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# A SINGULAR SPECTRAL IDENTITY AND INEQUALITY INVOLVING THE DIRICHLET INTEGRAL OF AN ORDINARY DIFFERENTIAL EXPRESSION

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## 1. INTRODUCTION

In this paper we establish an identity of the form

$$(1,1)\qquad \int_a^b \{p|f'|^2 + q|f|^2\} - |f(a)|^2 \cot\alpha = \int_{-\infty}^{\infty} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t)\ (f \in \mathscr{D}(\tau_\alpha))$$

associated in the singulaı case with the symmetric second-order differential expression $M$ defined by, for suitable functions $f$,

$$M[f] = w^{-1}(-(pf')' + qf) \text{ on } [a, b) \quad (' \equiv \mathrm{d}/\mathrm{d}x)\,,$$

wheıe $p$, $q$ and $w$ are given real-valued coefficients on the interval $[a, b)$, closed at $a$ but open at $b \leqq \infty$, of the real line $\mathbb{R}$. The function $\sigma_\alpha$ is a spectral distribution function associated with a self-adjoint operator $T_\alpha$ generated by the differential expression $M$ in the weighted Hilbert function-space $L^2_w[a, b)$, the elements in the domain of $T_\alpha$ being required to satisfy the real boundary condition

$$f(a)\cos\alpha + (pf')(a)\sin\alpha = 0$$

for some $\alpha \in [0, \pi)$. The function $F$ in $(1, 1)$ is the unitary transform of $f$ in the space $\mathscr{L}^2_\alpha$ (see Section 2). The expression on the left in $(1, 1)$ is a quadratic form that we shall denote by $\tau_\alpha$; its domain, $\mathscr{D}(\tau_\alpha)$, is a linear manifold in $L^2_w[a, b)$ such that the integrals in $(1, 1)$ are absolutely convergent. The integral on the left of $(1, 1)$ is called the *Dirichlet integral* of the differential expression $M$.

Our proof of this identity makes use of the theory of closed semibounded quadratic forms and their related self-adjoint operators, to be found in Friedrichs [9], [10] and Kato [14], the theory being due to the former.

We note here that the inequality

$$(1,2) \qquad \int_a^\infty \{|f'|^2 + q|f|^2\} - |f(a)|^2 \cot\alpha \geqq \int_{-\infty}^{\infty} t|F(t)|^2 \, d\sigma_\alpha(t),$$

where the integral on the left need only exist conditionally, $b = \infty$ and $p(x) = w(x) = 1$ $(x \in [a, \infty))$, is established when the spectrum of $T_\alpha$ is bounded below by Sears and Wray [19]. Their result is not contained in the results of this paper, and their methods are quite different from those employed below. An inequality of this kind is also established in the paper [18] of Putnam under more restrictive conditions on $q$ and $f$.

From the identity (1, 1) we shall deduce the inequality

$$(1,3) \qquad \int_a^b \{p|f'|^2 + q|f|^2\} \geqq |f(a)|^2 \cot\alpha + \mu_\alpha \int_a^b w|f|^2 \quad (f \in \mathscr{D}(\tau_\alpha)),$$

which in the case $\alpha = \pi/2$ (when the cotangent term vanishes) has been the subject of much study over the last decade; see the papers [1] and [2] of Amos and Everitt, and [3] and [4] of Bradley and Everitt. The constant $\mu_\alpha$ is the infimum of the spectrum of the operator $T_\alpha$. The reasons for studying (1, 3) are described in [3] and [2].

We outline the contents of the paper. In the next section the conditions required on the coefficients $p$, $q$ and $w$, and the statements of the main results to be proved, are to be found. In Section 3 we show that the quadratic form $\tau_\alpha$ is closed and bounded below. Section 4 contains details of the relationship between $\tau_\alpha$ and $T_\alpha$, and we there call upon a representation theorem for quadratic forms and self-adjoint operators in Kato [14]. Sections 5, 6 and 7 are devoted to the proofs of the results stated in Section 2.

## 2. STATEMENT OF RESULTS

We work with the interval $[a, b)$, with $-\infty < a < b \leqq \infty$. As usual, $L[a, b)$ and $L^2[a, b)$ denote the classical Lebesgue, complex integration spaces, AC absolute continuity and 'loc' a property to be satisfied on all compact sub-intervals of $[a, b)$. A symbol such as '$(f \in \Delta)$' is to be read as 'for all $f$ in the set $\Delta$'.

Throughout, the coefficients $p$, $q$ and $w$ are real-valued, Lebesgue measurable on $[a, b)$ and satisfy the basic conditions:

(2,1) (i) $p(x) > 0$ (almost all $x \in [a, b)$) and $p^{-1} \in L_{\text{loc}}[a, b)$;

(ii) $q \in L_{\text{loc}}[a, b)$;

(iii) $w(x) > 0$ (almost all $x \in [a, b)$) and $w \in L_{\text{loc}}[a, b)$.

The space $L^2_w[a, b)$, where $w$ satisfies (2,1) (iii), is the Hilbert space of functions $f : [a, b) \to \mathbb{C}$ (the complex field) that are Lebesgue measurable on $[a, b)$ and satisfy

$$\int_a^b w|f|^2 < \infty,$$

with the usual inner-product.

The conditions (2,1) allow us to consider the ordinary symmetric differential expression $M$ given by

$$M[f] = w^{-1}(-(pf')' + qf) \quad \text{on} \quad [a, b)\,. \tag{2,2}$$

Under the above basic conditions the linear differential equation

$$M[f] = \lambda f \quad \text{on} \quad [a, b)\,, \tag{2,3}$$

where the parameter $\lambda$ is complex, is regular at all points of $[a, b)$, i.e. if $c \in [a, b)$ then the initial value problem defined by (2,3) and the conditions

$$f(c) = \zeta\,, \quad (pf')(c) = \eta$$

can be solved for arbitrary complex numbers $\zeta$ and $\eta$ see [17, Section 16.1]. Although [17] has $w(x) = 1$ ($x \in [a, b)$), the results also hold in the case when a positive weight function $w$ satisfying (2,1) (iii) is introduced, provided that the space $L^2[a, b)$ is replaced by $L^2_w[a, b)$.

In this paper we are concerned with problems which are *singular at b* in the sense of [17, Section 15.1], i.e.

(2,4) either (i) $b = \infty$
or (ii) $b < \infty$ and at least one of $p^{-1}$, $q$ and $w$ is not in $L[a, b)$.

