho^\nu) = L_p(\Omega, \varrho^\mu)$

and

(4) $$W_p^s(\Omega; \varrho^\mu; \varrho^\nu) = \Big\{u : u \in L_p^{\text{loc}}(\Omega);\ \|u\|_{W_p{}^s(\Omega;\varrho^\mu;\varrho^\nu)} =$$

$$= \Big[\int_{\Omega\times\Omega} \sum_{|\alpha|=[s]} \frac{|\varrho^\mu(x)\, D^\alpha u(x) - \varrho^\mu(y)\, D^\alpha u(y)|^p}{|x-y|^{n+\{s\}p}}\, dx\, dy + \int_\Omega \varrho^{\nu p}(x)|u(x)|^p\, dx\Big]^{1/p} < \infty\Big\}$$

for $0 < s \neq$ integer, $s = [s] + \{s\}$, $[s]$ integer, $0 < \{s\} < 1$.

We introduce an equivalent norm in $W_p^s(\Omega; \varrho^\mu; \varrho^\nu)$: Let the assumptions of Definition 1 be fulfilled. Let further $0 \leqq t \leqq s$ and

$$\chi_t = \mu \frac{t}{s} + \nu \frac{s-t}{s} = \mu + (\nu - \mu)\frac{s-t}{s}\,.$$

Then

(5) $$(\|u\|^*_{W_p{}^s(\Omega;\varrho^\mu;\varrho^\nu)})^p = (\|u\|_{W_p{}^s(\Omega;\varrho^\mu;\varrho^\nu)})^p + \sum_{|\alpha|\leqq[s]} \int_\Omega \varrho^{\chi_{|\alpha|}p}(x)\, |D^\alpha u(x)|^p\, dx$$

defines an equivalent norm in $W_p^s(\Omega; \varrho^\mu; \varrho^\nu)$.

Definition 2. Let Ω be a bounded domain in R_n. Let $\varrho(x)$ be a weight-function in the sense of (1) and (2) for which, in addition, the identity $\varrho^{-1}(x) = d(x)$ holds near to $\partial\Omega$. Then we set for $1 < p < \infty$, $s \geqq 0$ and $\nu < \mu + s$:

(6) $$\mathscr{W}_p^s(\Omega; \varrho^\mu; \varrho^\nu) = \Big\{u : u \in L_p^{\text{loc}}(\Omega);\ \|u\|_{W_p{}^s(\Omega;\varrho^\mu;\varrho^\nu)} =$$

$$= \Big[\int_\Omega \big(\sum_{|\alpha|=s} \varrho^{\mu p}(x)\, |D^\alpha u(x)|^p + \varrho^{\nu p}(x)\, |u(x)|^p\big)\, dx\Big]^{1/p} < \infty\Big\}$$

if s is an integer. For $s = 0$ we put $\mu = \nu : \mathscr{W}_p^0(\Omega; \varrho^\mu; \varrho^\nu) = L_p(\Omega; \varrho^\mu)$

while

(7) $$\mathscr{W}_p^s(\Omega; \varrho^\mu; \varrho^\nu) = \Big\{u : u \in L_p^{\text{loc}}(\Omega);\ \|u\|_{W_p{}^s(\Omega;\varrho^\mu;\varrho^\nu)} =$$

$$= \Big[\int_{\Omega\times\Omega} \sum_{|\alpha|=[s]} \frac{|\varrho^\mu(x)\, D^\alpha u(x) - \varrho^\mu(y)\, D^\alpha u(y)|^p}{|x-y|^{n+\{s\}p}}\Big]^{1/p} + \|u\|_{W_p{}^{[s]}(\Omega;\varrho^\mu;\varrho^\nu)} < \infty\Big\}$$

for $s = [s] + \{s\}$, where $[s]$ is an integer and $0 < \{s\} < 1$.

$W_p^s(\Omega; \varrho^\mu; \varrho^\nu$; denotes the closure of $\bar{C}^\infty(\Omega)$ in $\mathscr{W}_p^s(\Omega; \varrho^\mu; \varrho^\nu)$ and by $\mathring{W}_p^s(\Omega; \varrho^\mu, \varrho^\nu)$ we mean the closure of $C_0^\infty(\Omega)$ in $\mathscr{W}_p^s(\Omega; \varrho^\mu; \varrho^\nu)$.

We note that (5) holds as well. Now we state embedding assertions analogous to the statements 1.2 (3) and 2.1 (6).

Proposition. (i) *Let* Ω *be an arbitrary domain and* $\varrho(x)$ *a weight-function in the sense of* (1) *and* (2). *Let* t, q, p, s, μ *and* ν *be real numbers with*

$$s \geqq t \geqq 0\,, \quad \infty > q \geqq p > 1\,, \quad s - \frac{n}{p} = t - \frac{n}{q}\,, \quad \nu > \mu + s\,.$$

Then

$$(8) \quad W_p^s(\Omega; \varrho^\mu; \varrho^\nu) \subseteq W_q^t(\Omega; \varrho^\chi; \varrho^\tau; \quad \text{with} \quad \chi = \mu \quad \text{and} \quad \tau = \mu\left(\frac{s-t}{s}\right) + \nu t\,.$$

(ii) *Let* Ω *be a bounded domain in* R_n *with a* C^∞*-boundary and let* $\varrho(x)$ *be a weight-function in the sense of* (1) *and* (2) *for which, in addition* $\varrho^{-1}(x) = d(x)$ *holds near to* $\partial\Omega$. *Let* t, q, p, s, μ *and* ν *be real number with*

$$s \geqq t \geqq 0\,, \quad \infty > q \geqq p > 1\,, \quad s - \frac{n}{p} > t - \frac{n}{q}\,.$$

