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ON o-IDEALS OF GROUPS OF DIVISIBILITY

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0. Introduction. For an integral domain A with the quotient field K, the group of divisibility $G(A)$ of A is a partially ordered factor group $K^*/U(A)$ of the multiplicative group K^* of K with respect to the group of units of A with $w_A(x) = x\,U(A) \leqq y\,U(A) = w_A(y)$ if and only if $yx^{-1} \in A$, where w_A is the canonical homomorphism of K^* onto $G(A)$. The study of divisibility of elements of A amounts essentially to the study of $G(A)$; it is well known that A is UFD if and only if $G(A)$ is a cardinal sum of copies of $\mathbf{Z}$, A is a GCD-domain if and only if $G(A)$ is lattice ordered etc.

Furthermore, several facts are known about relations between special subgroups of a group $G(A)$ and domains constructed by using A. In fact, there is a theorem that yields a correspondence between prime ideals of a valuation domain A and convex subgroups of $G(A)$ ([3]); a more general theorem establishes a correspondence between prime ideals of a Bezout domain A and prime l-ideals of $G(A)$ ([20]). These two theorems are generalized by the theorem of Mott [13], which yields a special bijection between saturated multiplicative systems in A and convex directed subgroups (i.e. *o-ideals*) of $G(A)$. Moreover, by means of this theorem it is possible to construct the group of divisibility of a quotient domain of A from the group $G(A)$. On the other hand, only for several elementary domains B constructed by using a domain A the construction of $G(B)$ from $G(A)$ is known. In Section 1 we show some facts about the group of divisibility of an intersection $\bigcap A_P$ of localizations of A and about the relation between o-ideals of $G(A)$ and $G(\bigcap A_P)$, and we deal with the so-called A-prime o-ideal of $G(A)$ that corresponds in Mott's bijection to the complement of a prime ideal in A.

Finally, in Section 2 we deal with topological groups of divisibility with topologies naturally induced from the topologies on the quotient fields and, especially, we deal with a "topological" version of Mott's bijection.

In this paper, all groups are abelian and all rings are integral domains. Following I. Kaplansky [6], we say that a ring A is a GCD-*domain* if each pair of nonzero elements of A has a greatest common divisor in A, i.e. $G(A)$ is a lattice ordered group (l-*group*). For a partially ordered group G we denote by $\mathfrak{O}(G)$ the set of o-ideals of G.

Let A be a ring, we denote by $\mathfrak{S}(A)$ the set of saturated multiplicative systems in A and by $\boldsymbol{m}_A$ (or shortly $\boldsymbol{m}$) the Mott's bijection between $\mathfrak{S}(A)$ and $\mathfrak{O}(G(A))$ defined by $\boldsymbol{m}_A(S) = \{w_A(s) - w_A(s') : s, s' \in S\}$, $\boldsymbol{m}_A^{-1}(H) = w_A^{-1}(H_+)$, where $H_+ = H^+ = \{\alpha \in H : \alpha \geqq 0\}$, $S \in \mathfrak{S}(A)$, $H \in \mathfrak{O}(G(A))$. If G_1, G_2 are partially ordered groups, a map $\sigma : G_1 \to G_2$ is called an *o-homomorphism* if it is a group homomorphism and $\sigma(G_1^+) \subseteq G_2^+$; it is called an *o-epimorphism* if it is a group epimorphism and $\sigma(G_1^+) = G_2^+$, and it is called an *o-isomorphism* if it is an o-epimorphism and a group isomorphism. The symbol $G_1 \cong G_2$ will denote the fact that there exists an o-isomorphism between G_1 and G_2. If A, B are rings with the same quotient field K, $A \subseteq B$, then by the canonical map $\sigma : G(A) \to G(B)$ we mean an o-homomorphism defined by $\sigma(w_A(x)) = w_B(x)$, $x \in K^*$. For any ring A and any $J \subseteq A$ we denote by J^* the set $J - \{0\}$.

1. o-ideals of some ring constructions. In this part we show several relations between ring constructions and their groups of divisibility. Some facts about these relations are in fact known; for example, if $S \in \mathfrak{S}(A)$, then $G(A_S) \cong G(A)/\boldsymbol{m}(A)$ ([13]); some facts are known about the group of divisibility of a composition of domains over a maximal ideal ([16], [14]). In this part we deal, particularly, with an intersection of localizations of A and with some basic facts about the group of divisibility of this intersection. It should be observed that in case A is a GCD-domain, this investigation is very easy. In fact, the following proposition holds.

Proposition 1.1. *Let A be a* GCD-*domain. Then the intersection of quotient rings of A is a quotient ring.*

Proof. Let $B = \bigcap A_{P_i}$ $(i \in I)$, where P_i are prime ideals of A. Let $S = \bigcap(A - P_i)$ $(i \in I)$. Then $A_S \subseteq B$. Let $z \in B$. Since $G(A)$ is an l-group, we have $w(z) = w(z)^+ - w(z)^-$, where $w = w_A$, $w(z)^+ = w(z) \vee 0$, $w(z)^- = -(w(z) \wedge 0)$ and $w(z)^+ \wedge w(z)^- = 0$. Then there exist $a, b \in A^*$ such that $z = ab^{-1}$, $w(z)^+ = w(a)$, $w(z)^- = w(b)$. Let $i \in I$, then for some $x_i \in A$, $y_i \in A - P_i$ we have $w(a) \leqq w(b) + w(x_i)$, $w(b) \leqq w(a) + w(y_i)$. Thus, $w(a) \leqq w(x_i)$, $w(b) \leqq w(y_i)$ and $x_i = ax_i'$, $y_i = by_i'$ for some $x_i', y_i' \in A$. Hence, $b \in A - P_i$ for every $i \in I$ and we obtain $z \in A_S$. Therefore, $B = A_S$.

We note that, in general, A_S constructed above is only the largest quotient ring of A contained in B. The family of rings with this property contains rings that are not intersections of localizations of A. In fact, for rings A, B with the same quotient field K we say that B is *well centred* on A, if $A \subseteq B$ and $B = A \,.\, U(B)$, i.e. the canonical map $G(A) \to G(B)$ is an o-epimorphism. The following lemma holds.

Lemma 1.2. *Let B be well centred on A. Then there exists the largest quotient ring of A contained in B.*

Proof. Let $\sigma : G(A) \to G(B)$ be the canonical map. Since σ is an o-epimorphism, there exists a convex subgroup H of $G(A)$ such that the factor ordered group $G(A)/H$ is o-isomorphic with $G(B)$. Let H^* be the *core* of H, i.e. $H^* = \{\alpha - \beta : \alpha, \beta \in H_+\}$. Then H^* is an o-ideal of $G(A)$ and for $S = \boldsymbol{m}_A^{-1}(H^*)$ we have $G(A_S) \cong G(A)/H^*$. Thus, $A_S \subseteq B$. If $A_{S'} \subseteq B$ for some $S' \in \mathfrak{S}(A)$, we have the canonical map $G(A_{S'}) \cong$ $\cong G(A)/\boldsymbol{m}_A(S') \to G(A)/H$ and it follows that $\boldsymbol{m}_A(S') \subseteq H$. Since $\boldsymbol{m}_A(S')$ is directed, we obtain $\boldsymbol{m}_A(S') \subseteq H^*$ and $A_{S'} \subseteq A_S$.