We now define the complex linear manifolds $\Delta$ and $\Delta'$ of $L^2_w[a, b)$ by

$$\Delta = \{f \in L^2_w[a, b) \mid f,\, pf' \in \mathrm{AC}_{\mathrm{loc}}[a, b) \text{ and } M[f] \in L^2_w[a, b)\}\,,$$

and

$$\Delta' = \{f \in L^2_w[a, b) \mid f \in \mathrm{AC}_{\mathrm{loc}}[a, b) \text{ and } p^{1/2}f',\ |q|^{1/2}f \in L^2[a, b)\}\,.$$

If $f, g \in \Delta$ then it follows from Green's formula that

$$\lim_{x \to b-} \{f(x)\,(p\bar{g}')(x) - (pf')(x)\,\bar{g}(x)\}$$

exists and is finite, the bar denoting complex conjugation; $M$ is said to be in the *limit-point* (LP) case at $b$ if this limit is zero for all $f, g \in \Delta$. Additionally, $M$ is said to be in the *strong limit-point* (SLP) case at $b$ if

$$\lim_{x \to b-} (pf')(x)\,\bar{g}(x) = 0 \quad (f, g \in \Delta)\,. \tag{2,5}$$

The differential expression $M$ is said to have the *Dirichlet* (D) *property* at $b$ if $\Delta \subset \Delta'$, i.e.

$$p^{1/2}f',\ |q|^{1/2}f \in L^2[a, b) \quad (f \in \Delta)\,. \tag{2,6}$$

(Note that we have $L^2$ here and not $L^2_w$.) The same expression is said to have the *conditional Dirichlet* (CD) *property at b* if

$$p^{1/2}f' \in L^2[a, b) \text{ and } \lim_{x \to b-} \int_a^x qf\bar{g} \text{ exists and is finite} \quad (f, g \in \Delta)\,. \tag{2,7}$$

For $M$ to be SLP, D or CD at $b$, the coefficients $p$, $q$ and $w$ must satisfy more than the basic conditions (2,1). Results of this kind may be found in Amos and Everitt [2], Evans [5], Everitt [7], Everitt, Hinton, Wong [6] and Kalf [13]. The existence of these results makes it reasonable to assume (2,5), (2,6) or (2,7) as conditions to be satisfied and so indirectly impose conditions on the coefficients $p$, $q$ and $w$.

The relationships between the LP, SLP, CD and D properties at $b$ are examined by Everitt [8], Kalf [13] and Kwong [15], [16] in the case $w(x) = 1$ $(x \in [a, b))$. In particular, we note that D at $b$ implies SLP at $b$ with a unit weight function $w$ (we assume hereafter that we have the singular case at $b$, i.e. condition (2,4) is to be satisfied). However, in the case of a general weight function this result is no longer true and the two properties (2,5) and (2,6) have to be proved separately. Clearly SLP at $b$ implies LP at $b$, and D at $b$ implies CD at $b$, for any $w$.

We now introduce the symmetric differential operator $T_\alpha$, which is defined by

$$\mathscr{D}(T_\alpha) = \{f \in \Delta \mid f(a) \cos \alpha + (pf')(a) \sin \alpha = 0\}, \tag{2,8}$$

and

$$T_\alpha f = M[f] \quad (f \in \mathscr{D}(T_\alpha)), \tag{2,9}$$

where $\alpha$ is a real number in the interval $[0, \pi)$. It is known that $T_\alpha$ is self-adjoint in $L^2_w[a, b)$ if and only if $M$ is LP at $b$; see [17, Section 18.3]. Whenever $T_\alpha$ occurs in this paper we shall assume that $M$ is LP at $b$.

When $T_\alpha$ is self-adjoint we say that $T_\alpha$ is *bounded below* in $L^2_w[a, b)$ if there is a real number $A$ such that

$$(T_\alpha f, f)_w \geqq A(f, f)_w \quad (f \in \mathscr{D}(T_\alpha)),$$

where $(\cdot, \cdot)_w$ represents the usual inner-product in $L^2_w[a, b)$. In this case we define $\mu_\alpha$ by

$$\mu_\alpha = \inf \{(T_\alpha f, f)_w \mid f \in \mathscr{D}(T_\alpha) \quad \text{and} \quad (f, f)_w = 1\}.$$

Then $\mu_\alpha$ is the infimum of the spectrum of $T_\alpha$; see Kato [14, p. 278].

The spectral distribution function $\sigma_\alpha$ mentioned in Section 1 may now be introduced. Let $\phi(\cdot, \lambda)$ be the (unique) solution of the differential equation $M[f] = \lambda f$ on $[a, b)$ satisfying the conditions

$$\phi(a, \lambda) = \sin \alpha, \quad (p\phi')(a, \lambda) = -\cos \alpha$$

for all complex $\lambda$, with $\alpha$ as in (2,8). It is known, see [17, Ch. VI], that there is a real non-decreasing function $\sigma_\alpha$ defined on the real line $\mathbb{R}$ which induces a unitary transformation from $L^2_w[a, b)$ onto the Hilbert space $\mathscr{L}^2_\alpha (\equiv \mathscr{L}^2_\alpha(\mathbb{R}))$, where (the integral being a Lebesgue-Stieltjes integral)

$$\mathscr{L}^2_\alpha = \left\{F : \mathbb{R} \to \mathbb{C} \mid F \text{ is Lebesgue measurable on } \mathbb{R} \text{ and } \int_{-\infty}^{\infty} |F(t)|^2 \, d\sigma_\alpha(t) < \infty\right\},$$

with the following properties. (Give $\mathscr{L}^2_\alpha$ the usual inner-product.)

If $f \in L^2_w[a, b)$ and $F \in \mathscr{L}^2_\alpha$ is its unitary transform then

$$F(t) = \lim_{s \to b-} \int_a^s f(x)\, \phi(x, t)\, w(x)\, \mathrm{d}x \tag{2,10}$$

in the norm of $\mathscr{L}^2_\alpha$ and, inversely,

$$f(x) = \lim_{\substack{s \to \infty \\ u \to -\infty}} \int_u^s F(t)\, \phi(x, t)\, \mathrm{d}\sigma_\alpha(t) \tag{2,11}$$

in the norm of $L^2_w[a, b)$; moreover,

$$\int_a^b w|f|^2 = \int_{-\infty}^{\infty} |F(t)|^2\, \mathrm{d}\sigma_\alpha(t)\,.$$

It will often be convenient to use lower-case letters to represent functions in $L^2_w[a, b)$ and the corresponding upper-case letters to represent their unitary transforms in $\mathscr{L}^2_\alpha$. When $M$ is LP at $b$ the spectral function $\sigma_\alpha$ is essentially unique for each $\alpha$, the image of $\mathscr{D}(T_\alpha)$ in $\mathscr{L}^2_\alpha$ under this unitary transformation is the manifold

$$\left\{F \in \mathscr{L}^2_\alpha \,\middle|\, \int_{-\infty}^{\infty} |t\, F(t)|^2\, \mathrm{d}\sigma_\alpha(t) < \infty\right\},$$

and the unitary transform of $T_\alpha f$ ($f \in \mathscr{D}(T_\alpha)$) in the sense of (2,10) is $E$, where $E(t) = t\, F(t)$ ($\sigma_\alpha$-almost all $t \in \mathbb{R}$) and $F$ is the transform of $f$. Furthermore, the spectrum of $T_\alpha$ in the LP case at $b$ is the complement, with respect to $\mathbb{R}$, of the union of all open sub-intervals of $\mathbb{R}$ in which $\sigma_\alpha$ is constant. Thus, if the spectrum of $T_\alpha$ has infimum $\mu_\alpha > -\infty$ then $\sigma_\alpha$ is constant in the open interval $(-\infty, \mu_\alpha)$.