Then

$$(9) \qquad \mathring{W}_p^s(\Omega; \varrho^\mu; \varrho^\nu) \subseteq \mathring{W}_q^t(\Omega; \varrho^\chi; \varrho^\tau) \quad \text{with} \quad \chi = \mu \quad \text{and} \quad \tau < \chi + t\,.$$

(iii) *Let the assumptions of* (ii) *be satisfied. Further, let*

$$\{s\} \neq \frac{1}{p}\,, \quad \mu + s \neq \frac{1}{p} + k \quad \text{with} \quad k = 0, 1, \ldots, [s] - 1\,,$$

$$\{t\} \neq \frac{1}{p}\,, \quad \chi + t \neq \frac{1}{p} + l \quad \text{with} \quad l = 0, 1, \ldots, [t] - 1\,.$$

Then (9) *holds for*

$$s - \frac{n}{p} \geqq t - \frac{n}{q}\,.$$

After these preliminaries we formulate

Theorem. (a) *Let* Ω *be an arbitrary domain in* R_n *and* $\varrho(x)$ *a weight-function in the sense of* (1) *and* (2). *If* $1 < p < \infty$, $\infty > s \geqq \mu$, $\nu > \mu + s$ *and if* $\varphi(x)$ *is a weight-function, we put*:

$$A_0 \;\; = L_p(\Omega, \varrho^\nu)\,, \quad A_2 = W_p^s(\Omega; \varrho^\mu; \varrho^\nu)\,, \quad (Au)(x) = \varphi(x)\, u(x)\,,$$

$$D(A) = \{u : u \in L_p(\Omega, \varrho^\nu)\,; \quad \varphi u \in L_p(\Omega, \varrho^\nu)\,,$$

$$A_1 \;\; = [D(A), \|\cdot\|_A] = L_p(\Omega, \varrho^\nu \varphi)\,.$$

For functions $\varphi(x)$ *belonging to a class* $K_1^{(m)}$ *with* $m \geqq s$, *Grisvard's conditions are satisfied and*

(10) $$(L_p(\Omega, \varrho^\nu),\ L_p(\Omega, \varrho^\nu \varphi) \cap W_p^s(\Omega; \varrho^\mu; \varrho^\nu))_{\theta,p} = $$
$$= L_p(\Omega, \varrho^\nu \varphi^\theta) \cap W_p^{s\theta}(\Omega; \varrho^\tau; \varrho^\nu)$$

holds with $\tau = \theta(\mu - \nu) + \nu$.

(b) *Let* $\Omega \subset R_n$ *be an arbitrary domain and let* $\varrho(x)$ *be a weight-function in the sense of* (1) *and* (2). *Let the Banach spaces* A_0, A_1 *and* A_2 *be the same as in* (a). *For weight-functions* $\varphi(x)$ *belonging to a class* $K_2^{(m)}$ *which* $m \geqq s$, (10) *holds.*

(c) *Let* Ω *be a bounded domain in* R_n *with a* C^∞*-boundary. Let* $\varrho(x)$ *be the same weight-function as in Definition 2 and let* $\varphi(x)$ *be a weight-function. If* p, s, μ *and* ν *are real numbers with* $1 < p < \infty$, $0 \leqq s < \infty$, $\nu < \mu + s$, $\{s\} \neq 1/p$, $\mu + s \neq (1/p) + k$ *for* $k = 0, 1, \ldots, [s] - 1$, *we put:*

$$A_0 = L_p(\Omega, \varrho^\nu), \quad A_2 = \mathring{W}_p^s(\Omega; \varrho^\mu; \varrho^\nu), \quad (Au)(x) = \varphi(x)\, u(x),$$
$$A_1 = L_p(\Omega, \varrho^\nu \varphi).$$

For weight-functions $\varphi(x)$ *belonging to a class* $K_3^{(m)}$ *with* $m \geqq s$ *Grisvard's conditions are satisfied and*

(11) $$(L_p(\Omega, \varrho^\nu),\ L_p(\Omega, \varrho^\nu \varphi) \cap \mathring{W}_p^s(\Omega; \varrho^\mu; \varrho^\nu))_{\theta,p} =$$
$$= L_p(\Omega, \varrho^\nu \varphi^\theta) \cap W_p^{s\theta}(\Omega; \varrho^\mu; \varrho^{\mu + s\theta p}).$$

Remark 1. In the case that the conditions of (c) are satisfied we can replace in (a) and (b) the spaces $W_p^s(\Omega; \varrho^\mu; \varrho^\nu)$ by $\mathring{W}_p^s(\Omega; \varrho^\mu; \varrho^\nu)$, so that an analogous formula to formula (10) holds.

Remark 2. Using the same argument as in Section 1.1 we put the statements (a) and (b) together obtaining the following

Corollary. *Let* Ω *be an arbitrary domain in* R_n *and* $\varrho(x)$ *a weightfunction in the sense of* (1) *and* (2). *Let* A_0, A_1 *and* A_2 *be the same Banach spaces as in Theorem* (a). *Let* $\varphi(x)$ *be a weight-function satisfying*

(1) $$\frac{1}{\varphi(x)} \in L^\infty(\Omega),$$

(2) *for all* $1 \leqq |\alpha| \leqq m$ *there exist numbers* q_α, $n/|\alpha| \leqq q_\alpha \leqq \infty$, *such that*

$$\frac{D^\alpha \varphi}{\varphi} \in L_{q_\alpha}(\Omega).$$

Then (10) *holds.*

3. THE INTERPOLATION $(W_p^m(\Omega, \sigma_0),\ W_p^m(\Omega, \sigma_1) \cap W_p^r(\Omega, \sigma_0))_{\theta,p},\ r \geqq m$

In this section we describe the spaces

$$(1) \qquad (W_p^m(\Omega, \sigma_0),\ W_p^m(\Omega, \sigma_1) \cap W_p^r(\Omega, \sigma_0))_{\theta,p}, \quad r, m \text{ integers}, \quad r \geqq m \geqq 0,$$

for certain classes of pairs $(\sigma_0(x), \sigma_1(x))$ of weight-functions. First we present some preliminaries.