Let S be a torsion-free cancellation additive semigroup. Then the semigroup ring of S over a ring A is the set $A[S]$ of formal polynomials $a_1X^{\alpha_1} + \ldots + a_nX^{\alpha_n}$, $\alpha_i \in S$, $a_i \in A$, with addition and multiplication naturally defined.

Proposition 1.3. *Let $A \subseteq B$ be rings with the quotient field K. Then the following conditions are equivalent.*

(1) *B is well centred on A and $w_A(U(B)) \in \mathfrak{O}(G(A))$.*

(2) *B is a quotient ring of A.*

(3) *$B[S]$ is well centred on $A[S]$.*

(4) *$B[X]$ is well centred on $A[X]$.*

Proof. (1) $\Rightarrow$ (2). We set $S' = U(B) \cap A$. Then $A_{S'} \subseteq B$. Let $x \in B^*$. Then there exist $a \in A$, $u \in U(B)$ such that $x = au$. Since $w_A(U(B))$ is directed, there exists $j \in U(B)$ such that $w_A(j) \geqq w_A(u^{-1})$, 0. Hence, $j \in S'$, $x = abj^{-1} \in A_{S'}$ for some $b \in A$. Therefore, $B = A_{S'}$.

(2) $\Rightarrow$ (3). Let $B = A_N$, where $N \in \mathfrak{S}(A)$ and let $b_1X^{\alpha_1} + \ldots + b_nX^{\alpha_n} \in B[S]$, $b_i \in B$, $\alpha_i \in S$. There exist $a_1, \ldots, a_n \in A$, $s \in N$, such that $b_i = a_is^{-1}$. By [4]; 4.2, $s^{-1}X^\beta \in U(B[S])$, where β has an additive inverse in S. Then $b_1X^{\alpha_1} + \ldots + b_nX^{\alpha_n} =$ $s^{-1}X^\beta(a_1X^{\alpha_1 - \beta} + \ldots + a_nX^{\alpha_n - \beta})$ and $B[S]$ is well centred on $A[S]$.

(3) $\Rightarrow$ (4). Trivial.

(4) $\Rightarrow$ (1). It is clear that B is well centred on A. Let $i \in U(B)$, then $iX + 1_A \in B[X]$. Hence, there exist $j \in U(B)$, $a_1, a_0 \in A$, such that $iX + 1_A = ja_1X + ja_0$ and 0, $w_A(i) \geqq w_A(j)$. Therefore, $w_A(U(B)) \in \mathfrak{O}(G(A))$.

J. Rachůnek [18] shows that the family $\mathfrak{O}(G)$ is a complete lattice with the ordering by inclusion, where for any $\{H_i : i \in I\} \subseteq \mathfrak{O}(G)$, the infimum $\inf\{H_i : i \in I\}$ is the core of the convex subgroup $\bigcap H_i$ $(i \in I)$. The following proposition shows the relation between infimum in $\mathfrak{O}(G(A))$ and a certain ring construction on A.

Proposition 1.4. *Let $\emptyset \neq \{H_i : i \in I\} \subseteq \mathfrak{O}(G(A))$, $H = \inf\{H_i : i \in I\}$. Then $G(A)/H \cong G(A_{\cap S_i})$, where $S_i = \boldsymbol{m}_A^{-1}(H_i)$.*

Proof. Let $x \in \bigcap S_i$ $(i \in I)$. Then $w_A(x) \in H_i^+$ and we have $w_A(x) \in H^+$. Conversely, for $w_A(x) \in H^+$ we have $x \in S_i$ for every $i \in I$. Hence, $w_A(\bigcap S_i\, (i \in I)) = H^+$ and $\boldsymbol{m}_A(\overline{\bigcap S_i}) = H$, where $\overline{\bigcap S_i}$ = saturation of $\bigcap S_i$.

Now, let for a ring A, $B = \bigcap A_{S_i}\,(i \in I)$, where $S_i \in \mathfrak{S}(A)$; $H_i = \boldsymbol{m}_A(S_i)$, $H = \bigcap H_i\,(i \in I)$. Then the following holds.

Theorem 1.5. *The group of divisibility $G(B)$ of B is o-isomorphic with the group $(G(A)/H, \leqq)$, where $\leqq$ is an ordering on $G(A)/H$ stronger than the factor one. $G(B)$ is o-isomorphic with $G(A)/H$ with the factor ordering if and only if B is well centred on A.*

Proof. We set $S = \bigcap S_i\,(i \in I)$ and let $H' = \boldsymbol{m}_A(S)$. (By Prop. 1.4. $H' = \inf\{H_i : i \in I\}$.) Let $\sigma : G(A_S) \cong G(A)/H' \to G(B)$ be the canonical map. Then the canonical map τ of $G(B)$ into the product $\Pi G(A)/H_i\,(i \in I)$ defined by $\tau(w_B(x)) = (w_A(x) + H_i)_i$ is an o-isomorphism into. There exists an o-homomorphism and a bijection ϱ such that the following diagram commutes,

$$\begin{array}{ccccccc} G(A)/H' & \xrightarrow{\sigma} & G(B) & \xrightarrow{\tau} & \{(w_A(x) + H_i)_i : x \in K^*\} & \subseteq & \Pi G(A)/H_i \\ \big\downarrow & & & & \big\uparrow \varrho & & \\ & \longrightarrow & G(A)/T & \longrightarrow & & & \end{array}$$

where K is the quotient field of A and $T/H' = \ker(\tau\sigma)$. Then $T = \{w_A(x) : x \in K^*, \tau\sigma(w_A(x) + H') = \tau\, w_B(x) = (w_A(x) + H_i)_i = (H_i)_i\} = \bigcap H_i\,(i \in I) = H$. On the group $G(A)/H$ we define an order relation $\leqq$ setting $(G(A)/H)_+ = \varrho^{-1}(\tau(G(B)_+))$. Then $(G(A)/H, \leqq)$ is o-isomorphic with $G(B)$ and $\leqq$ is stronger than the factor ordering. If B is well centred on A, B is well centred on A_S and σ is an o-epimorphism. Then the factor ordered group $(G(A)/H')\ker(\tau\sigma) \cong G(A)/H$ is o-isomorphic with $G(B)$. Conversely, if $G(A)/H \cong G(B)$, the canonical map is an o-epimorphism.

It should be observed that there exists a domain $B = \bigcap A_P$ such that B is not a quotient ring of A and it is well centred on A. We use the example 4.1 of [4a]. Let G be a countable weak direct sum of the additive group of integers, lexicographically ordered. Let k be a field and let k_0 be a subfield over which k is algebraic. We consider $x_1, x_2, \ldots, x_n, \ldots$ elements of an extension field of k which are algebraically independent over k. Then for $ax_1^{r_1} \ldots x_n^{r_n} \in k[x_1, \ldots]^*$ we set $w(ax_1^{r_1} \ldots x_n^{r_n}) = (r_1, \ldots, r_n, 0, \ldots) \in G$; $w(f(x))$ = minimum value of the nonzero monomials occuring in $f(x)$, $f(x) \in k[x_1, \ldots]^*$; and $w(f/g) = w(f) - w(g)$ for $f/g \in k(x_1, \ldots) = K$. Then w is a valuation in K and in [4a] it is proved that $R_w = k + M_w$ is an intersection of quotient rings of a domain $D = k_0 + M_w$, where M_w is the maximal ideal of R_w. Moreover, for $g = (g_1, g_2, \ldots) \in G_+$, $\operatorname{supp}(g) = \{i_1, \ldots, i_n\}$, we have $X = x_{i_1}^{g_1} \ldots x_{i_n}^{g_n} \in D$ and $w(X) = g$. Thus, R_w is well centred on D and it is easy to see that R_w is not a quotient ring of D.