These results concerning the properties of $\sigma_\alpha$ are all to be found essentially in [17, Ch. VI]; the introduction of the weight function $w$ does not entail additional difficulties.

Now let the linear manifold $\mathscr{D}(\tau_\alpha)$ in $L^2_w[a, b)$ be given by

$$\mathscr{D}(\tau_\alpha) = \{f : [a, b) \to \mathbb{C} \mid f \in \Delta' \quad \text{and} \quad f(a) = 0 \quad \text{if} \quad \alpha = 0\}\,, \tag{2,12}$$

i.e. $f \in \mathscr{D}(\tau_\alpha)$ if and only if $f \in L^2_w[a, b)$, $f \in \mathrm{AC}_{\mathrm{loc}}\,[a, b)$, $p^{1/2}f'$, $|q|^{1/2} f \in L^2[a, b)$ and $f(a) = 0$ if $\alpha = 0$, where $\alpha$ is as in (2,8). We define the sesquilinear form $\tau_\alpha : \mathscr{D}(\tau_\alpha) \times \mathscr{D}(\tau_\alpha) \to \mathbb{C}$ by

$$\tau_\alpha[f, g] = \int_a^b \{pf'\bar{g}' + qf\bar{g}\} - f(a)\,\bar{g}(a) \cot \alpha \quad (f, g \in \mathscr{D}(\tau_\alpha))$$

if $\alpha \neq 0$, the cotangent term being omitted if $\alpha = 0$. We shall also write $\tau_\alpha[f, f] = \tau_\alpha[f]$ ($f \in \mathscr{D}(\tau_\alpha)$), i.e.

$$\tau_\alpha[f] = \int_a^b \{p|f'|^2 + q|f|^2\} - |f(a)|^2 \cot \alpha \quad (f \in \mathscr{D}(\tau_\alpha))\,, \tag{2,13}$$

this being the quadratic form associated with $\tau_\alpha[\cdot, \cdot]$. It should be noted that the integrals in the definitions above are absolutely convergent, from the definition of $\mathscr{D}(\tau_\alpha)$.

Now we state the principal result of this paper as

**Theorem 1.** *Let the differential expression $M$ on $[a, b)$ be defined by* (2,2) *and let the coefficients $p$, $q$ and $w$ satisfy the basic conditions* (2,1); *let the condition* (2,4) *hold so that $M$ is singular at $b$. Let the linear manifold $\mathscr{D}(\tau_\alpha)$ of $L^2_w[a, b)$ be defined by* (2,12).

*Suppose additionally that the coefficients $p$, $q$ and $w$ are so chosen that*

(2,14) (i) *there is a non-negative constant $\gamma$ such that*

$$q(x) \geqq -\gamma\, w(x) \quad (\textit{almost all } x \in [a, b));$$

(ii) *$M$ satisfies the Dirichlet condition at $b$; and*

(iii) *the following condition is satisfied*

$$\lim_{x \to b-} (pf')(x)\, \bar{g}(x) = 0 \quad (f \in \Delta,\ g \in \Delta').$$

*Then the following identity holds*

$$(2,15)\quad \int_a^b \{p|f'|^2 + q|f|^2\} - |f(a)|^2 \cot\alpha = \int_{-\infty}^{\infty} t|F(t)|^2\, \mathrm{d}\sigma_\alpha(t) \quad (f \in \mathscr{D}(\tau_\alpha)),$$

*where $\sigma_\alpha$ is the above distribution function and $F$ is the unitary transform of $f$ in the sense of* (2,10); *the cotangent term is absent in the case $\alpha = 0$. The integrals in* (2,15) *are absolutely convergent.*

*In addition, the operator $T_\alpha$ defined by* (2,9) *is self-adjoint and bounded below in $L^2_w[a, b)$.*

Proof. This is given in Section 5.

**Remark 1.** Note that $\mathscr{D}(\tau_\alpha) = \Delta'$ if $\alpha \neq 0$.

**Remark 2.** If conditions (2,14) (ii) and (iii) hold then $M$ is in particular SLP at $b$, hence LP at $b$, and so $T_\alpha$ is self-adjoint in $L^2_w[a, b)$.

**Remark 3.** The condition (2,14) (iii) is quite reasonable and is discussed in Section 4. Conditions (2,14) (ii) and (iii) both hold, for example, if (2,14) (i) holds and $p^{-1} \in L[a, b)$, $w \notin L[a, b)$. This follows from the discussion in Section 4 since by [2, Theorem 1] $M$ is then D at $b$.

Another set of conditions sufficient for both of (2,14) (ii) and (iii) to hold is: (2,14) (i) and in addition

$$p^{-1} \notin L[a, b) \quad \text{and} \quad \int_a^b w(x) \left(\int_a^x p^{-1}(t)\, \mathrm{d}t\right)^\eta \mathrm{d}x = \infty,$$

for some constant $\eta \in (0, 2]$. Again we call on the discussion in Section 4 since $M$ is D at $b$ by [13, p. 199].

See also [1, Lemma 2].

**Remark 4.** That the condition $f(a) = 0$ if $\alpha = 0$ (see (2,12)) is necessary to the identity (2,15) follows from the example in [19, p. 206] in which $\alpha = 0$, $f(a) = 1$, the integral on the left of (2,15) converges and the integral on the right has the value $\infty$.

**Remark 5.** Theorem 1 contains the similar, but much less general, result of Putnam [18, p. 785]. In [18] the result is stated as an inequality which is now seen to be an equality.

**Remark 6.** Hinton [12, Theorem 2(i)] establishes a version of (2,15) for differential expressions of order $2n$. For the case $n = 1$, and in our notation, [12] requires in particular that $p$ have a continuous derivative, $w$ be continuous and that $q$ be bounded on compact sub-intervals of $[a, b)$, from which it is clear that Theorem 1 is not contained in the result in [12].

We now consider an inequality that follows from Theorem 1.

**Corollary.** *Let all the conditions of Theorem 1 hold (in particular, let $M$ be singular at $b$). Then if $\mu_\alpha \in \mathbb{R}$ is the infimum of the spectrum of $T_\alpha$ we have*

$$\int_a^b \{p|f'|^2 + q|f|^2\} \geqq |f(a)|^2 \cot \alpha + \mu_\alpha \int_a^b w|f|^2 \quad (f \in \mathscr{D}(\tau_\alpha)), \tag{2,16}$$

*there being no cotangent term in the case $\alpha = 0$. If $\mu_\alpha$ is an eigenvalue of $T_\alpha$ then there is equality in (2,16) if and only if $f$ is a corresponding eigenfunction of $T_\alpha$. If $\mu_\alpha$ is in the continuous spectrum of $T_\alpha$ then there is equality in (2,16) if and only if $f$ is null on $[a, b)$; however the inequality is then best possible in the sense that if $\varepsilon > 0$ is chosen arbitrarily there is a function $f \in \mathscr{D}(T_\alpha)$ such that $(f, f)_w = 1$ and*

$$\int_a^b \{p|f'|^2 + q|f|^2\} < |f(a)|^2 \cot \alpha + \mu_\alpha + \varepsilon. \tag{2,17}$$

Proof. This is given in Section 6.