3.1. THE SPACES $(W_p^m(\Omega, \sigma_0),\ W_p^m(\Omega, \sigma_1))_{\theta,p}$

Let Ω be an arbitrary domain in R_n and let $\sigma_0(x)$ and $\sigma_1(x)$ be weight-functions. For $m = 0, 1, 2, \ldots$ and $1 < p < \infty$ the spaces $W_p^m(\Omega, \sigma)$ are defined as in Section 2.1. Favini showed in [1] that the spaces $(W_p^m(\Omega, \sigma_0), W_p^m(\Omega, \sigma_1))_{\theta,p}$ satisfy the embedding assertion

$$(1) \qquad (W_p^m(\Omega, \sigma_0),\ W_p^m(\Omega, \sigma_1))_{\theta,p} \subseteq W_p^m(\Omega, \sigma_\theta), \quad 0 < \theta < 1, \quad 1 < p < \infty, \quad \sigma_\theta = \sigma_0^{1-\theta}\sigma_1^\theta.$$

We will show that under certain growth conditions on the pair $(\sigma_0(x), \sigma_1(x))$ of weight-functions the inverted embedding holds as well. To this aim we need a proposition from the interpolation theory of positive operators; for the proposition we refer to H. Triebel [16], Ch. 1.

Definition. Let Λ be a closed operator acting in A with a dense domain of definition $D(\Lambda)$. The operator Λ is said to be *positive*, if $(-\infty, 0]$ is contained in the resolvent set of Λ and if there exists a number c, $c \geqq 0$, such that

$$(2) \qquad \|(\Lambda - tE)^{-1}\| \leqq \frac{c}{1 + |t|}; \quad t \in (-\infty, 0].$$

(An equivalent inequality to (2) is $\|(t\Lambda + E)^{-1}\| \leqq c/(1 + t)$, $t \in [0, \infty)$.)

Proposition. *Let Λ be a positive operator acting in A. Let m be a natural number, $0 < \theta < 1$, and let $1 < p < \infty$. Then*

$$(3) \qquad (A, D(\Lambda^m))_{\theta,p} = \{a : a \in A \|a\|^* = \|t^{\theta m}[\Lambda(\Lambda + tE)^{-1}]^m\, a\|_{L_p^*(A)} < \infty\}$$

is valid, when putting

$$(4) \qquad \|v(t)\|_{L_p^*(A)} = \left(\int_0^\infty \|v(t)\|_A^p \frac{dt}{t}\right)^{1/p}$$

with the usual modifications for $p = \infty$.

After this preliminaries we present

Theorem 1. *Let Ω be an arbitrary domain in R_n and let m be a natural number. We assume that $1 < p < \infty$ and that $\sigma_0(x)$ $\sigma_1(x)$ are weight-functions satisfying*

$$\mathrm{D}^\alpha\left(\frac{\sigma_1}{\sigma_0}\right) \leqq c\,\frac{\sigma_1}{\sigma_0}\,, \quad 0 < |\alpha| \leqq m\,, \tag{5}$$

and

$$\frac{\sigma_0}{\sigma_1} \in L^\infty(\Omega)\,. \tag{6}$$

Then the operator $(\Lambda u)(x) = (\sigma_1(x)/\sigma_0(x))\,u(x)$, $D(\Lambda) = \{u : u \in W_p^m(\Omega, \sigma_0),$ $(\sigma_1/\sigma_0)\,u \in W_p^m(\Omega, \sigma_0)\}$, is a positive operator acting in $W_p^m(\Omega, \sigma_0)$. Further, $D(\Lambda) = W_p^m(\Omega, \sigma_1)$ up to equivalent norms and for $0 < \theta < 1$, $1 < p < \infty$, the following identity holds:

$$(W_p^m(\Omega, \sigma_0), W_p^m(\Omega, \sigma_1))_{\theta,p} = W_p^m(\Omega, \sigma_\theta)\,, \quad \text{where} \quad \sigma_\theta = \sigma_0^{1-\theta}\sigma_1^\theta\,. \tag{7}$$

Proof. First step: We show that $D(\Lambda) = W_p^m(\Omega, \sigma_1)$. The inclusion $D(\Lambda) \subseteq W_p^m(\Omega, \sigma_1)$:Let $u(x) \in D(\Lambda) = \{u : u \in W_p^m(\Omega, \sigma_0),\ (\sigma_1/\sigma_0)\,u \in W_p^m(\Omega, \sigma_0)\}$, which is equipped with the norm

$$\|u\|_{D(\Lambda)} = \|u\|_{W_p{}^m(\Omega,\sigma_0)} + \left[\sum_{0\leqq|\alpha|\leqq m} \left\|\mathrm{D}^\alpha\left(\frac{\sigma_1}{\sigma_0}\right)\right\|^p_{L_p(\Omega,\sigma_0)}\right]^{1/p}.$$