The intersection of localizations is a special case of the so called *generalized quotient ring* of A (g.q.r.) with respect to a *generalized multiplicative system* $\mathcal{S}$ (g.m.s.) of A (see [17]). Recall that a g.m.s. of A is a family $\mathcal{S}$ of non-empty subsets of A such that $\{0\} \notin \mathscr{S}$ and for any $X_1, X_2 \in \mathscr{S}$, $X_1 . X_2 = \{\sum_{i=1}^{n} a_i b_i : a_i \in X_1, b_i \in X_2, n \in Z_+\} \in \mathscr{S}$ holds. A g.q.r. of A with respect to a g.m.s. $\mathscr{S}$ in A is a ring

$$A_{\mathscr{S}} = \{x \in K : \exists J \in \mathscr{S} \text{ such that } x . J^* \subseteq A\} ,$$

where K is the quotient field of A. If $B = \bigcap A_{P_i} (i \in I)$, we have $B = A_{\mathscr{S}}$ for $\mathscr{S} = \{J : J \subseteq A, J \neq \{0\}, J \nsubseteq P_i \text{ for every } i \in I\}$.

We say that $E \subseteq A$ *misses* $\mathscr{S}$ ($\mathscr{S}$ a g.m.s. of A) if $J^* \nsubseteq E$ for every $J \in \mathscr{S}$.

Proposition 1.6. *Let $S \in \mathfrak{S}(A)$ and let $\mathscr{S}$ be a g.m.s. of A that consists of ideals of A. Then S misses $\mathscr{S}$ if and only if $S_{\mathscr{S}} = \{x \in K^* : \exists J \in \mathscr{S}$ such that $x . J^* \subseteq S\} = \emptyset$.*

Proof. Let S miss $\mathscr{S}$. We suppose that there exists $x \in S_{\mathscr{S}} \cap A$. Then for some $J \in \mathscr{S}$ we have $x . J^* \subseteq S$, hence for every $a \in J^*$ we have $a^{-1} = x(xa)^{-1} \in A_S$ and $a \in U(A_S) \cap A = S$, $J^* \subseteq S$, a contradiction. Further, we suppose that there exists $x \in S_{\mathscr{S}}$. Then for some $J \in \mathscr{S}$ we have $x . J^* \subseteq S$. Let $b \in J^*$, then $xf \in S$, $b . J^* \subseteq J^*$ and $(xb) . J^* \subseteq x . J^* \subseteq S$ and $xb \in S_{\mathscr{S}} \cap A$, a contradiction. Hence, $S_{\mathscr{S}} = \emptyset$. The converse is trivial.

Now, let $\mathscr{S}$ be a g.m.s. of A and let $S \subseteq A$. We say that S is *large with respect to $\mathscr{S}$, if for every $J \in \mathscr{S}$, $J \cap S \neq \emptyset$* holds.

Proposition 1.7. *Let $\mathscr{S}$ be a g.m.s. of A. Then the maps $S \mapsto U((A_{\mathscr{S}})_S) \cap A_{\mathscr{S}} = \operatorname{sat}_{\mathscr{S}} S$, $\mathbf{S} \mapsto \mathbf{S} \cap A$, are mutually inverse bijections between the sets of saturated multiplicative systems of A, $A_{\mathscr{S}}$, respectively, that are large with respect to $\mathscr{S}$. Furthermore,*

$$(A_{\mathscr{S}})_{\mathbf{S}} = A_{\mathbf{S} \cap A} , \quad A_S = (A_{\mathscr{S}})_{\operatorname{sat}_{\mathscr{S}} S} .$$

Proof. Let $S \in \mathfrak{S}(A)$ be large with respect to $\mathscr{S}$ and let $x \in \operatorname{sat}_{\mathscr{S}} S \cap A$. Then $x^{-1} \in (A_{\mathscr{S}})_S$, $x^{-1} = as^{-1}$ for some $a \in A_{\mathscr{S}}$, $s \in S$. Then there exists $J \in \mathscr{S}$ such that $a . J^* \subseteq A$. Let $s' \in J \cap S$, then $x^{-1} = (as') . (ss')^{-1} \in A_S$ and $x \in U(A_S) \cap A = S$. Hence, $\operatorname{sat}_{\mathscr{S}} S \cap A = S$. Let $xs^{-1} \in (A_{\mathscr{S}})_S$, $x \in A_{\mathscr{S}}$, $s \in S$. Then for some $J \in \mathscr{S}$ we have $x . J^* \subseteq A$ and again, for some $s' \in J \cap S$ we obtain $xs^{-1} = (xs') . (ss')^{-1} \in A_S$ and $(A_{\mathscr{S}})_{\operatorname{sat}_{\mathscr{S}} S} = (A_{\mathscr{S}})_S = A_S$. Let $\mathbf{S} \in \mathfrak{S}(A_{\mathscr{S}})$ be large with respect to $\mathscr{S}$, $S = \mathbf{S} \cap A$. Since $S \subseteq U(A_{\mathbf{S} \cap A}) \cap A = U(A_{\mathbf{S} \cap A}) \cap A_{\mathscr{S}} \cap A = S$, we have $S \in \mathfrak{S}(A)$. Let $y \in \mathbf{S}$, then there exists $J \in \mathscr{S}$ such that $y . J^* \subseteq A$. Let $s \in J \cap \mathbf{S} = J \cap (\mathbf{S} \cap A)$. Then $ys \in A \cap \mathbf{S}$ and $y^{-1} = s(ys)^{-1} \in A_S$. Thus, $\mathbf{S} \subseteq U(A_S)$. Further, $A_{\mathscr{S}} \subseteq A_S$ and $(A_{\mathscr{S}})_{\mathbf{S}} \subseteq A_S$. Hence, $(A_{\mathscr{S}})_{\mathbf{S}} = A_S = (A_{\mathscr{S}})_S = (A_{\mathscr{S}})_{\operatorname{sat}_{\mathscr{S}} S}$ and we obtain $\mathbf{S} = \operatorname{sat}_{\mathscr{S}} (\mathbf{S} \cap A)$.