**Remark 1.** The results in this Corollary in the case $\alpha = \pi/2$ should be compared with those of Amos and Everitt [1, Theorem 2] and [2, Theorem 4]. An examination of the conditions assumed in these papers shows that in neither case does the theorem contain the above Corollary, while also the Corollary contains neither of the cited theorems. The treatment of the cases of equality in (2,16) and the proof that it is best possible, to be given below, differ from those in [1] and [2] in that we now work in the space $\mathscr{L}_\alpha^2$ rather than $L_w^2[a, b)$ and make no use of the methods of the calculus of variations.

**Remark 2.** If one is concerned with the minimisation of the Dirichlet integral on the left of the inequality in (2,16), with $f \in \Delta'$, then it is clear from (2,16) that only in the case $\alpha = \pi/2$ do the spectral properties of the self-adjoint operator $T_\alpha$ suffice to determine a best possible inequality of the form

$$\int_a^b \{p|f'|^2 + q|f|^2\} \geqq \varkappa \int_a^b w|f|^2 \quad (f \in \Delta');$$

in this case $\varkappa = \mu_{\pi/2}$ is best possible.

**Remark 3.** The fact that

$$\int_a^b \{p|f'|^2 + q|f|^2\} \geqq \mu_0 \int_a^b w|f|^2$$

holds for all $f \in \Delta'$ with $f(a) = 0$ answers the question raised by Bradley and Everitt [3, p. 309].

To accompany Theorem 1 we have a result which includes a pointwise integral expansion formula for functions in $\mathscr{D}(\tau_\alpha)$.

**Theorem 2.** *Let all the conditions of Theorem* 1 *hold. Then if* $f \in L_w^2[a, b)$ *is such that*

$$\int_{-\infty}^{\infty} t|F(t)|^2 \, d\sigma_\alpha(t) < \infty,$$

*where F is its unitary transform in the sense of* (2,10), *then* $f \in \mathscr{D}(\tau_\alpha)$,

$$f(x) = \int_{-\infty}^{\infty} F(t)\, \phi(x, t)\, d\sigma_\alpha(t) \quad (x \in [a, b)),$$

*with uniform convergence on compact sub-intervals of* $[a, b)$, *and*

$$\lim_{s\to\infty} \int_a^b r|f' - f_s'|^2 = 0,$$

*where*

$$f_s(x) = \int_{-\infty}^{s} F(t)\, \phi(x, t)\, d\sigma_\alpha(t) \quad (x \in [a, b),\ s \in \mathbb{R}).$$

Proof. This is given in Section 7.

**Remark.** As in the case of Theorem 1 this result should be compared with the corresponding part of [12, Theorem 2]. The proof given below is similar to that in [11] but makes use of the closedness of $\tau_\alpha$.

**Corollary.** *Let all the conditions of Theorem* 1 *hold; then a necessary and sufficient condition for* $f \in L_w^2[a, b)$ *to be in the domain* $\mathscr{D}(\tau_\alpha)$ *of the quadratic form* $\tau_\alpha$ *is that the unitary transform F of f satisfy*

$$\int_{-\infty}^{\infty} |t|\, |F(t)|^2 \, d\sigma_\alpha(t) < \infty.$$

Proof. This follows from the results obtained in Theorems 1 and 2 above.

**Remark.** This Corollary should be seen in the light of the fact that, in the case when the lower bound $\mu_\alpha$ is non-negative, the domain $\mathscr{D}(\tau_\alpha)$ of the quadratic form is precisely that of the square-root $T_\alpha^{1/2}$ of the operator $T_\alpha$; see [14, Theorem 2.23, p. 331, and formula (5,16), p. 356], bearing in mind the connection between $T_\alpha$ and $\tau_\alpha$, which is discussed in Section 4 below.

## 3. PROPERTIES OF THE QUADRATIC FORM $\tau_\alpha$

Let the quadratic form $\tau_\alpha$ be as defined in (2,13). In this section we show that under certain conditions $\tau_\alpha$ is bounded below and closed; the terminology is that of Kato [14, Ch. VI, § 1].

We say that $\tau_\alpha$ is *bounded below* if there is a real number $\eta_\alpha$ such that

$$\tau_\alpha[f] \geqq \eta_\alpha(f, f)_w \quad (f \in \mathscr{D}(\tau_\alpha)) . \tag{3,1}$$

The quadratic form is said to be *closed* if for any sequence $(f_n)_{n=1}^\infty \subset \mathscr{D}(\tau_\alpha)$ such that $(f_n)$ converges in the norm of $L_w^2[a, b)$ to some function $f \in L_w^2[a, b)$, as $n \to \infty$, and $\tau_\alpha[f_m - f_n] \to 0$, as $m, n \to \infty$, we have $f \in \mathscr{D}(\tau_\alpha)$ and $\tau_\alpha[f_n - f] \to 0$, as $n \to \infty$.

The following lemma is needed, to cope with the term $|f(a)|^2 \cot \alpha$ in $\tau_\alpha[f]$.

**Lemma 1.** *Let the coefficients p, q and w satisfy the basic conditions* (2,1). *Then given* $\varepsilon > 0$ *there is a constant* $A_\varepsilon > 0$ *such that*

$$|f(a)|^2 \leqq \varepsilon \int_a^b p|f'|^2 + A_\varepsilon \int_a^b w|f|^2 \quad (f \in \Delta') . \tag{3,2}$$

*The constant* $A_\varepsilon$ *does not depend on* $f$.

Proof. Let $f \in \Delta'$. We have

$$f(a) = f(x) - \int_a^x f' \quad (x \in [a, b))$$

because of the local absolute continuity of $f$, whence

$$|f(a)|^2 \leqq 2|f(x)|^2 + 2\left(\int_a^x |f'|\right)^2 \leqq \tag{3,3}$$

$$\leqq 2|f(x)|^2 + 2\int_a^x p^{-1} \int_a^x p|f'|^2 \leqq 2|f(x)|^2 + 2\int_a^x p^{-1} \int_a^b p|f'|^2,$$

by the Cauchy-Schwarz inequality and the positivity of $p$. Now choose $\varepsilon > 0$ and then any $k > a$ such that

$$\int_a^k p^{-1} < \varepsilon/2 . \tag{3,4}$$

We now multiply the inequality (3,3) by $w(x)$ and integrate over $[a, k]$. This gives

$$|f(a)|^2 \int_a^k w(x)\,dx \leqq 2 \int_a^k w(x)\,|f(x)|^2\,dx + \int_a^k w(x)\,dx \int_a^x p^{-1} \int_a^b p|f'|^2 \leqq$$

$$\leqq 2 \int_a^b w(x)\,|f(x)|^2\,dx + \int_a^k w(x)\,dx \int_a^x p^{-1} \int_a^b p|f'|^2\,,$$

and the inequality in (3,2) follows on division by the positive integral

$$\int_a^k w(x)\,dx$$

since, by (3,4),

$$\int_a^k w(x)\,dx \int_a^x p^{-1} \leqq \frac{\varepsilon}{2} \int_a^k w(x)\,dx\,.$$

**Lemma 2.** *Let the coefficients $p$, $q$ and $w$ satisfy the basic conditions* (2,1) *and suppose that $q$ satisfies condition* (2,14) (i). *Then the quadratic form $\tau_\alpha$ is bounded below and closed.*

Proof. It follows from (3,2) and (2,14) (i) that if we choose $\varepsilon > 0$ then there is a constant $B_\varepsilon > 0$ such that

$$(3,5) \qquad |f(a)|^2 \leqq \varepsilon \int_a^b \{p|f'|^2 + q|f|^2\} + B_\varepsilon \int_a^b w|f|^2 \quad (f \in \Delta')\,,$$

where $B_\varepsilon$ is independent of $f$.