Then

$$\mathrm{D}^\alpha\left(\frac{\sigma_1}{\sigma_0}\,u\right) = \frac{\sigma_1}{\sigma_0}\,\mathrm{D}^\alpha u + \sum_{|\gamma|\geqq 1} c_\gamma D^\gamma\left(\frac{\sigma_1}{\sigma_0}\right)\mathrm{D}^{\alpha-\gamma}u \tag{8}$$

is valid. Consequently,

$$\|u\|_{W_p{}^m(\Omega,\sigma_1)} = \sum_{0\leqq|\alpha|\leqq m} \|\mathrm{D}^\alpha u\|^p_{L_p(\Omega,\sigma_1)} =$$

$$= \sum_{0\leqq|\alpha|<m} \|\mathrm{D}^\alpha u\|^p_{L_p(\Omega,\sigma_1)} + \sum_{|\alpha|=m} \left\|\frac{\sigma_1}{\sigma_0}\,\mathrm{D}^\alpha u\right\|^p_{L_p(\Omega,\sigma_0)} \leqq$$

$$\leqq \sum_{0\leqq|\alpha|\leqq m-1} \|\mathrm{D}^\alpha u\|^p_{L_p(\Omega,\sigma_1)} + c\sum_{|\alpha|=m} \left\|\mathrm{D}^\alpha\left(\frac{\sigma_1}{\sigma_0}\,u\right)\right\|^p_{L_p(\Omega,\sigma_0)} +$$

$$+ c\sum_{|\gamma|\geqq 1} \left\|\mathrm{D}^\gamma\left(\frac{\sigma_1}{\sigma_0}\right)\mathrm{D}^{\alpha-\gamma}u\right\|^p_{L_p(\Omega,\sigma_0)} \leqq$$

$$\leqq \sum_{|\alpha|=m} \left\|\mathrm{D}^\alpha\left(\frac{\sigma_1}{\sigma_0}\,u\right)\right\|^p_{L_p(\Omega,\sigma_0)} + c' \sum_{0\leqq|\alpha|\leqq m-1} \|\mathrm{D}^\alpha u\|^p_{L_p(\Omega,\sigma_1)}\,.$$

By iteration we get $\|u\|_{W_p{}^m(\Omega,\sigma_1)} \leqq c\|u\|_{D(\Lambda)}$, where c is a positive constant.

The inclusion $W_p^m(\Omega, \sigma_1) \subseteq D(A)$:

We have to show

$$(9) \quad \left\| D^\alpha \left(\frac{\sigma_1}{\sigma_0} u \right) \right\|^p_{L_p(\Omega,\sigma_0)} \leqq c_\alpha \|u(x)\|^p_{W_p{}^m(\Omega,\sigma_1)}, \quad 0 \leqq |\alpha| \leqq m, \quad u \in W_p^m(\Omega, \sigma_1).$$

Using (8) we have

$$(10) \quad \left\| D^\alpha \left(\frac{\sigma_1}{\sigma_0} u \right) \right\|^p_{L_p(\Omega,\sigma_0)} \leqq c \|D^\alpha u\|^p_{L_p(\Omega,\sigma_1)} +$$

$$+ c \sum_{|\gamma| \geqq 1} \left\| \frac{D^\gamma \left(\frac{\sigma_1}{\sigma_0} \right)}{\frac{\sigma_1}{\sigma_0}} D^{\alpha-\gamma} u \frac{\sigma_1}{\sigma_0} \right\|^p_{L_p(\Omega,\sigma_0)} \leqq c \sum_{0 \leqq |\gamma| \leqq |\alpha|} \|D^\gamma u\|_{L_p(\Omega,\sigma_1)} = c \|u\|_{W_p{}^{|\alpha|}(\Omega,\sigma_1)}.$$

Second step: We recall Theorem 1.1. The properties (5) and (6) of $\sigma_1(x)/\sigma_0(x)$ correspond to the properties 1.1 (1) and 1.1 (2) of a weight-function $\varphi(x)$ belonging to a class $K_1^{(m)}$. Thus, the proof of the positivity of the operator A is obtained by obvious modifications of the proof of Theorem 1.1.

Third step: To show

$$(W_p^m(\Omega, \sigma_0), W_p^m(\Omega, \sigma_1))_{\theta,p} = W_p^m(\Omega, \sigma_0^{1-\theta}\sigma_1^\theta).$$

The inclusion $(W_p^m(\Omega, \sigma_0), W_p^m(\Omega, \sigma_1))_{\theta,p} \subseteq W_p^m(\Omega, \sigma_0^{1-\theta}\sigma_1^\theta)$ was proved by A. Favini [1]. We show the converse inclusion. Using the above Proposition we have to show that any function $u(x) \in W_p^m(\Omega, \sigma_0^{1-\theta}\sigma_1^\theta)$ satisfies

$$(11) \quad u \in W_p^m(\Omega, \sigma_0),$$

$$(12) \quad \int_0^\infty t^{\theta p} \| \|A(A + tE)^{-1} u\|^p_{W_p{}^m(\Omega,\sigma_0)} \frac{dt}{t} \leqq c\|u\|^p_{W_p{}^m(\Omega,\sigma_0{}^{1-\theta}\sigma_1{}^\theta)}.$$

For an arbitrary function $u(x) \in W_p^m(\Omega, \sigma_0^{1-\theta}\sigma_1^\theta)$ we have

$$(13) \quad \int_\Omega |\sigma_0 D^\alpha u|^p \, dx = \int_\Omega |\sigma_0^{1-\theta}\sigma_1^\theta D^\alpha u|^p \left(\frac{\sigma_0}{\sigma_1}\right)^{\theta p} dx \leqq c \|D^\alpha u\|^p_{L_p(\Omega,\sigma_0{}^{1-\theta}\sigma_1{}^\theta)}$$

so that $u(x)$ belongs to $W_p^m(\Omega, \sigma_0)$.