By using this proposition it is possible to give a new proof of the following well known proposition (see [3]): If $\{P_i : i \in I\}$ is a family of prime ideals of A such that there is no containment relation among distinct members of the set P_i and each prime ideal of A contained in $\bigcup P_i$ is contained in some P_i, then $\bigcap A_{P_i}(i \in I) = A_{\cap(A-P_i)}$. In fact, let $S = \bigcap(A - P_i)(i \in I)$, $\mathcal{S} = \{J : J$ ideal of A, $J \nsubseteq P_i$ for every $i \in I\}$. Then $\bigcap A_{P_i} = A_{\mathcal{S}}$. Suppose that there exists $J \in \mathcal{S}$ such that $J \cap S = \emptyset$. Then by Krull's theorem there exists a prime ideal P of A such that $J \subseteq P$, $P \cap S = \emptyset$. Hence, $P \subseteq \bigcup P_i$ and $P \subseteq P_i$ for some $i \in I$. Hence, $J \subseteq P_i$, a contradiction. Thus, S is large with respect to $\mathcal{S}$. Since $S \subseteq U(A_{\mathcal{S}})$, we have $\mathrm{sat}_{\mathcal{S}} S = U(A_{\mathcal{S}})$ and

$$\bigcap A_{P_i}(i \in I) = A_{\mathcal{S}} = (A_{\mathcal{S}})_{\mathrm{sat}_{\mathcal{S}} S} = A_S = A_{\cap(A-P_i)}\,.$$

Using Proposition 1.7 we can show a certain relation between special o-ideals of $G(A)$ and $G(\bigcap A_{P_i})$. Let $\{P_i : i \in I\}$ be a set of prime ideals of A, $H_i = \boldsymbol{m}_A(A - P_i)$, $B = \bigcap A_{P_i}(i \in I)$, $\sigma : G(A) \to G(B)$ the canonical map and let

$$\mathfrak{O}_1 = \{H : H \in \mathfrak{O}(G(A)),\ \exists i \in I,\ H_i \subseteq H\}\,,$$

$$\mathfrak{O}_2 = \{\bar{H} : \bar{H} \in \mathfrak{O}(G(B)),\ \exists i \in I,\ \sigma(H_i) \subseteq \bar{H}\}\,.$$

Proposition 1.8. *The map $H \mapsto \sigma(H)$ is a bijection between $\mathfrak{O}_1$ and $\mathfrak{O}_2$ and for $H \in \mathfrak{O}_1$, $G(A)/H \cong G(B)/\sigma(H)$ holds.*

Proof. For $\mathcal{S} = \{J : J \subseteq A,\ J \subseteq P_i$ for every $i \in I\}$ we have $B = A_{\mathcal{S}}$. We denote $\bar{\mathfrak{O}}_1 = \{S : S \in \mathfrak{S}(A),\ S$ is large with respect to $\mathcal{S}\}$, $\bar{\mathfrak{O}}_2 = \{S : S \in \mathfrak{S}(B),\ S$ is large with respect to $\mathcal{S}\}$. By Prop. 1.7 there is a bijection $\beta : \bar{\mathfrak{O}}_1 \to \bar{\mathfrak{O}}_2$ such that $A_S = (A_S)_{\beta(S)}$, $S \in \bar{\mathfrak{O}}_1$. We denote $\boldsymbol{m}_1 = \boldsymbol{m}_A \,|\, \bar{\mathfrak{O}}_1$, $\boldsymbol{m}_2 = \boldsymbol{m}_B \,|\, \bar{\mathfrak{O}}_2$. Then $\boldsymbol{m}_i$ is a bijection between $\bar{\mathfrak{O}}_i$ and $\mathfrak{O}_i$. In fact, we suppose that for some $S \in \bar{\mathfrak{O}}_1$, $\boldsymbol{m}_1(S) \notin \mathfrak{O}_1$ holds. Then for every $i \in I$ we may find an element $\alpha_i = w_A(a_i) \in H_i^+ - \boldsymbol{m}_1(S)_+ = H_i^+ - w_A(S)$. Thus, $J = \{a_i : i \in I\} \in \mathcal{S}$ and $J \cap S = \emptyset$, a contradiction. Conversely, let $\boldsymbol{m}_1(S) \in \mathfrak{O}_1$. Then for some $i \in I$ we have $H_i \subseteq \boldsymbol{m}_1(S)$ and it follows that $A - P_i \subseteq S$. Thus, $S \in \bar{\mathfrak{O}}_1$. Analogously, for $S \in \bar{\mathfrak{O}}_2$ there exists $i \in I$ such that $A - P_i \subseteq S$. Let $\alpha \in H_i$, $\alpha = \alpha_1 - \alpha_2$, for $\alpha_i \in H_i^+$, $\alpha_i = w_A(a_i)$. Since $a_i \in A - P_i$, we have $\sigma(\alpha) = w_B(a_1) - w_B(a_2) \in \boldsymbol{m}_2(S)$ and $\boldsymbol{m}_2(S) \in \mathfrak{O}_2$. Conversely, for $S \in \mathfrak{S}(B)$, $\boldsymbol{m}_2(S) \in \mathfrak{O}_2$, there exists $i \in I$ with $\sigma(H_i) \subseteq \boldsymbol{m}_2(S)$. Then for $a \in A - P_i$ we have $w_A(a) \in H_i$, $w_B(a) = \sigma\, w_A(a) \in \sigma(H_i^+) \subseteq \boldsymbol{m}_2(S)_+ = w_B(S)$ and $A - P_i \subseteq S$, $S \in \bar{\mathfrak{O}}_2$.

Thus, $\boldsymbol{m}_2\beta\boldsymbol{m}_1^{-1}$ is a bijection between $\mathfrak{O}_1$ and $\mathfrak{O}_2$ and by Prop. 1.7, $G(A)/H \cong G(A_{\boldsymbol{m}_1^{-1}(H)}) = G(B_{\beta\boldsymbol{m}_1^{-1}(H)}) \cong G(B)/\boldsymbol{m}_2\beta\boldsymbol{m}_1^{-1}(H)$. Then $H = w_A(U(A_{\boldsymbol{m}_1^{-1}(H)})) = w_B(U(B_{\beta\boldsymbol{m}_1^{-1}(H)}))$ and we obtain $\sigma(H) = \sigma w_A(U(B_{\beta\boldsymbol{m}_1^{-1}(H)})) = w_B(U(B_{\beta\boldsymbol{m}_1^{-1}(H)})) = \boldsymbol{m}_2\beta\boldsymbol{m}_1^{-1}(H)$.

Corollary 1.9. *For $S \in \bar{\mathfrak{O}}_1$ we have*

$$\mathrm{sat}_B S = U(B_S) \cap B =$$

$$= \{x \in B : \exists y \in \mathrm{sat}_B S \text{ such that } xy^{-1} \in U(A_{P_i}) \text{ for every } i \in I\}\,.$$

Proof. Let $H = \boldsymbol{m}_1(S)$. Then $\mathrm{sat}_B S = w_B^{-1}(\sigma(H)_+) = \{x \in B : \exists y \in U(A_S) \cap B$

such that $w_A(xy^{-1}) \in H_i$ for every $i \in I\} = \{x \in B : \exists y \in \mathrm{sat}_B S$ such that $xy^{-1} \in$ $\in U(A_{P_i})$ for every $i \in I\}$.

It should be observed that if $\mathcal{S}_1, \mathcal{S}_2$ are g.m.s. in A such that $A_{\mathcal{S}_1} = A_{\mathcal{S}_2}$ and $S \in$ $\in \mathfrak{S}(A)$ is large with respect to $\mathcal{S}_1$, it does not follow, in general, that S is large with respect to $\mathcal{S}_2$. In fact, let A be a discrete rank one valuation ring with the maximal ideal M. Since $M^2 = M$, $\mathcal{S}_1 = \{A, M\}$, $\mathcal{S}_2 = \{\{1\}\}$ are g.m. systems in A and $A_{\mathcal{S}_1} = A = A_{\mathcal{S}_2}$. But, for $S = A - M$, S is large with respect to $\mathcal{S}_2$ and it is not large with respect to $\mathcal{S}_1$.

J. Rachůnek [19] shows that for any 2-isolated partially ordered group G (i.e. $g + g \geqq 0$, $g \in G$ implies $g \geqq 0$) and any subset $H \subseteq G_+$ there is the smallest o-ideal $C(H)$ in G containing H. The following lemma shows that the same is true for groups of divisibility without the assumption mentioned above. For $H \subseteq G_+$ we denote by $[H]$ the subsemigroup of G_+ generated by H.