We now obtain the boundedness below of $\tau_\alpha$. If $\cot\alpha > 0$ we obtain from (3,5)

$$\tau_\alpha[f] \geqq (1 - \varepsilon \cot\alpha) \int_a^b \{p|f'|^2 + q|f|^2\} - B_\varepsilon \cot\alpha \int_a^b w|f|^2 \quad (f \in \mathscr{D}(\tau_\alpha))\,,$$

from which we deduce (3,1) with $\eta_\alpha = -\gamma + \cot\alpha(\gamma\varepsilon - B_\varepsilon)$ if we employ (2,14) (i) and take $\varepsilon > 0$ sufficiently small to ensure that $1 - \varepsilon\cot\alpha > 0$. If, on the other hand, either $\alpha = 0$ or $\cot\alpha \leqq 0$ then (3,1) follows immediately from (2,14) (i) with $\eta_\alpha = -\gamma$. This analysis suggests that the lower bound of $\tau_\alpha$ (i.e. the supremum of the set of real numbers $\eta_\alpha$ satisfying (3,1)) becomes very large negative as $\alpha$ approaches 0, with a "discontinuity" at 0 where it cannot be less than $-\gamma$.

We now establish that $\tau_\alpha$ is closed. Let $(f_n)_{n=1}^\infty$ be a sequence of functions in $\mathscr{D}(\tau_\alpha)$ such that

$$(3,6) \qquad \int_a^b w|f_n - f|^2 \to 0 \quad (n \to \infty)\,,$$

for some $f \in L^2_w[a, b)$ and

$$(3,7) \qquad \tau_\alpha[f_m - f_n] \to 0 \quad (m, n \to \infty)\,.$$

We must show that $f \in \mathscr{D}(\tau_\alpha)$ and $\tau_\alpha[f_n - f] \to 0$ $(n \to \infty)$.

With $q_+ = \max(q, 0)$, $q_- = \max(-q, 0)$ we have, from (2,14) (i), $0 \leqq q_-(x) \leqq \leqq \gamma\, w(x)$ $(x \in [a, b))$ and so

$$(3,8) \qquad \int_a^b q_-|f_m - f_n|^2 \to 0 \quad (m, n \to \infty)$$

in view of (3,6). If $\varepsilon > 0$ is chosen arbitrarily, it follows from (3,5) that

$$(1 - \varepsilon \cot \alpha)\,|f_m(a) - f_n(a)|^2 \leqq \varepsilon \tau_\alpha[f_m - f_n] + B_\varepsilon \int_a^b w|f_m - f_n|^2$$

for all positive integers $m$ and $n$. Since once again we may choose $\varepsilon$ sufficiently small to give $1 - \varepsilon \cot \alpha > 0$ we now deduce from (3,6) and (3,7) that

$$(3,9) \qquad |f_m(a) - f_n(a)|^2 \to 0 \quad (m, n \to \infty)\,.$$

Let us write temporarily

$$(3,10) \qquad k = \lim_{n \to \infty} f_n(a)\,,$$

the existence of $k$ being a consequence of (3,9). It now follows from (3,7)–(3,9) that

$$(3,11) \qquad \int_a^b p|f_m' - f_n'|^2 \to 0\,, \quad \int_a^b q_+|f_m - f_n|^2 \to 0 \quad (m, n \to \infty)\,.$$

Hence there is a measurable function $g : [a, b) \to \mathbb{C}$ such that

$$(3,12) \qquad \int_a^b p|g|^2 < \infty \quad \text{and} \quad \int_a^b p|f_n' - g|^2 \to 0 \quad (n \to \infty)\,.$$

If now $x \in [a, b)$ we have

$$\left|\int_a^x f_n' - \int_a^x g\right|^2 \leqq \left(\int_a^x |f_n' - g|\right)^2 \leqq \int_a^x p^{-1} \int_a^x p|f_n' - g|^2$$

for all positive integers $n$, by the Cauchy-Schwarz inequality, and so by (3,12)

$$(3,13) \qquad \lim_{n \to \infty} \int_a^x f_n' = \int_a^x g\,.$$

Using

$$f_n(x) = f_n(a) + \int_a^x f_n'$$

plus (3,10) and (3,13), we see that

$$h(x) = \lim_{n \to \infty} f_n(x)$$

exists for all $x \in [a, b)$ and that the convergence is uniform on compact sub-intervals of $[a, b)$. From (3,6) we then obtain $h(x) = f(x)$ (almost all $x \in [a, b)$) and may therefore redefine $f$ if necessary by $f(x) = h(x)$ $(x \in [a, b))$. It is apparent that

$k = f(a)$ and

$$f \in \mathrm{AC}_{\mathrm{loc}}[a, b), \quad f'(x) = g(x) \quad (\text{almost all } x \in [a, b)), \tag{3,14}$$
$$f(a) = 0 \quad \text{if} \quad \alpha = 0, \quad |f_n(a) - f(a)| \to 0 \quad (n \to \infty),$$

and, from (3,12),

$$\int_a^b p|f'|^2 < \infty \quad \text{and} \quad \int_a^b p|f_n' - f'|^2 \to 0 \quad (n \to \infty). \tag{3,15}$$

Now, as a result of (3,11) there is a measurable function $j : [a, b) \to \mathbb{C}$ such that

$$\int_a^b q_+|j|^2 < \infty \quad \text{and} \quad \int_a^b q_+|f_n - j|^2 \to 0 \quad (n \to \infty). \tag{3,16}$$

Since $f_n \to f$ uniformly on $[a, x]$ $(x \in [a, b))$ as $n \to \infty$, we have

$$\int_a^x q_+|f_n - f|^2 \to 0 \quad (n \to \infty) \tag{3,17}$$

for any fixed $x \in [a, b)$, and so

$$\left\{\left(\int_a^x q_+|f|^2\right)^{\frac{1}{2}} - \left(\int_a^x q_+|j|^2\right)^{\frac{1}{2}}\right\}^2 \leqq \int_a^x q_+|f - j|^2 \leqq 2\int_a^x q_+|f - f_n|^2 + 2\int_a^x q_+|f_n - j|^2$$

for any positive integer $n$. Since the last expression tends to zero as $n \to \infty$, by (3,16) and (3,17), we have

$$\int_a^x q_+|f|^2 = \int_a^x q_+|j|^2 \quad \text{and} \quad \int_a^x q_+|f - j|^2 = 0 \quad (x \in [a, b)),$$

whence

$$\int_a^b q_+|f|^2 < \infty \quad \text{and} \quad \int_a^b q_+|f - j|^2 = 0. \tag{3,18}$$

Hence, from (3,16) and (3,18), we find that

$$\int_a^b q_+|f_n - f|^2 \to 0 \quad (n \to \infty). \tag{3,19}$$

Results corresponding to (3,18) and (3,19) with $q_+$ replaced by $q_-$ hold for the same reasons and thus we obtain

$$\int_a^b |q|\,|f|^2 < \infty \quad \text{and} \quad \int_a^b q|f_n - f|^2 \to 0 \quad (n \to \infty). \tag{3,20}$$

The closedness of $\tau_\alpha$ now follows from (3,14), (3,15) and (3,20).
This completes the proof of Lemma 2.