To show (12) we write the integral on the left hand side as

$$\int_0^\infty t^{\theta p} \sum_{0 \leqq |\alpha| \leqq m} \left\| D^\alpha \left(\frac{\frac{\sigma_1}{\sigma_0}}{\frac{\sigma_1}{\sigma_0} + t} \right) u \right\|^p_{L_p(\Omega,\sigma_0)} \frac{dt}{t}.$$

We consider only two summands. First summand: $\alpha = (0, \ldots, 0) = \alpha_0$;

$$\int_0^\infty t^{\theta p} \left\| \frac{\frac{\sigma_1}{\sigma_0}}{\frac{\sigma_1}{\sigma_0} + t} u \right\|^p_{L_p(\Omega,\sigma_0)} \frac{dt}{t} =$$

$$= \int_0^1 t^{\theta p} \left\| \frac{\frac{\sigma_1}{\sigma_0}}{\frac{\sigma_1}{\sigma_0} + 1} u \right\|^p_{L_p(\Omega,\sigma_0)} \frac{dt}{t} + \int_1^\infty t^{\theta p} \left\| \frac{\frac{\sigma_1}{\sigma_0}}{\frac{\sigma_1}{\sigma_0} + 1} u \right\|^p_{L_p(\Omega,\sigma_0)} \frac{dt}{t} \leqq$$

$$\leqq \int_0^1 t^{\theta p} \|u\|^p_{L_p(\Omega,\sigma_0)} \frac{dt}{t} + \int_1^\infty t^{\theta p} \left\| \frac{\frac{\sigma_1}{\sigma_0}}{\left(\frac{\sigma_1}{\sigma_0}\right)^{1-\theta+\varepsilon} t^{\theta-\varepsilon}} u \right\|^p_{L_p(\Omega,\sigma_0)} \frac{dt}{t} \leqq$$

$$\leqq c_1 \|u\|^p_{L_p(\Omega,\sigma_0)} + \int_1^\infty t^{-\varepsilon p} \left\| \left(\frac{\sigma_0}{\sigma_1}\right)^\varepsilon \sigma_0^{1-\theta} \sigma_1^\theta u \right\|^p_{L_p(\Omega)} \frac{dt}{t} \leqq c_2 \|u\|^p_{L_p(\Omega,\sigma_0{}^{1-\theta}\sigma_1{}^\theta)},$$

where ε is a suitable real number,

$$0 < \varepsilon < \theta .$$

Second summand: $\alpha = (0, \ldots, 0, 1) = \alpha_1$;

$$(14) \quad \int_0^\infty t^{\theta p} \left\| \frac{\partial}{\partial x_n} \left(\frac{\frac{\sigma_1}{\sigma_0}}{\frac{\sigma_1}{\sigma_0} + t} u \right) \right\|^p_{L_p(\Omega,\sigma_0)} \frac{dt}{t} \leqq \int_0^\infty t^{\theta p} \left(\left\| \frac{\frac{\partial}{\partial x_n}\left(\frac{\sigma_1}{\sigma_0}\right)}{\frac{\sigma_1}{\sigma_0} + t} u \right\|^p_{L_p(\Omega,\sigma_0)} + \right.$$

$$\left. + \left\| \frac{\sigma_1}{\sigma_0} \frac{\frac{\partial}{\partial x_n}\left(\frac{\sigma_1}{\sigma_0}\right)}{\left(\frac{\sigma_1}{\sigma_0} + t\right)^2} u \right\|^p_{L_p(\Omega,\sigma_0)} + \left\| \frac{\frac{\sigma_1}{\sigma_0}}{\frac{\sigma_1}{\sigma_0} + t} \frac{\partial}{\partial x_n} u \right\|^p_{L_p(\Omega,\sigma_0)} \right) \frac{dt}{t} .$$

With the aid of the above decomposition of $[0, \infty)$ and using the properties (5) and (6) of $\sigma_1(x)/\sigma_0(x)$, we further estimate (14) and get the desired result. In the same way the summands

$$\int_0^1 t^{\theta p} \left\| D^\alpha \left(\frac{\frac{\sigma_1}{\sigma_0}}{\frac{\sigma_1}{\sigma_0} + t} u \right) \right\|^p_{L_p(\Omega,\sigma_0)} \frac{dt}{t}$$

with $\alpha \neq \alpha_0, \alpha_1$ can be estimated, so that the theorem is proved.

Remark 1. From the second step in the proof we deduce an embedding assertion for weight Sobolev spaces:

Corollary. *Let Ω be an arbitrary domain in R_n. Let m be an integer and $\sigma(x)$ a weight-function with*

$$\frac{D^\alpha \sigma}{\sigma} \in L^\infty(\Omega) \quad for \quad 0 < |\alpha| \leqq m\,.$$

Then

$$W_p^s(\Omega, \sigma) \subseteq W_q^t(\Omega, \sigma)$$

holds provided $\infty > q \geqq p > 1$, $0 \leqq t \leqq s \leqq m < \infty$ and

$$s - \frac{n}{p} \geqq t - \frac{n}{q}\,.$$

Proof. Let us assume that s is an integer. Then $u \in W_p^s(\Omega, \sigma)$ implies that $\sigma u \in \in W_p^s(\Omega)$. The embedding assertion 1.2 (3) for unweighted Sobolev spaces implies that $\sigma u \in W_q^t(\Omega)$, so that $u \in W_q^t(\Omega. \sigma)$. By means of interpolation we extend the result to real values s, $s \neq$ integer.

Remark 2 (Examples). All weight-functions $\sigma_0(x)$ and $\sigma_1(x)$ (in the sense of Section 1.1) whose quotient $\sigma_1(x)/\sigma_0(x)$ belongs to a class $K_1^{(m)}$ are examples of pairs $(\sigma_0(x), \sigma_1(x))$ of weight-functions in the sense of Theorem 1. So we refer to Remark 1.1.2.