Lemma 1.10. *Let $G = G(A)$ be a group of divisibility of A, $\emptyset \neq H \subseteq G_+$. Then there exists the smallest o-ideal $C(H)$ in G containing H. Furthermore,*

$$C(H)_+ = \{\alpha \in G_+ : \exists \beta \in [H] \text{ with } \beta \geqq \alpha\} .$$

Proof. Let $S' = w_A^{-1}([H])$, $S = U(A_{S'}) \cap A$. Then $S \in \mathfrak{S}(A)$ and we set $C(H) =$ $= \boldsymbol{m}_A(S)$. It is clear that $H \subseteq C(H)$. For the o-ideal $H' \in O(G)$ such that $H \subseteq H'$ and for $\alpha = w_A(x) \in C(H)_+$ we have $x \in S$ and there are $a \in A$, $s \in S'$ such that $xa = s$. Then $w_A(s) \geqq w_A(x) \geqq 0$ and $\alpha \in H'$. The rest is clear.

Proposition 1.11. *Let $\emptyset \neq \{H_i : i \in I\} \subseteq \mathfrak{O}(G(A))$, $S_i = \boldsymbol{m}_A^{-1}(H_i)$, $S = \{s_{i_1} \dots s_{i_n} :$ $: i_1, \dots, i_n \in I,\ s_{i_t} \in S_{i_t}\}$, $H = \{\alpha_{i_1} + \dots + \alpha_{i_n} : i_1, \dots, i_n \in I,\ \alpha_{i_t} \in H_{i_t}\}$. Then there exists the smallest o-ideal F of $G(A)$ containing H and for this o-ideal, $G(A_S) \cong$ $\cong G(A)/F$ holds.*

Proof. It is clear that H is a directed subgroup of $G(A)$. By Lemma 1.10 there exists the smallest o-ideal F of $G(A)$ containing H_+, where $F_+ = \{\alpha \in G(A)_+ :$ $: \exists \beta \in H_+$ such that $\beta \geqq \alpha\}$. Let $S' = U(A_S) \cap A$. Then for $x \in S'$ there exist $\alpha_{i_t} \in H_{i_t}^+$, $t = 1, \dots, n$, such that $w_A(x) \leqq \alpha_{i_1} + \dots + \alpha_{i_n}$ and it follows that $w_A(x) \in F_+$. Analogously, we obtain $F_+ \subseteq w_A(S')$. Then $F = \boldsymbol{m}_A(S')$ and $G(A_S) = G(A_{S'}) \cong$ $\cong G(A)/F$. It is clear that F is the smallest o-ideal in $G(A)$ containing H.

J. L. Mott [13] introduced the notion of prime o-ideal in a partially ordered group G in the following way: $H \in \mathfrak{O}(G)$ is *prime* if G/H is totally ordered. If $G = G(A)$ is a group of divisibility and $H \in \mathfrak{O}(G)$ is prime, a ring $A_{\boldsymbol{m}^{-1}(H)}$ is a valuation ring, since $G(A_{\boldsymbol{m}^{-1}(H)}) \cong G/H$ is totally ordered. Hence, $\boldsymbol{m}^{-1}(H) = A - P$ for a prime ideal P of A. This property naturally leads to the following definition. An o-ideal H of $G(A)$ is A-*prime*, if there exists a prime ideal P of A such that $\boldsymbol{m}_A^{-1}(H) = A - P$. Several properties of A-prime o-ideals are investigated in [10], where the main tool in studying these properties is a special partially ordered group endowed with a multivalued addition, called a *d-group* (see [15]). The following multivalued addition $\oplus_A$ on $G(A)$ has received considerable attention: For $\alpha, \beta, \gamma \in G(A)$, $\alpha \in \beta \oplus_A \gamma$ if and only if there are $a, b, c \in K^*$, $u_1, u_2 \in U(A)$ (K is the quotient field of A) such that $\alpha = w_A(a)$, $\beta = w_A(b)$, $\gamma = w_A(c)$, $a = bu_1 + cu_2$. If A is a GCD-domain, it is possible to define another multivalued addition $\oplus_m$ in the following way: $\alpha \oplus_m \beta = \{\gamma \in G(A) : \alpha \wedge \beta = \alpha \wedge \gamma = \beta \wedge \gamma\}$. For any CGD-domain A and for every $\alpha, \beta \in G(A)$ we have $\alpha \oplus_A \beta \subseteq \alpha \oplus_m \beta$ and it should be observed that the converse inclusion does not hold in general (see [10]).

We note that since every prime l-ideal of a group of divisibility of a GCD-domain A is a prime o-ideal it follows that every prime l-ideal of $G(A)$ is A-prime. (Here $H \in \mathfrak{O}(G)$ is a prime l-ideal of an l-group G if $\alpha, \beta \in G_+$, $\alpha \wedge \beta \in H$ imply $\alpha \in H$ or $\beta \in H$.)

Lemma 1.12. *Let A be a* GCD-*domain such that $\oplus_A = \oplus_m$. Then every A-prime o-ideal in $G(A)$ is a prime l-ideal and A is a Bezout domain. The converse implication is not valid in general.*

Proof. Let $\oplus_A = \oplus_m$ and let H be an A-prime o-ideal in $G(A)$. Then for a prime ideal P of A, $G(A_P) \cong G(A)/H$. By [10]; Prop. 7, H is a prime d-convex subgroup of a d-group $(G(A), \oplus_A) = (G(A), \oplus_m)$ (for definition, see [15]), i.e. the facts $\alpha, \beta \in G(A)_+$, $(\alpha \oplus_m \beta) \cap H \neq \emptyset$, imply, $\alpha \in H$ or $\beta \in H$. Let $\alpha, \beta \in G(A)_+$, $\alpha \wedge \beta \in H$. Then $\alpha \wedge \beta \in (\alpha \oplus_m \beta) \cap H$ and H is a prime l-ideal in $G(A)$. To prove that A is a Bezout domain we need to show that A is a Prüfer domain. Let P be a prime ideal of A. Then $H = \boldsymbol{m}_A(A - P)$ is A-prime and according to the proof presented above, H is a prime l-ideal of $G(A)$. Thus, $G(A_P) \cong G(A)/H$ is totally ordered and A_P is a valuation ring.

To show that the fact that every A-prime o-ideal of $G(A)$ is a prime l-ideal does not imply $\oplus_A = \oplus_m$ we set $A = Z_{(2)}$. Then since $0 \notin 0 \oplus_A 0$, $0 \in 0 \oplus_m 0$, we have $\oplus_A \neq \oplus_m$ and the set of A-prime o-ideals of $G(A) = Z$ is $\{Z, \{0\}\}$, i.e. the set of prime l-ideals of $G(A)$.

We note that the notion of an A-prime o-ideal of $G(A)$ is based essentially on A. In fact, let A be a GCD-domain that is not a Bezout one. Then there exists a prime ideal P of A such that A_P is not a valuation ring, i.e. $H = \boldsymbol{m}_A(A - P)$ is an A-prime o-ideal that is not a prime l-ideal of $G = G(A)$. Let B be a Bezout domain constructed in [16], such that $G(B) = G$. Then by [10]; Lemma 2, $\oplus_B = \oplus_m$ holds. Since H is not a prime l-ideal in G, H is not B-prime by Lemma 1.12.

