Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 2, 212–220

Persistent URL: http://dml.cz/dmlcz/101873

Czechoslovak Mathematical Journal, 33 (108) 1983, Praha

PONTRYAGIN DUALITY FOR CONVERGENCE GROUPS OF UNIMODULAR CONTINUOUS FUNCTIONS

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(Received November 11, 1980)

The P_c-duality theory of convergence groups of continuous unimodular functions has been extensively investigated by E. Binz. Using universal coverings he studied $\mathscr{C}_c(X, \boldsymbol{S})$ in [2] and proved its P_c-reflexity for a large class of topological spaces X, including e.g. all CW-complexes. With similar methods he proved in [3] for a compact, connected $\mathscr{C}^\infty$-manifold X and any $0 \leqq k \leqq \infty$ the P_c-reflexivity of $C^k(X, \boldsymbol{S})$ endowed with the usual topology. It is the purpose of this paper to study the topological elements of the former class. Actually we will show for every locally compact convergence space X that the topological group $\mathscr{C}_c(X, \boldsymbol{S})$ is P_c-reflexive. The proof for the compact case can be easily modified to give a simpler proof of the cited result of E. Binz in [3].

We denote by $\boldsymbol{S}$ the compact topological group $\boldsymbol{R}/\boldsymbol{Z}$ and by $\nu : \boldsymbol{R} \to \boldsymbol{S}$ the natural projection. Given a convergence group G – always assumed to be commutative – the symbol $\Gamma_c G$ stands for the character group of G, i.e. the group of all continuous group homomorphisms from G into $\boldsymbol{S}$ endowed with the continuous convergence structure. Furthermore $\varkappa_G : G \to \Gamma_c\Gamma_c G$ is defined by $\varkappa_G(x)(\varphi) = \varphi(x)$ for all $x \in G$ and all $\varphi \in \Gamma G$. Now $\varkappa_G$ is a continuous group homomorphism and G is called P_c-reflexive if $\varkappa_G$ is an isomorphism, i.e. a homeomorphism. Given convergence spaces X and Y we denote by $\mathscr{C}_c(X, Y)$ the convergence space of all continuous maps from X to Y, again endowed with the continuous convergence structure. Particularly they set $\mathscr{C}_c(X) = \mathscr{C}_c(X, \boldsymbol{R})$ and for a convergence vector space E the subspace of $\mathscr{C}_c(E)$ consisting of all linear continuous functionals is denoted by $\mathscr{L}_c E$. If X is a locally compact convergence space then both $\mathscr{C}_c(X)$ and $\mathscr{C}_c(X, \boldsymbol{S})$ are topological, namely they carry the topology of uniform convergence on the compact subsets of X.

Finally, if $f : X \to Y$ is a continuous map between convergence spaces we define the continuous maps
$$f^* : \mathscr{C}_c(Y) \to \mathscr{C}_c(X)$$
and
$$f' : \mathscr{C}_c(Y, \boldsymbol{S}) \to \mathscr{C}_c(X, \boldsymbol{S})$$
by sending g to $f \circ g$ for all $f \in \mathscr{C}(Y)$ and $f \in \mathscr{C}(Y, \boldsymbol{S})$, respectively. In order to simplify

the notion any restriction of f^* and f' will be denoted by the same symbol. So is in particular for any continuous group homomorphism $h : G \to H$ the map $h' : \Gamma_c H \to \Gamma_c G$ also a continuous group homomorphism.

For further definitions and properties of convergence spaces and the continuous convergence structure we refer the reader to [1].

In order to prove the P_c-reflexivity of $\mathscr{C}_c(X, \boldsymbol{S})$ for all locally compact spaces we shall first show the compact case. To this end we follow an idea of Binz in [2] and [3] and use the map

$$v_X : \mathscr{C}_c(X) \to \mathscr{C}_c(X, \boldsymbol{S})$$

defined by

$$v_X(f) = v \circ f$$

for all $f \in \mathscr{C}(X)$. Actually we show that the image of v_X is P_c-reflexive for all compact topological spaces X. With the help of a group theoretical theorem of Baer we will then see that this topological group is an open, direct summand of $\mathscr{C}_c(X, \boldsymbol{S})$, whence this case follows. The locally compact case is then reduced to the compact one.