After these preliminaries we formulate

Theorem 2. *Let Ω be an arbitrary domain in R_n and let $\sigma_0(x)$ and $\sigma_1(x)$ be weight-functions satisfying (5) and (6). For $1 < p < \infty$, natural numbers m and r, $r > m$, we put $A_0 = W_p^m(\Omega, \sigma_0)$, $A_2 = W_p^r(\Omega, \sigma_0)$,*

$$(Au)(x) = \frac{\sigma_1(x)}{\sigma_0(x)}\, u(x)\,, \quad D(A) = \left\{u : u \in W_p^m(\Omega, \sigma_0)\,, \frac{\sigma_1}{\sigma_0}\, u \in W_p^m(\Omega, \sigma_0)\,.\right.$$

Then

$$(15) \qquad A_1 = [D(A), \|\cdot\|_A] = W_p^m(\Omega, \sigma_1)\,,$$

Grisvard's conditions are satisfied and

$$(16) \qquad (W_p^m(\Omega, \sigma_0),\ W_p^m(\Omega, \sigma_1) \cap W_p^r(\Omega, \sigma_0))_{\theta,p} =$$
$$= W_p^m(\Omega, \sigma_0^{1-\theta}\sigma_1^\theta) \cap (W_p^m(\Omega, \sigma_0),\ W_p^r(\Omega, \sigma_0))_{\theta,p}\,.$$

Proof, (15) follows immediately from Theorem 1. The A_0-condition and the A_2-condition can be proved in the same way as the positivity of the operator A in Theorem 1. Hence

(17) $$(W_p^m(\Omega, \sigma_0),\ W_p^m(\Omega, \sigma_1) \cap W_p^r(\Omega, \sigma_0))_{\theta,p} =$$
$$= (W_p^m(\Omega, \sigma_0),\ W_p^m(\Omega, \sigma_1))_{\theta,p} \cap (W_p^m(\Omega, \sigma_0),\ W_p^r(\Omega, \sigma_0))_{\theta,p} .$$

The first space on the right hand side of (17) is described in Theorem 1, for the other we refer to H. Triebel [16], Sec. 3.3.

4. EXAMPLES AND COUNTER-EXAMPLES OF THE NON-COMMUTATIVE INTERPOLATION

H. Triebel in [15] gave examples and counter-examples to (*), obtained independently of all the general results mentioned. We extend this result to the spaces introduced by P. Grisvard in [5] and [6], but we use the version of H. Triebel [16], Ch. 4. The spaces under consideration are spaces between $W_p^m(\Omega)$ and $\mathring{W}_p^m(\Omega)$ and are determined by differential operators. They are of interest in the theory of differential operators.

Next we give without proof some statements we use throughout this section. We refer to P. Grisvard [5, 6] and to H. Triebel [16], for details and for proofs.

Let Ω be a bounded domain in R_n with a C^∞-boundary. Let $d(x)$ be the distance from $x \in \Omega$ to the boundary $\partial\Omega$. Then we denote by $L_{p,\nu}(\Omega) = L_{p,\nu}$ the space

(1) $$L_{p,\nu}(\Omega) = \{u : \|u\|_{L_{p,\nu}} = \left(\int_\Omega d(x)^{\nu p} |u(x)|^p \,\mathrm{d}x\right)^{1/p} < \infty .$$

We define spaces $W_p^{s(l)}(\Omega)$, characterized by boundary conditions.

Definition. Let Ω be a bounded domain in R_n with a C^∞-boundary. Further, let $1 < p < \infty$, $0 < \theta < 1$; let m be an integer and $l = 0, 1, 2, \ldots$.

(a) Let us assume that $m\theta - (1/p) \neq j$ for each number j; $j = 0, 1, \ldots, l$. Then

(3) $$(L_p(\Omega),\ W_p^{m(l)}(\Omega))_{\theta,p} = W_p^{m\theta(l)}(\Omega) .$$

(b) If there is $k \in \{0, \ldots, l\}$, such that $k = m\theta - (1/p)$, then

(4) $$(L_p(\Omega),\ W_p^{m(l)}(\Omega))_{\theta,p} =$$
$$= \left\{u : u \in W_p^{m\theta(l)}(\Omega) \int_\Omega d(x)^{-1} |\mathrm{D}^\alpha u(x)|^p \,\mathrm{d}x < \infty \quad \forall\alpha,\ |\alpha| = k\right\} .$$

Remark 1. We give another formulation of the statements (a) and (b):

(a) (i) for $l = 0, 1, 2, \ldots$ and $\theta \in \left(0, \dfrac{1}{pm}\right)$ we have

(5) $$(L_p(\Omega),\ W_p^{m(l)}(\Omega))_{\theta,p} = W_p^{m\theta}(\Omega) ;$$

(ii) for $l < m - 1$ and $\theta \in \left(\frac{1 + rp}{mp}, \frac{1 + (r+1)p}{mp}\right)$, $r = 0, \ldots, l - 1$,

or

$\theta \in \left(\frac{1 + rp}{mp}, 1\right)$, $r = l$, we have

(6) $$(L_p(\Omega), W_p^{m(l)}(\Omega))_{\theta,p} = \{u : u \in W_p^{m\theta}(\Omega),\ D^\alpha u(x)|_{\partial\Omega} = 0,\ |\alpha| \leqq r\}\ ;$$

(iii) if $l \geqq m - 1$ and $\theta \in \left(\frac{1 + rp}{mp},\ \min\left(1, \frac{1 + (r+1)p}{mp}\right)\right)$,

$r = 0, \ldots, m - 1$; then (6) again holds;

(b) for $\theta = \frac{1 + rp}{mp}$, $r = 0, \ldots, \min(l, m - 1)$, we have

(7) $$(L_p(\Omega), W_p^{m(l)}(\Omega))_{\theta,p} = \{u : u \in W_p^{m\theta}(\Omega),\ D^\alpha u(x)|_{\partial\Omega} = 0\,,\ |\alpha| < r,\ D^\alpha u(x) \in L_{p,-1/p},\ |\alpha| = r\}\,.$$

Further, we need two well-known assertions:

(8) For $1 < p < \infty$, $0 < \theta < 1$, ν real, the identity $(L_p(\Omega), L_{p,\nu})_{\theta,p} = L_{p,\theta\nu}$ holds.