We note that via the notion of d-group the following problem can be solved: Does there exist a group of divisibility G and a nondirected convex subgroup H of G such that the factor group G/H is a group of divisibility? The answer is in affirmative as the following example shows.

First, we note that if $G = G(A)$ is a group of divisibility of a domain A and H

is a convex subgroup of G such that $H \,.\, G_+ \oplus_A H \,.\, G_+ = H \,.\, G_+$, the factor group G/H is a d-group with respect to the multivalued addition $\oplus'$ defined by

$$g_1 H \oplus' g_2 H = (g_1 H \oplus_A g_2 H)/H$$

and the canonical map $\sigma : G \to G/H$ satisfies the condition $\sigma(g_1 \oplus_A g_2) \subseteq \sigma(g_1) \oplus' \oplus' \sigma(g_2)$. Now, $w = \sigma w_A$ is a semi-valuation on the quotient field K of A with the value group G/H. In fact, for $a, b, c \in K^*$, $a + b \in K^*$, such that $w(c) \leqq w(a), w(b)$, and from the fact $w_A(a + b) \in w_A(a) \oplus_A w_A(b)$ we obtain $w(a + b) \in w(a) \oplus' w(b)$ and hence $w(a + b) \geqq w(c)$ (see the properties of d-groups in [15]). Hence, by [16], w is a semi-valuation on K and G/H is a group of divisibility. The following example of a domain A such that there exists a non directed d-convex subgroup in $(G(A), \oplus_A)$ is taken from [15].

Let $A = \mathbf{Z}[X, Y]$, where $\mathbf{Z}$ is the ring of rational integers and let w be an (X, Y)-adic valuation on the quotient field K of A, $G = G(A)$. We set

$$S = \{f \in A^* : \text{for every irreducible polynomial } p \text{ in } A \text{ such that}$$
$$w_A(p) \leqq w_A(f) \text{ we have } w(p) \geqq 1\},$$
$$H = \{w_A(fg^{-1}) : f, g \in S,\ w(f) = w(g)\}.$$

In [15] it is proved that H is a nondirected d-convex subgroup in $(G, \oplus_A)$ and it follows that G/H is a group of divisibility.

In what follows we denote by $\mathfrak{P}(G(A))$ the set of A-prime o-ideals of $G(A)$.

Lemma 1.13. $\bigcap H (H \in \mathfrak{P}(G(A))) = \{0\}$. $\{0\} \in \mathfrak{P}(G(A))$ *if and only if A is quasi-local.*

Proof. Since $A = \bigcap A_P$ (P a prime ideal of A), it follows that the canonical map $\sigma : G(A) \to \Pi G(A)/H (H \in \mathfrak{P}(G(A)))$ defined by $\sigma(w_A(x)) = (w_H(x))_H$, $x \in K^*$, is an injection, where $w_H : K \to G(A_P) \cong G(A)/H$, $H = \boldsymbol{m}_A(A - P)$, is a semi-valuation associated with A_P. Hence, $\bigcap H = \{0\}$. The rest is clear.

Proposition 1.14. *Let $\{H_i : i \in I\} \subseteq \mathfrak{P}(G(A))$ be such that*

1) *there are no containment relations between distinct H_i,*
2) *for every $H \in \mathfrak{P}(G(A))$ such that $\inf\{H_i : i \in I\} \subseteq H$ there exists $i \in I$ such that $H_i \subseteq H$.*

Then $\inf\{H_i : i \in I\} = \bigcap H_i (i \in I)$ in $\mathfrak{O}(G(A))$.

Proof. Let $P_i = A - \boldsymbol{m}_A^{-1}(H_i)$. Then by [3]; $B = \bigcap A_{P_i} = A_{\cap(A - P_i)}$. Let $H = \inf H_i$. By Prop. 1.4, $G(A)/H \cong G(A_{\cap(A-P_i)})$ and since B is well centred on A, Theorem 1.5 yields $G(A)/H \cong G(A)/(\bigcap H_i)$. Therefore, $H = \bigcap H_i (i \in I)$.

Corollary 1.15. *Let $H_1, \ldots, H_n \in \mathfrak{P}(G(A))$. Then $\bigcap_{i=1}^{n} H_i$ is an o-ideal in $G(A)$.*

Proposition 1.16. *Let* $\alpha_1, \ldots, \alpha_n \in G(A)_+$. *Then the smallest o-ideal in* $G(A)$, *containing* $\alpha_1, \ldots, \alpha_n$ *is A-prime if and only if the smallest o-ideal in* $G(A)$ *containing* $\alpha_1 + \ldots + \alpha_n$ *is A-prime.*

Proof. The smallest o-ideal $C(H)$ in $G(A)$ containing $H \subseteq G(A)_+$ is A-prime if and only if the family $\{P : P \in \operatorname{Spec} A, P \cap w_A^{-1}(H) = \emptyset\}$ admits the largest element. The proposition then follows from the fact that for $a_i \in w_A^{-1}(\alpha_i)$, $i = 1, \ldots, n$, $\{P \in \operatorname{Spec} A : w_A^{-1}(\{\alpha_1, \ldots, \alpha_n\}) \cap P = \emptyset\} = \{P \in \operatorname{Spec} A : a_1 \cdot \ldots \cdot a_n \notin P\} = \{P \in \operatorname{Spec} A : w_A^{-1}(\alpha_1 + \ldots + \alpha_n) \cap P = \emptyset\}$.

In the theory of lattice ordered groups, a well known notion is that of a *value* of an element $g \neq 0$ of an l-group G, i.e. the largest l-ideal H_g in G not containing g. It is well known that H_g is a prime l-ideal. Analogously, in the theory of partially ordered groups it is possible to define the value of an element, namely, the *value* of an element g of a partially ordered group G is the largest o-ideal H_g in G such that $g \notin H_g$. It should be observed that not every value is a prime o-ideal. In fact, let $G = (\mathbf{Z}, +, \lesssim)$, where $a \lesssim b$ if and only if $b - a$ is an even nonnegative number. Then G is a directed partially ordered group and it is easy to see that $H = \{2k : k \in G\}$ is the unique nonzero o-ideal in G. Then H is a value of $1 \in G$ and since G/H is not totally ordered, H is not prime.

If G is a group of divisibility, we may say something more about the value of an element.

First, we say that a prime ideal P in a ring A is *isolated*, if $\bigcup P'$ (P' is a prime ideal of A, $P' \subset P$) $\subset P$.

Proposition 1.17. *An o-ideal H in* $G(A)$ *is a value of an element of* $G(A)_+$ *if and only if H is A-prime and* $A - m_A^{-1}(H)$ *is an isolated prime ideal of A. Every element of* $G(A)$ *has a value.*

Proof. Let H be a value of $\alpha = w_A(a) \in G(A)_+$, $S = m_A^{-1}(H)$. Let P be a maximal ideal of A such that $P \cap S = \emptyset$, $a \in P$, and let $H' = \boldsymbol{m}_A(A - P)$. Then $\alpha \notin H'$, $H \subseteq H'$ and hence $H = H'$. Thus, H is A-prime. We suppose that $P = \bigcup P'$ (P' a prime ideal of A, $P' \subset P$). Then for $a \in P$ there exists a prime ideal $P' \subset P$ with $a \in P'$. For $H'' = \boldsymbol{m}_A(A - P')$ we have $\alpha \notin H''$, $H \subset H''$, a contradiction. Therefore, P is isolated. Conversely, let H be an A-prime o-ideal of $G(A)$ such that $P = A - m_A^{-1}(H)$ is isolated. Let $a \in P - \bigcup P'$ (P' a prime ideal of A, $P' \subset P$). Then $\alpha = w_A(a) \notin H$. For $H' \in \mathfrak{O}(G(A))$ such that $H \subset H'$, $S = m_A^{-1}(H')$, we have $S = \bigcap (A - P_i)$, where $P_i = \{P : P \text{ a prime ideal of } A, P \cap S = \emptyset\}$. Since $A - P \subset A - P_i$ for every i, we obtain $a \notin P_i$ for every i and $a \in S$. Thus, $\alpha \in H'$ and H is a value of α. The rest is clear.