The connection between $\mathscr{C}_c(X)$ and $\mathscr{C}_c(X, \boldsymbol{S})$ is given by

Lemma 1. *For all compact convergence spaces X the map v_X is open and therefore $\mathscr{C}_c(X)/\ker v_X$ is isomorphic to $v_X(\mathscr{C}_c(X))$.*

Proof. Set for $\varepsilon > 0$

$$T_\varepsilon = \{f \in \mathscr{C}(X) : f(X) \subset [-\varepsilon, \varepsilon]\} \quad \text{and}$$
$$T'_\varepsilon = \{f \in \mathscr{C}(X, \boldsymbol{S}) : f(X) \subset v([-\varepsilon, \varepsilon])\},$$

then $\{T_\varepsilon : \varepsilon > 0\}$ and $\{T'_\varepsilon : \varepsilon > 0\}$ are zero-neighborhood bases of $\mathscr{C}_c(X)$ and $\mathscr{C}_c(X, \boldsymbol{S})$, respectively. But evidently for $\varepsilon < \frac{1}{2}$

$$v_X(T_\varepsilon) = T'_\varepsilon,$$

proving the statement.

Because of lemma 1 we want to study the convergence group $\mathscr{C}_c(X)/\ker v_X$ in the sequel. In order to simplify our notation we set for all convergence spaces X

$$G_X = \ker v_X = \{f \in \mathscr{C}(X) : f(X) \subset \boldsymbol{Z}\}$$

and denote by $\pi : \mathscr{C}_c(X) \to \mathscr{C}_c(X)/G_X$ the natural projection. We know from [4] that

$$\pi' : \Gamma_c(\mathscr{C}_c(X)/G_X) \to G_X^\perp := \{\varphi \in \Gamma\,\mathscr{C}_c(X) : \varphi(G_X) = \{0\}\}$$

is an isomorphism and by [5]

$$T : \mathscr{L}_c\,\mathscr{C}_c(X) \to \Gamma_c\,\mathscr{C}_c(X) \quad \varphi \mapsto v \circ \varphi$$

is also an isomorphism. So we finally get an isomorphism

$$T^{-1} \circ \pi' : \Gamma_c(\mathscr{C}_c(X)/G_X) \to T^{-1}(G_X^\perp) = \{\varphi \in \mathscr{L}\,\mathscr{C}_c(X) : \varphi(G_X) \subset \boldsymbol{Z}\},$$

and we set therefore

$$H_X = \{\varphi \in \mathscr{L}\,\mathscr{C}_c(X) : \varphi(G_X) \subset \boldsymbol{Z}\}\,.$$

Then $\Gamma_c(\mathscr{C}_c(X)/G_X)$ is isomorphic to H_X and we we want to calculate it at least in a very special case.

Definition. For a topological space X and a clopen (i.e. closed and open) set $U \subset X$ we denote by c_U the characteristic function of U and for $\varphi \in \mathscr{L}\,\mathscr{C}_c(X)$ we say that U is φ-irreducible if for all clopen sets $V \subset U$ we have $\varphi(c_V) \in \{0, \varphi(c_U)\}$.

The announced description of H_X is now prepared by the propositions 1 and 2 and given in proposition 3:

Proposition 1. *Assume X to be a compact topological space and let $\varphi \in H_X$. Then X can be represented as the finite union of pairwise disjoint φ-irreducible sets.*

Proof. Since X is compact it is enough to show that every point in X is contained in a φ-irreducible set. Assume that this is wrong for $x_0 \in X$, then one can construct a decreasing sequence (U_n) of clopen neighborhoods of x_0 such that $\varphi(c_{U_n}) \neq \varphi(c_{U_{n+1}})$ for all $n \in \boldsymbol{N}$. Since $c_{U_n} - c_{U_{n+1}} \in G_X$ we have $\varphi(c_{U_n} - c_{U_{n+1}}) \in \boldsymbol{Z}$ and therefore there are $z_n \in \{-1, 1\}$ such that

$$\varphi(z_n(c_{U_n} - c_{U_{n+1}})) \in \boldsymbol{N} \quad \text{for all} \quad n \in \boldsymbol{N}\,.$$