(9) For $1 < p < \infty$, $0 < \theta < 1$, $m = 0, 1, \ldots$, $(L_p(\Omega), W_p^m(\Omega))_{\theta,p} = W_p^{m\theta}(\Omega)$ holds.

We also need the following

Lemma. *Let $\Omega \subset R_n$ be a bounded domain with a C^∞-boundary.*

(a) *Let $1 < p < \infty$, let m be a natural number and l an integer, $0 \leqq l \leqq m - 1$. Then $W_p^{m(l)}(\Omega) = W_p^m(\Omega) \cap L_{p,\nu}$ holds provided ν satisfies $-m \leqq \nu$, $-(l + 1) - 1/p < \nu \leqq -l - 1/p$.*

(b) *For $1/p < s \leqq 1$ and $-s \leqq \nu \leqq -1/p$ we have*

$$W_p^{s(0)}(\Omega) = \mathring{W}_p^s(\Omega) = W_p^s(\Omega) \cap L_{p,\nu}\,,$$

for $0 \leqq s < 1/p$ we have

$$W_p^s(\Omega) = W_p^{s(0)} = \mathring{W}_p^s = W_p^s \cap L_{p,\nu} \quad \text{with} \quad -s \leqq \nu\,.$$

Remark 2. Combining the statements (a) and (b) of Lemma we conclude: Let $1 < p < \infty$, let s be a positive number, $s = [s] + \{s\}$, $[s]$ = integer and $0 \leqq \{s\} < 1$, then for $l = 0, \ldots, [s] - 1$ the identity $W_p^{s(l)}(\Omega) = W_p^s(\Omega) \cap L_{p,\nu}$ holds provided ν satisfies

$$-(l + 1) - \frac{1}{p} < \nu \leqq -l - \frac{1}{p}, \quad -s \leqq \nu\,.$$

If $s = [s] + \{s\}$ with $\{s\} > 1/p$, then $W_p^{s([s])}(\Omega) = W_p^s(\Omega) \cap L_{p,\nu}$ for ν satisfying

$$-s \leqq \nu \leqq -[s] - \frac{1}{p}.$$

Remark 3. The interpolation couples (A_0, A_1) and (A_0, A_2) always satisfy $(A_0, A_1 \cap A_2)_{\theta,p} \subseteq (A_0, A_1)_{\theta,p} \cap (A_0, A_2)_{\theta,p}$ because $\max(K(t, a; A_0, A_1), K(t, a; A_0, A_2)) \leqq c\, K(t, a; A_0, A_1 \cap A_2)$, $a \in A_0 + (A_1 \cap A_2)$, where $K(t, a; A_0, A_1)$ is the K-functional of the real K-method:

$$K(t, a; A_0, A_1) = \inf_{\substack{a = a_0 + a_1 \\ a_0 \in A_0, a_1 \in A_1}} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1.$$

Remark 4. For $0 < s \leqq 1$ and $\max(-s, -1/p)$ we have

$$W_p^s(\Omega) \subset L_{p,\nu}.$$

After these preliminaries we can formulate

Theorem. *Let Ω be a bounded domain in R_n with a C^∞-boundary, $d(x)$ denotes the distance from $x \in \Omega$ to $\partial\Omega$. Let $1 < p < \infty$ and let m be a natural number.*

(a) *Let ν be a real number satisfying either $\nu \leqq -m$ or $\nu > -1/p$. Then*

$$(L_p(\Omega), W_p^m(\Omega) \cap L_{p,\nu})_{\theta,p} = (L_p(\Omega), W_p^m(\Omega))_{\theta,p} \cap (L_p(\Omega), L_{p,\nu})_{\theta,p} \tag{10}$$

provided $0 < \theta < 1$.

(b) *Let* $\max(-(l+1) - (1/p), -m) < \nu \leqq -l - (1/p)$, $l \in \{0, \ldots, m-1\}$.

(i) *In addition, let there exist numbers k, $k \in \{0, \ldots, l-1\}$, with*

$$\nu < \frac{-(1 + kp)\, mp}{1 + (k+1)\, p};$$

let k_0 denote the maximum of these numbers. Then (10) holds for

$$\theta \in \left(0, \frac{1}{mp}\right) \cup \left[-\frac{1}{p\nu}, \frac{1+p}{mp}\right) \cup \left[\frac{-1-p}{p\nu}, \frac{1+2p}{mp}\right) \cup \ldots$$

$$\ldots \cup \left[\frac{-1-k_0 p}{p\nu}, \frac{1 + (k_0+1)\, p}{mp}\right) \cup \left[\frac{-1-lp}{p\nu}, 1\right) = J_1.$$

(ii) *If there do not exist numbers k described in (i), then (10) holds for*

$$\theta \in \left[0, \frac{1}{mp}\right) \cup \left[\frac{-1-lp}{p}, 1\right) = J_2.$$

(c) *Under the same conditions on* ν *as in* (b) (i) *or in* (b) (ii), (10) *does not hold for*

$$\theta \in (0, 1) \setminus J_1 \quad \text{or} \quad \theta \in (0, 1) \setminus J_2 , \quad \text{respectively} ;$$

(a) *and* (b) *give examples of non-commutative interpolation while* (c) *gives a counter-example of non-commutative interpolation.*

Proof. First step: for $\nu \leqq m$ we refer to H. Triebel [7], Sect. 3.4.2. In virtue of (8), (9) and Remarks 3 and 4, formula (10) is immediately clear for $\nu > -1/p > > -1$.