2. Topological groups of divisibility. In [11] we introduced the notion of a topological group of divisibility in the following way. Let A be an integral domain with the quotient field K and let $\mathcal{T}$ be a topology on K such that $(K, \mathcal{T})$ is a topological field. Then the factor topological group $K^*/U(A)$ of a topological group $(K^*, \mathcal{T} \mid K^*)$ is called a *topological group of divisibility* of A and in this case we write $G(A) = (K, \mathcal{T}, A)$. In this section we show a "topological" version of [13]; Theorem 2.1, for topological groups of divisibility. We use the following notation. The symbol $\mathfrak{S}_0(A)$ $(\mathfrak{S}_c(A))$ denotes the set of elements of $\mathfrak{S}(A)$ that are open (closed) in a topological group $(K^*, \mathcal{T} \mid K^*)$ and the symbol $\mathfrak{D}_o(G(A))$ $(\mathfrak{D}_c(G(A)))$ denotes the set of elements of $\mathfrak{D}(G(A))$ that are open (closed) in $G(A) = (K, \mathcal{T}, A)$.

The proof of the following lemma is due to B. Šmarda.

Lemma 2.1. *Let G be a topological partially ordered group and let H be a directed subgroup of G such that H_+ is closed in G. Then H is closed in G.*

Proof. Let $\bar{H}$ be a closure of H in G and let $g \in \bar{H}$. Then for any neighbourhood $\mathcal{U}$ of zero in G there exists a neighbourhood $\mathcal{V}$ of zero such that $-\mathcal{V} \subseteq \mathcal{U}$. Since $(g + \mathcal{V}) \cap H \neq \emptyset$, there exists $v \in \mathcal{V}$ such that $g + v = h \in H$. Since H is directed, we may find elements $h_1, h_2 \in H_+$ such that $h = h_1 - h_2$. Thus, $-h_1 + g + v = -h_2$ and $-v - g + h_1 = h_2 \in H_+$, $-v + (-g + h_1) \in (\mathcal{U} + (-g + h_1)) \cap H_+$. Hence $-g + h_1 \in \bar{H}_+ = H_+$, where $\bar{H}_+$ is the closure of H_+ in G. Therefore, $-g \in H_+ - h_1 \subseteq H$ and H is closed in G.

The proof of the following lemma is straightforward and will be omitted.

Lemma 2.2. *Let A, B be rings with the same quotient field K and for a subgroup H of $G(A)$ let there exist a group isomorphism ϱ such that $w_B = \varrho \cdot \varphi \cdot w_A$, where φ is the canonical map of $G(A)$ onto $G(A)/H$. Let $G(A) = (K, \mathcal{T}, A)$, $G(B) = (K, \mathcal{T}, B)$. Then $G(B)$ is homeomorphic with the factor topological group $G(A)/H$.*

Proposition 2.3. *Let $G(A) = (K, \mathcal{T}, A)$ and let $S \in \mathfrak{S}(A)$. Then $G(A_S) = (K, \mathcal{T}, A_S)$ is homeomorphic with the factor topological group $G(A)/\boldsymbol{m}_A(S)$.*

The proof follows directly by Lemma 2.2.

In the next theorem we set $\boldsymbol{m}_0 = \boldsymbol{m}_A \mid \mathfrak{S}_0(A)$, $\boldsymbol{m}_c = \boldsymbol{m}_A \mid \mathfrak{S}_c(A)$.

Theorem 2.4. *Let $G(A) = (K, \mathcal{T}, A)$. Then $\boldsymbol{m}_0(\boldsymbol{m}_c)$ is a bijection between $\mathfrak{S}_0(A)$ $(\mathfrak{S}_c(A))$ and $\mathfrak{D}_o(G(A))$ $(\mathfrak{D}_c(G(A)))$ if and only if A^* is open (closed) in K^*. If A^* is open in K^*, then $\mathfrak{S}_o(A) = \mathfrak{S}(A)$, $\mathfrak{D}_o(G(A)) = \mathfrak{D}(G(A))$.*

Proof. Let A^* be open in K^*. Then $U(A) = A^* \cap (A^*)^{-1}$ (where $(A^*)^{-1} = \{x^{-1} : x \in A^*\}$) is open in K^* and $G(A) = (K, \mathcal{T}, A)$ is a discrete space, $\mathfrak{D}_o(G(A)) = \mathfrak{D}(G(A))$. Let $S \in \mathfrak{S}(A)$, then since $\boldsymbol{m}_A(S)$ is open in $G(A)$ and w_A is continuous, we obtain that $S = w_A^{-1}(\boldsymbol{m}_A(S)) \cap A^*$ is open in A^*; hence $S \in \mathfrak{S}_o(A)$. Thus, $\mathfrak{S}_o(A) = \mathfrak{S}(A)$ and $\boldsymbol{m}_o = \boldsymbol{m}_A$. Conversely, let $\boldsymbol{m}_o$ be a bijection, then $\boldsymbol{m}_A(A^*) = G(A) \in \mathfrak{D}_o(G(A))$ and it follows that A^* is open in K^*.

Let A^* be closed in K^*, $S \in \mathfrak{S}_c(A)$, $H = \boldsymbol{m}_A(S)$, and let $\alpha = w_A(x) \in \overline{H}_+$ (the closure of H_+ in $G(A)$). Let $\mathscr{U}$ be a neighbourhood of x in K^*, then there exists $z \in K^*$ such that $w_A(z) \in w_A(\mathscr{U}) \cap H_+$ and for some $s \in S$, $a \in \mathscr{U}$, $i, j \in U(A)$ we have $z = sj = ai$. Since S is a saturated multiplicative system, we obtain $zi^{-1} \in \mathscr{U} \cap S$ and $x \in \bar{S}$, the closure of S in K^*. Since S is closed, we have $\alpha \in H_+$ and H_+ is closed in $G(A)$. By Lemma 2.1, $H \in \mathfrak{O}_c(G(A))$. Further, let $H \in \mathfrak{O}_c(G(A))$, $S = \boldsymbol{m}_A^{-1}(H)$. Since $w_A \mid A : : A^* \to G(A)_+$ is continuous, $S = w_A^{-1}(H_+)$ is closed in A^* and S is closed in K^*. Therefore, $\boldsymbol{m}_c$ is the required bijection. Conversely, if $\boldsymbol{m}_c$ is a bijection, then the fact that $\boldsymbol{m}_A(A^*) = G(A) \in \mathfrak{O}_c(G(A))$ implies that A^* is closed in K^*.