Setting for all $k \in \boldsymbol{N}$

$$f_k = \sum_{n=1}^{k} z_n(c_{U_n} - c_{U_{n+1}})$$

we have $\varphi(f_k) \geqq k$ and

$$\|f_k\| = \max\{|f_k(x)| : x \in X\} = 1\,.$$

But since φ is a continuous linear functional on the Banach space $\mathscr{C}_c(X)$ we have

$$|\varphi(f_k)| \leqq \|\varphi\|\,\|f_k\| = \|\varphi\| \quad \text{for all} \quad k \in \boldsymbol{N}\,.$$

This contradiction completes the proof.

Proposition 2. *For any compact topological space X having a base of clopen sets, any $\varphi \in H_X$ and any non-empty φ-irreducible clopen set $U \subset X$ there is an $x_0 \in U$ such that*

$$\varphi(f) = \varphi(c_U)f(x_0) \quad \textit{for all} \quad f \in \mathscr{C}(X) \quad \textit{with} \quad f(X \setminus U) = \{0\}\,.$$

Proof. As is well-known there is a regular, signed Borel-measure μ_φ on X such that

$$\varphi(f) = \int_X f(x)\,\mathrm{d}\mu_\varphi(x) \quad \text{for all} \quad f \in \mathscr{C}(X)\,.$$

If $\varphi(c_U) = 0$ then $\mu_\varphi(V) = 0$ for all clopen sets $V \subset U$ implying $\mu_\varphi \mid U = 0$, since X has a basis of clopen sets. In this case we get for any $x_0 \in U$ and any $f \in \mathscr{C}(X)$ with

$f(X \setminus U) = \{0\}$:

$$\varphi(f) = \int_X f(x)\,\mathrm{d}\mu_\varphi(x) = \int_U f(x)\,\mathrm{d}\mu_\varphi(x) + \int_{X\setminus U} f(x)\,\mathrm{d}\mu_\varphi(x) = 0\,.$$

On the other hand, if $\varphi(c_U) \neq 0$, the measure

$$A \mapsto \frac{1}{\varphi(c_U)}\,\mu_\varphi(A) \quad \text{for all Borel-sets} \quad A \subset U$$

is a regular Borel-measure on U taking only the values 0 and 1. Since U is compact it is realcompact and therefore there is an $x_0 \in U$ such that for all Borel-sets $A \subset U$

$$\frac{1}{\varphi(c_U)}\,\mu_\varphi(A) = \begin{cases} 0 & \text{if} \quad x_0 \in U \setminus A\,, \\ 1 & \text{if} \quad x_0 \in A\,, \end{cases}$$

implying $\varphi(f) = \varphi(c_U) f(c_U) f(x_0)$ for all $f \in \mathscr{C}(X)$ vanishing on $X \setminus U$.

The following definition will be needed in the sequel:

Definition 2. Foı any convergence space X we define

$$i_X : X \to \mathscr{L}_c\,\mathscr{C}_c(X) \quad \text{by} \quad i_X(x)(f) = f(x) \quad \text{for all} \quad x \in X \quad \text{and all} \quad f \in \mathscr{C}(X)$$

and

$$j_X : X \to \Gamma_c\,\mathscr{C}_c(X, \boldsymbol{S}) \quad \text{by} \quad j_X(x)(f) = f(x) \quad \text{for all} \quad x \in X \quad \text{and} \quad f \in \mathscr{C}(X, \boldsymbol{S})\,.$$

Both i_X and j_X are continuous and we have

Proposition 3. *If X is a compact topological space having a base of clopen sets then $H_X = \langle i_X(X) \rangle$, the group generated by $i_X(X)$.*

Proof. Evidently $\langle i_X(X) \rangle \subset H_X$, so take any $\varphi \in H_X$. By proposition 1 there are pairwise disjoint, non-empty φ-irreducible sets $U_1, \ldots, U_k \subset X$ such that $X = U_1 \cup \ldots \cup U_k$. By proposition 2 we can find $x_n \in U_n$ with

$$\varphi(f) = \varphi(c_{U_n}) f(x_n) \quad \text{for all} \quad n \in \{1, \ldots, k\} \quad \text{and all}$$
$$f \in \mathscr{C}(X) \quad \text{with} \quad f(X \setminus U_n) = \{0\}\,.$$