## 4. THE RELATION BETWEEN $T_\alpha$ AND $\tau_\alpha$

Now we show how the operator $T_\alpha$ and the quadratic form $\tau_\alpha$ are related in the case where $M$ is singular at $b$.

Firstly, if $M$ is CD at $b$ and $f \in \Delta$, $g \in \Delta'$, then an integration by parts gives

$$(4,1) \qquad (M[f], g)_w = -k + (pf')(a)\,\bar{g}(a) + \int_a^b \{pf'\bar{g}' + qf\bar{g}\},$$

where the integral is Cauchy-Lebesgue and

$$k = \lim_{x \to b-} (pf')(x)\,\bar{g}(x),$$

the limit being finite but not necessarily zero.

There are several situations in which $k = 0$ $(f \in \Delta,\ g \in \Delta')$, i.e. condition (2,14) (iii) holds. If $M$ is CD at $b$ and $w \notin L[a, b)$ then the argument of [2, p. 25] gives this result. If $p^{-1} \notin L[a, b)$ and $M$ is CD at $b$ we may call upon the argument of [15, pp. 203–204] (which does not require $b < \infty$ or a unit weight function $w$). As we are considering the singular case these non-integrability conditions are quite reasonable. As a final example, CD plus the condition $p(x) \leqq K\,w(x)$ (almost all $x \in [a, b)$), for some constant $K > 0$, gives $p^{1/2}g \in L^2[a, b)$ $(g \in \Delta')$ and hence $k = 0$ $(f \in \Delta,$ $g \in \Delta')$.

Note that one of our conditions, see (2,14) (ii), is that $M$ is D at $b$; this, of course, implies that $M$ is CD at $b$ and, consequently, the remarks made in the previous paragraph are valid.

As a result of (4,1) and the boundary condition at $a$ satisfied by the functions in $\mathscr{D}(T_\alpha)$ we have

**Lemma 3.** *If $M$ is* D *at $b$ and if condition* (2,14) (iii) *of Theorem* 1 *is satisfied, then $\mathscr{D}(T_\alpha) \subset \mathscr{D}(\tau_\alpha)$ and*

$$(4,2) \qquad (T_\alpha f, g)_w = \tau_\alpha[f, g] \quad (f \in \mathscr{D}(T_\alpha),\ g \in \mathscr{D}(\tau_\alpha)).$$

(*Note that* (2,14) (ii) *and* (iii) *imply that $M$ is* SLP *at $b$ and hence that $T_\alpha$ is self-adjoint.*)

Suppose now that the conditions of Lemma 2 hold, so that $\tau_\alpha$ is closed (and bounded below). Using the terminology of Kato [14, Ch. VI, § 1] we then say that a linear sub-manifold $\mathscr{E}$ of $\mathscr{D}(\tau_\alpha)$ is a *core* of $\tau_\alpha$ if the restriction of $\tau_\alpha$ to $\mathscr{E}$ has closure $\tau_\alpha$, i.e. $\mathscr{D}(\tau_\alpha)$ is the set of all functions $f \in L^2_w[a, b)$ for which there exists a sequence $(f_n)_{n=1}^\infty \subset \mathscr{E}$ such that $f_n \to f$ in the norm of $L^2_w[a, b)$ as $n \to \infty$ and $\tau_\alpha[f_n - f_m] \to 0$ as $m, n \to \infty$; for such an $f$ one has $\tau_\alpha[f_n] \to \tau_\alpha[f]$ as $n \to \infty$.

We now state

**Lemma 4.** *Let all the conditions of Theorem* 1 *hold. Then* (4,2) *holds, the operator $T_\alpha$ is self-adjoint and bounded below with the same lower bound as has $\tau_\alpha$, and $\mathscr{D}(T_\alpha)$ is a core of $\tau_\alpha$.*

Proof. The conditions of Theorem 1 give (4,2) and the self-adjointness of $T_\alpha$ (see Remark 2 after the theorem). By Lemma 2 $\tau_\alpha$ is closed and bounded below, and since $\mathscr{D}(\tau_\alpha)$ is dense in $L^2_w[a, b)$ (this result follows from standard arguments) there is, by [14, Theorems 2.1 and 2.6, pp. 322–323], a self-adjoint operator $S_\alpha : \mathscr{D}(S_\alpha) \subset \subset L^2_w[a, b) \to L^2_w[a, b)$ such that

(4,3) (i) $\mathscr{D}(S_\alpha) \subset \mathscr{D}(\tau_\alpha)$ and $\tau_\alpha[f, g] = (S_\alpha f, g)_w$ $(f \in \mathscr{D}(S_\alpha),\ g \in \mathscr{D}(\tau_\alpha))$,

(ii) $\mathscr{D}(S_\alpha)$ is a core of $\tau_\alpha$,

(iii) if $f \in \mathscr{D}(\tau_\alpha)$, $h \in L^2_w[a, b)$ and $\tau_\alpha[f, g] = (h, g)_w$ holds for all $g$ belonging to a core of $\tau_\alpha$, then $f \in \mathscr{D}(S_\alpha)$ and $S_\alpha f = h$, and

(iv) $S_\alpha$ is bounded below and has the same lower bound as has $\tau_\alpha$.

From (4,2) and (4,3) (iii) we find that $\mathscr{D}(T_\alpha) \subset \mathscr{D}(S_\alpha)$ and $T_\alpha f = S_\alpha f$ $(f \in \mathscr{D}(T_\alpha))$, i.e. $S_\alpha$ is a self-adjoint extension of the self-adjoint operator $T_\alpha$. Hence $S_\alpha = T_\alpha$ and the proof is complete.

## 5. PROOF OF THEOREM 1

Let all the conditions of Theorem 1 hold. By Lemma 4, $T_\alpha$ is self-adjoint, bounded below and has the same lower bound as $\tau_\alpha$; let the lower bound be $\mu_\alpha$. The spectral distribution function $\sigma_\alpha$ is then constant in the interval $(-\infty, \mu_\alpha)$ and so integrals with respect to $d\sigma_\alpha(t)$ over $\mathbb{R}$ may be replaced by integrals over $[\mu_\alpha, \infty)$.