Corollary 2.5. *For $S \in \mathfrak{S}_c(A)$, $U(A_S)$ is closed in K^*. If A^* and $U(A_S)$ are closed in K^*, then $S \in \mathfrak{S}_c(A)$ and $\mathfrak{S}_o(A) \subseteq \mathfrak{S}_c(A)$.*

Proof. Let $S \in \mathfrak{S}_c(A)$. From the proof of 2.4 it follows that $\boldsymbol{m}_A(S) \in \mathfrak{O}_c(G(A))$. By Prop. 2.3, $G(A)/\boldsymbol{m}_A(S)$ is homeomorphic with $K^*/U(A_S)$. Since $G(A)/\boldsymbol{m}_A(S)$ is a T_2-space, $K^*/U(A_S)$ is a T_2-space and $U(A_S)$ is closed in K^*. The rest is clear.

It is well known that every valuation w on the field K with a value group G_w defines a field topology $\mathscr{T}_w$ on K with the sets $\mathscr{U}_{w,\alpha} = \{x \in K^* : w(x) > \alpha\} \cup \{0\}$, $\alpha \in G_w^+$, as a base of the neighbourhoods of zero in K. On the other hand, Matlis [7] introduced the notion of an *A-topology* $\mathscr{T}_A$ in a quotient field K of A, where the set of ideals $a \,.\, A$, $a \in A^*$, is a base for the open neighbourhoods of zero in K. Then $(K, \mathscr{T}_A)$ is a topological ring. It is easy to see that the topology $\mathscr{T}_A$ may be defined by using semi-valuations. In fact, for $\alpha \in G(A)_+$ we set $\mathscr{U}_{w,\alpha} = \{x \in K^* : w(x) > \alpha\} \cup \cup \{0\}$, $\mathscr{U}'_{w,\alpha} = \{x \in K^* : w(x) \geqq \alpha\} \cup \{0\}$, $w = w_A$. If $A \neq K$, for every $\alpha \in G(A)_+$ there exists $\beta \in G(A)$, $\beta > \alpha$, and $\mathscr{U}'_{w,\beta} \subseteq \mathscr{U}_{w,\alpha} \subseteq \mathscr{U}'_{w,\alpha}$. For every $\alpha, \beta \in G(A)_+$ there exists $\gamma > \alpha, \beta$ and $\mathscr{U}_{w,\gamma} \subseteq \mathscr{U}_{w,\alpha} \cap \mathscr{U}_{w,\beta}$ and $a \,.\, A = \mathscr{U}'_{w,w(a)}$ for every $a \in A^*$. Hence, the topology $\mathscr{T}_w$ with the base $\{\mathscr{U}_{w,\alpha} : \alpha \in G(A)_+\}$ equals the topology with the base $\{\mathscr{U}'_{w,\alpha} : \alpha \in G(A)_+\}$ as well as the A-topology on K.

The idea of the following proposition is the same as that of [1]; Ch. 6, § 5, Prop. 1.

Proposition 2.6. *Let A be a quasi-local domain. Then $(K, \mathscr{T}_A)$ is a topological field and $G(A) = (K, \mathscr{T}_A, A)$ is a discrete space.*

Proof. Let $x, y \in K^*$, $\alpha \in G(A)_+$, be such that $w(x - y) > \alpha + 2\,w(y)$, $w(y)$, where $w = w_A$. Then $w(x^{-1} - y^{-1}) > \alpha$. In fact, since $w(x - y) > w(y)$ and A is quasi-local, it follows by [16] that $w(y) = w(x)$. Since $x^{-1} - y^{-1} = x^{-1}(y - x)\,y^{-1}$, we obtain $w(x^{-1} - y^{-1}) = w(x - y) - w(x) - w(y) = w(x - y) - 2\,w(y) > \alpha$. Let $x_0 \in K^*$, $\beta > \alpha + 2\,w(x_0)$, $w(x_0)$, 0, and let $y \in (x_0 + U_{w,\beta})^{-1}$. Then for some $x \in x_0 + \mathscr{U}_{w,\beta}$, $y = x^{-1}$ holds and $w(x - x_0) > \beta > \alpha + 2\,w(x_0), w(x_0)$ and $w(x^{-1} - x_0^{-1}) > \alpha$. Thus, $(x_0 + \mathscr{U}_{w,\beta})^{-1} \subseteq x_0^{-1} + \mathscr{U}_{w,\alpha}$ and $(K, \mathscr{T}_A)$ is a topological field. Since $w^{-1}(\{0\}) \supseteq 1 + \mathscr{U}_{w,0}$, $G(A)$ is a discrete space.

It should be observed that, in general, $(K, \mathscr{T}_A)$ is not a topological field. In fact, for

$A = \mathbf{Z}$, $K = \mathbf{Q}$, the fact that there exists no $y \in A$ such that $(2/3 + y \,.\, A)^{-1} \subseteq$ $\subseteq 3/2 + 2 \,.\, A$, implies that $(K, \mathcal{T}_A)$ is not a topological field.

Proposition 2.7. *Let $\{w_i : i \in I\}$ be the family of valuations on the field K with value groups G_i that are nonnegative on A and let $A = \bigcap R_i \,(i \in I)$, where $R_i = w_i^{-1}(G_i^+) \cup \{0\}$, $\mathcal{T} = \sup\{\mathcal{T}_{w_i} : i \in I\}$. Let $G(A) = (K, \mathcal{T}, A)$. Then $\mathfrak{D}(G(A)) = \mathfrak{D}_c(G(A))$.*

Proof. Let $H \in \mathfrak{D}(G(A))$, $S = w_A^{-1}(H_+)$. Then $S = \bigcap(A - P)$ (P a prime ideal in A, $P \cap S = \emptyset$). For every such P we may find $w_P \in \{w_i : i \in I\}$ such that $P = \{x \in A : w_P(x) > 0\}$ (see [3]; Theorem 16.5). Since $A^* \cap (x + \mathcal{U}_{w_P,0}) \subseteq P$ for every $x \in P$, it follows that P is open in A^* and S is closed in A^*. Since $A^* = \bigcap R_i^*$ is closed in $(K^*, \mathcal{T} \mid K^*)$, we obtain $S \in \mathfrak{S}_c(A)$ and by Theorem 2.4, H is closed in $G(A)$.

Let A be a ring, $\{R_i : i \in I\}$ a set of quasi-local rings in the quotient field K of A such that $A = \bigcap R_i$. Let $w_i = w_{R_i}$ and let $\sigma : G(A) \to \Pi G(R_i)$ $(i \in I)$ be the canonical map.

Proposition 2.8. *$G(A) = (K, \sup\{\mathcal{T}_{w_i} : i \in I\}, A)$ is homeomorphic with $\sigma(G(A))$, where $\sigma(G(A))$ inherits its topology from the product of discrete topologies on $\Pi G(R_i)$ if and only if for every open neighbourhood $\mathcal{U}$ of 1 in K^* there exist $i_1, \ldots, i_n \in I$ such that $U(R_{i_1}) \cap \ldots \cap U(R_{i_n}) \subseteq \mathcal{U} \,.\, U(A)$.*

The proof directly follows from the fact that for every open neighbourhood $\mathcal{U}$ of 1 in K^*, $\mathcal{U} \,.\, U(A)$ is an open neighbourhood of 1 in K^* and $w_A^{-1}(w_A(\mathcal{U} \,.\, U(A)) = \mathcal{U} \,.\, U(A)$.

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