Second step: We prove (b): Let

$$\max\left(-(l+1) - \frac{1}{p}, -m\right) < \nu \leqq -l - \frac{1}{p} , \; l \in \{0, \ldots, m-1\} .$$

Using (5), (8) and (9) and Remarks 3 and 4, we find that formula (10) holds for $0 < \theta < 1/mp$ not only for (b) (i) but for (b) (ii) as well. If $(-1 - lp)/\nu p < \theta < 1$, we obtain on the left hand side in (10)

$$(L_p(\Omega), W_p^m(\Omega) \cap L_{p,\nu})_{\theta,p} = W_p^{m\theta}(\Omega) \cap \{u : D^\alpha u|_{\partial\Omega} = 0; \; |\alpha| \leqq l\} .$$

There we used the facts that $W_p^m(\Omega) \cap L_{p,\nu} = W_p^{m(l)}(\Omega)$ and $(-1 - lp)/\nu p > > (1 + lp)/mp$ and therefore Proposition (a) and Remark 1 (ii), (iii) are applicable. Further, $\theta\nu < -(1/p) - l$. Thus, using (8), (9) and Remark 2, we conclude that the right hand side of (10) equals the left hand side. This proves (10) not only for (b) (i) but for (b) (ii) as well. Now, let the assumptions of (b) (i) be satisfied and let

$$\frac{-1 - rp}{\nu p} \leqq \theta < \frac{1 + (r+1)\,p}{mp} , \quad r = 0, \ldots, l - 1 .$$

For the left hand side of (10) we obtain because of Lemma (a) and (b):

$$(L_p^m(\Omega), W_p^m(\Omega) \cap L_{p,\nu})_{\theta,p} = (L_p(\Omega), W_p^{m(l)}(\Omega))_{\theta,p} =$$
$$= W_p^m(\Omega) \cap \{u : D^\alpha u|_{\partial\Omega} = 0 \; \forall\alpha, \; 0 \leqq |\alpha| \leqq r\} .$$

For ν we have

$$\nu < \frac{-(1 + kp)\,m}{(1 + (k+1)\,p)} < -\frac{(1 + (k-1)p)\,m}{1 + kp} < \ldots < \frac{-m}{1 + p}$$

and so

$$\frac{-1 - rp}{\nu p} < \frac{1 + (r+1)\,p}{m} , \quad r = 0, \ldots, k_0 .$$

The relations $m\theta > r + (1/p)$ and $\theta\nu \leqq -(1/p) - r$, $r = 0, \ldots, k_0$, are always satisfied. Thus, using (8), (9) and Remarks 2 and 3, we conclude that the left hand side of (10) equals the right hand side.

Third step: We prove (c): Let

$$\theta = \frac{1 + rp}{mp} \quad \text{with} \quad r = 0, \ldots, k_0$$

in the case (b) (i) and $r = 0$ in the case (b) (ii). Using the statement (a) of Lemma and (7), we obtain for the left hand side in (10):

$$(L_p(\Omega), W_p^m(\Omega) \cap L_{p,v})_{\theta,p} = (L_p(\Omega), W_p^{m(l)}(\Omega))_{\theta,p} =$$

$$= W_p^{m\theta}(\Omega) \cap \{u : \mathrm{D}^\alpha u|_{\partial\Omega} = 0;\ |\alpha| < r;\ \mathrm{D}^\alpha u \in L_{p,-1/p};\ |\alpha| = r\} .$$

With the aid of (8) and (9) we obtain on the right hand side of (10) the space $W_p^{m\theta}(\Omega) \cap$ $\cap\ L_{p,\theta v}$, where $\theta v > -(1/p) - r$; thus, this space is different from $(L_p(\Omega), W_p^{m(l)}(\Omega))_{\theta,p}$.

Now, let

$$\frac{1 + rp}{mp} < \theta < \frac{-1 - rp}{vp}, \quad r = 0, \ldots, k_0 ,$$

in the case (b) (i).

On the left hand side in (10) we obtain as above

$$(L_p(\Omega), W_p^{m(l)}(\Omega))_{\theta,p} = W_p^{m\theta}(\Omega) \cap \{u : \mathrm{D}^\alpha u|_{\partial\Omega} = 0;\ |\alpha| \leqq r\} .$$

Because of $\theta v > -(1/p) - r$ and $m\theta > r + (1/p)$ this space differs from the space on the right hand side.

Now, let

$$\frac{1 + (k_0 + 1)\, p}{mp} < \theta < \frac{-1 - lp}{mp}$$

in the case (b) (i), and

$$\frac{1}{mp} < \theta < \frac{-1 - lp}{mp}$$

in the case (b) (ii).

Then $\theta v > -((1/p) + (k_0 + 1))$ in the case (b) (i) and $\theta v > -1/p$ in the case (b) (ii). The space arising on the left hand side of (10) in the case (b) (i) is a subspace of

$$W_p^{m\theta}(\Omega) \cap \{u : \mathrm{D}^\alpha u|_{\partial\Omega} = 0;\ |\alpha| \leqq k_0 + 1\}$$

while in the case (b) (ii) it is a subspace of

$$W_p^{m\theta}(\Omega) \cap \{u : u|_{\partial\Omega} = 0\} .$$

The spaces arising on the right hand side of (10) are not subspaces of this space, which proves (c).

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