If $f \in \mathscr{D}(T_\alpha)$ and $F \in \mathscr{L}^2_\alpha$ is its unitary transform then, as we observed in Section 2, the unitary transform of $T_\alpha f$ is $E$, where $E(t) = t\,F(t)$ ($\sigma_\alpha$ – almost all $t \in R$). Hence, by (4,2),

$$\tau_\alpha[f] = (T_\alpha f, f)_w = \int_{-\infty}^{\infty} t|F(t)|^2 \, d\sigma_\alpha(t) \quad (f \in \mathscr{D}(T_\alpha)) . \tag{5,1}$$

This is (2,15) restricted to $\mathscr{D}(T_\alpha)$, and the results of Section 4 enable us to extend it to all of $\mathscr{D}(\tau_\alpha)$, as we now proceed to show.

By Lemma 4 $\mathscr{D}(T_\alpha)$ is a core of $\tau_\alpha$, and so if we choose any $f \in \mathscr{D}(\tau_\alpha)$ there is a sequence $(f_n)_{n=1}^\infty \subset \mathscr{D}(T_\alpha)$ such that $f_n \to f$ in the norm of $L^2_w[a, b)$ as $n \to \infty$,

$$\tau_\alpha[f_m - f_n] \to 0 \quad (m, n \to \infty) \tag{5,2}$$

and

$$\tau_\alpha[f] = \lim_{n\to\infty} \tau_\alpha[f_n] . \tag{5,3}$$

If $F_n$ denotes the unitary transform of $f_n$, for all $n$, then

$$\int_{-\infty}^{\infty} |F_n(t) - F(t)|^2 \, d\sigma_\alpha(t) \to 0 \quad (n \to \infty) \tag{5,4}$$

and an application of (5,1) to $f_n$ gives us

$$\tau_\alpha[f] = \lim_{n\to\infty} \int_{-\infty}^{\infty} t|F_n(t)|^2 \, d\sigma_\alpha(t) , \tag{5,5}$$

in view of (5,3).

Suppose that $\mu_\alpha < 0$. Then if $s \leqq 0$ we have

$$\int_{\mu_\alpha}^{s} |t|\, |F_n(t) - F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \leqq -\mu_\alpha \int_{\mu_\alpha}^{s} |F_n(t) - F(t)|^2 \,\mathrm{d}\sigma_\alpha(t)$$

and so by (5,4)

$$\int_{\mu_\alpha}^{s} |t|\, |F_n(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \to \int_{\mu_\alpha}^{s} |t|\, |F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \quad (n \to \infty),$$

i.e.

$$\int_{\mu_\alpha}^{s} t|F_n(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \to \int_{\mu_\alpha}^{s} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \quad (n \to \infty).$$

Similarly,

$$\int_{0}^{s} t|F_n(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \to \int_{0}^{s} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \quad (n \to \infty)$$

for any $s \in [0, \infty)$. We have therefoie

(5,6) $$\int_{\mu_\alpha}^{s} t|F_n(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \to \int_{\mu_\alpha}^{s} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \quad (n \to \infty),$$

for any $s \in [\mu_\alpha, \infty)$, where we no longer need to assume that $\mu_\alpha < 0$. Then if $s \in$ $\in [\mu_\alpha, \infty)$ and $s \geqq 0$ we find

$$\int_{\mu_\alpha}^{s} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) = \lim_{n\to\infty} \int_{\mu_\alpha}^{s} t|F_n(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \leqq \lim_{n\to\infty} \int_{\mu_\alpha}^{\infty} t|F_n(t)|^2 \,\mathrm{d}\sigma_\alpha(t) = \tau_\alpha[f],$$

in view of (5,5), and so

$$\int_{-\infty}^{\infty} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) < \infty .$$

Next we have

$$\tau_\alpha[f_m - f_n] = \int_{\mu_\alpha}^{\infty} t|F_m(t) - F_n(t)|^2 \,\mathrm{d}\sigma_\alpha(t)$$

and it follows from this, (5,2) and (5,4) that

$$\int_{\nu_\alpha}^{\infty} t|F_m(t) - F_n(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \to 0 \quad (m, n \to \infty),$$

where $\nu_\alpha = \max(\mu_\alpha, 1)$. There must therefore, by completeness, be a measurable function $G : [\nu_\alpha, \infty) \to \mathbb{C}$ such that

$$\int_{\nu_\alpha}^{\infty} t|G(t)|^2 \,\mathrm{d}\sigma_\alpha(t) < \infty ,$$

and

(5,7) $$\int_{\nu_\alpha}^{\infty} t|F_n(t) - G(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \to 0 \quad (n \to \infty).$$

Then we have

$$\int_{\nu_\alpha}^{s} t|F_n(t) - G(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \to 0 \quad (n \to \infty)$$

for any $s > \nu_\alpha$, and since also, by (5,4),

$$\int_{\nu_\alpha}^{s} t|F_n(t) - F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \to 0 \quad (n \to \infty)$$

we have $F(t) = G(t)$ ($\sigma_\alpha$-almost all $t \in [\nu_\alpha, \infty)$).

Hence, from (5,7), we obtain

$$(5,8) \qquad \int_{\nu_\alpha}^{\infty} t|F_n(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \to \int_{\nu_\alpha}^{\infty} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \quad (n \to \infty).$$

From (5,5), (5,6) and (5,8) we deduce finally that

$$\tau_\alpha[f] = \int_{-\infty}^{\infty} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t).$$

This completes the proof of Theorem 1.

## 6. PROOF OF THE COROLLARY TO THEOREM 1

Suppose that all the conditions of Theorem 1 are satisfied. Then we have the identity in (2,15), and since

$$\int_{-\infty}^{\infty} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) = \int_{\mu_\alpha}^{\infty} t|F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) \geqq \mu_\alpha \int_{\mu_\alpha}^{\infty} |F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) = \mu_\alpha \int_a^b w|f|^2 \quad (f \in \mathscr{D}(\tau_\alpha)),$$

where $\mu_\alpha$ is the lower bound of $T_\alpha$ and $F$ is the unitary transform of $f$, we obtain easily the inequality in (2,16).

We now obtain all the cases of equality in (2,16), as follows.

From (2,15) and the above discussion it is clear that there is equality in (2,16) for $f \in \mathscr{D}(\tau_\alpha)$ if and only if

$$\int_{\mu_\alpha}^{\infty} (t - \mu_\alpha)\, |F(t)|^2 \,\mathrm{d}\sigma_\alpha(t) = 0\,,$$

where $F$ is the unitary transform of $f$, i.e. $(t - \mu_\alpha)\,|F(t)|^2 = 0$ ($\sigma_\alpha$-almost all $t \in (-\infty, \infty)$). In addition, the unitary transform of $(T_\alpha - \mu_\alpha) f$ is $(t - \mu_\alpha)\, F(t)$, if $f \in \mathscr{D}(T_\alpha)$.

From known results, $\mu_\alpha$ is in the continuous spectrum of $T_\alpha$ if and only if $\sigma_\alpha$ is continuous and strictly increasing at $\mu_\alpha$, whereas $\mu_\alpha$ is an eigenvalue of $T_\alpha$ if and only if $\sigma_\alpha$ has a jump discontinuity at $\mu_\alpha$.

Suppose $\mu_\alpha$ is an eigenvalue of $T_\alpha$. From the above considerations it follows that if $f \in \mathscr{D}(T_\alpha)$ is an eigenfunction of $T_\alpha$ corresponding to $\mu_\alpha$ with unitary transform $F$, then $(t - \mu_\alpha)\, F(t)$ is $\sigma_\alpha$-null and so $(t - \mu_\alpha)\,|F(t)|^2$ is also $\sigma_\alpha$-null, which means that with this choice of $f$ the inequality in (2,16) becomes an equality. Conversely, if $f \in \mathscr{D}(\tau_\alpha)$ gives equality in (2,16), then its unitary transform $F$ must satisfy $F(t) = 0$ for $\sigma_\alpha$-almost all $t \neq \mu_\alpha$. Hence, from the discussion of $\mathscr{D}(T_\alpha)$ in Section 2, $f$ is contained in $\mathscr{D}(T_\alpha)$ and, from above, $f$ is in fact an eigenfunction of $T_\alpha$ corresponding to $\mu_\alpha$. This follows from the fact that $(t - \mu_\alpha) F(t)$ must be $\sigma_\alpha$-null.

Next, suppose that $\mu_\alpha$ is in the continuous spectrum of $T_\alpha$. From known results about Lebesgue-Stieltjes measures, the singleton set $\{\mu_\alpha\}$ has zero $\sigma_\alpha$-measure in this case. Hence if $f \in \mathscr{D}(\tau_\alpha)$ gives equality in (2,16), its unitary transform $F$ must be $\sigma_\alpha$-null, which means that $f$ is null in $L^2_w[a, b)$. Clearly this argument is reversible.

This completes the discussion of the cases of equality in (2,16).

We now suppose that $\mu_\alpha$ is in the continuous spectrum of $T_\alpha$ and show that then (2,16) is best possible as explained in the statement of the Corollary. Choose $\varepsilon > 0$ arbitrarily. We construct a function $F$ which is the unitary transform of a function $f \in \mathscr{D}(T_\alpha)$ which satisfies

$$(f, f)_w = 1 \quad \text{and} \quad 0 \leqq \tau_\alpha[f] - \mu_\alpha < \varepsilon .$$

This is equivalent to (2,17).

Since $\sigma_\alpha$ is continuous at $\mu_\alpha$ there exists a $\delta > 0$ such that $\delta < \varepsilon$, $\sigma_\alpha$ is continuous at $\mu_\alpha + \delta$ and

$$0 < \sigma_\alpha(\mu_\alpha + \delta) - \sigma_\alpha(\mu_\alpha) < \varepsilon .$$

(This last quantity is positive because $\mu_\alpha$ is in the spectrum of $T_\alpha$). Now define $F : [\mu_\alpha, \infty) \to C$ by

$$F(t) = \begin{cases} \{\sigma_\alpha(\mu_\alpha + \delta) - \sigma_\alpha(\mu_\alpha)\}^{-1/2} & (\mu_\alpha \leqq t \leqq \mu_\alpha + \delta), \\ 0 \quad (t > \mu_\alpha + \delta) . \end{cases} \tag{6,1}$$

Clearly $F$ satisfies

$$\int_{\mu_\alpha}^{\infty} |F(t)|^2 \, d\sigma_\alpha(t) = 1 , \tag{6,2}$$

i.e. $F \in \mathscr{L}^2_\alpha$, and

$$\int_{\mu_\alpha}^{\infty} |t \, F(t)|^2 \, d\sigma_\alpha(t) < \infty ,$$

so that if $F$ is the unitary transform of $f$ then $f \in \mathscr{D}(T_\alpha)$ and $(f, f)_w = 1$. Finally, we have from (2,15), (2,16), (6,1) and (6,2),

$$0 \leqq \tau_\alpha[f] - \mu_\alpha = \int_{\mu_\alpha}^{\infty} t |F(t)|^2 \, d\sigma_\alpha(t) - \mu_\alpha \int_{\mu_\alpha}^{\infty} |F(t)|^2 \, d\sigma_\alpha(t) =$$

$$= \int_{\mu_\alpha}^{\mu_\alpha + \delta} (t - \mu_\alpha) \, |F(t)|^2 \, d\sigma_\alpha(t) < \delta \int_{\mu_\alpha}^{\mu_\alpha + \delta} |F(t)|^2 \, d\sigma_\alpha(t) = \delta < \varepsilon .$$

This completes the proof of the Corollary.

## 7. PROOF OF THEOREM 2

Assume that all the conditions of Theorem 1 hold and choose any $f \in L^2_w[a, b)$ such that

$$\int_{\mu_\alpha}^{\infty} t|F(t)|^2 \, d\sigma_\alpha(t) < \infty ,$$

$F$ being as usual the unitary transform of $f$. Most of the proof resembles the argument of [11, pp. 305–306]. Reasoning similarly, we find that if

$$f_s(x) =^{\text{def.}} \int_{\mu_\alpha}^{s} F(t)\,\phi(x, t)\, d\sigma_\alpha(t) \quad (x \in [a, b),\ s \in [\mu_\alpha, \infty))$$

then $f_s \in \mathscr{D}(T_\alpha)$ $(s \in [\mu_\alpha, \infty))$, $f_s(x) \to f(x)$ $(s \to \infty)$ uniformly on compact subintervals of $[a, b)$ and $f_s$ has unitary transform $F_s$ $(s \in [\mu_\alpha, \infty))$, where

$$F_s(t) = \begin{cases} F(t) & t \in [\mu_\alpha, s] , \\ 0 & t \in (s, \infty) . \end{cases} \tag{7,1}$$

By (2,11) we have $f_s \to f$ $(s \to \infty)$ in the norm of $L^2_w[a, b)$. Also, if $u > s$ then by (2,15) and (7,1)

$$\tau_\alpha[f_u - f_s] = \int_s^u t|F(t)|^2 \, d\sigma_\alpha(t) ,$$

whence $\tau_\alpha[f_u - f_s] \to 0$ $(u, s \to \infty)$. Since $\tau_\alpha$ is closed we obtain $f \in \mathscr{D}(\tau_\alpha)$ and $\tau_\alpha[f - f_s] \to 0$ $(s \to \infty)$, i.e.

$$\int_a^b \{p|f' - f_s'|^2 + q|f - f_s|^2\} - |f(a) - f_s(a)|^2 \cot \alpha \to 0$$

$(s \to \infty)$. Since $f_s(a) \to f(a)$ and, by (2,14) (i),

$$\int_a^b q_-|f - f_s|^2 \leqq \gamma \int_a^b w|f - f_s|^2 \to 0 \quad (s \to \infty)$$

we see that

$$\int_a^b p|f' - f_s'|^2 \to 0 \quad (s \to \infty) .$$

This completes the proof of Theorem 2.


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