This implies for all $f \in \mathscr{C}(X)$:

$$\varphi(f) = \varphi\Big(\sum_{n=1}^{k} f c_{U_n}\Big) = \sum \varphi(f c_{U_n}) = \sum \varphi(c_{U_n}) f(x_n) = \Big(\sum \varphi(c_{U_n})\, i_X(x_n)\Big)(f)\,.$$

Since $c_{U_n} \in G_X$ we have $\varphi(c_{U_n}) \in \boldsymbol{Z}$ and are ready.

Before we prove the main result of the first part we notice

Lemma 2. *For any compact topological space X having a base of clopen sets $v_X : \mathscr{C}_c(X) \to \mathscr{C}_c(X, \boldsymbol{S})$ is surjective.*

Proof. Take $g \in \mathscr{C}(X, \boldsymbol{S})$, then to any point $x \in X$ there is a clopen neighborhood U_x

of x such that $g \mid U_x$ is not surjective. By the compactness of X this implies the existence of pairwise disjoint clopen sets $U_1, \ldots, U_k$ such that $g \mid U_n$ is not surjective and $X = U_1 \cup \ldots \cup U_k$. Then choose functions $f_n \in \mathscr{C}(U_n)$ such that $v \circ f_n = g \mid U_n$ and define

$$f : X \to \boldsymbol{R} \quad \text{by} \quad f(x) = f_n(x) \quad \text{if} \quad x \in U_n .$$

Then clearly $f \in \mathscr{C}(X)$ and $v_X(f) = g$.

Theorem 1. *For any compact topological space X the group H_X has the extension property in $\mathscr{L}_c\mathscr{C}_c(X)$, i.e. any character of H_X can be lifted to a character of $\mathscr{L}_c\mathscr{C}_c(X)$.*

Proof. Define an equivalence relation $\sim$ on X by $x \sim y$ if and only if $(x \in U \Leftrightarrow y \in U$ for all clopen $U \subset X)$ set $Y = X/\sim$ and $\varrho : X \to Y$ the projection. Since Y is compact and has point-separating clopen sets it has a basis of clopen sets. We get the following commutative diagramme:

$$\begin{array}{ccc} X & \xrightarrow{\quad \varrho \quad} & Y \\ \Big\downarrow{\scriptstyle i_X} & & \Big\downarrow{\scriptstyle i_Y} \\ \mathscr{L}_c\mathscr{C}_c(X) & \xrightarrow{\varrho^{**}} & \mathscr{L}_c\mathscr{C}_c(Y) \end{array}$$

The rest of the proof splits into three parts:

(i) claim: $H_X = i_X(X) + \ker \varrho^{**}$

proof: For all $\varphi \in \ker \varrho^{**}$ and $f \in G_X$ there is a $g \in \mathscr{C}(Y)$ with $g \circ \varrho = f$ and so

$$\varphi(f) = \varphi(g \circ \varrho) = \varphi(\varrho^*(g)) = \varphi \circ \varrho^*(g) = \varrho^{**}(\varphi)(g) = 0 ,$$

showing "$\supset$". On the other hand, for all $\varphi \in H_X$ we have

$$\varrho^{**}(\varphi)(g) = \varphi(\varrho^*(g)) = \varphi(g \circ \varrho) = 0 \quad \text{for all} \quad g \in G_Y ,$$

and so $\varrho^{**}(\varphi) \in H_Y$. Now by proposition 3 there are $z_1, \ldots, z_k \in \boldsymbol{Z}$ and $y_1, \ldots, y_k \in Y$ with

$$\varrho^{**}(\varphi) = \sum_{n=1}^{k} z_n \, i_Y(y_n) .$$

Since ϱ is surjective $y_n = \varrho(x_n)$ for some $x_n \in X$ and so

$$\varrho^{**}(\varphi) = \sum z_n \, i_Y(\varrho(x_n)) = \sum z_n \, \varrho^{**}(i_X(x_n)) = \varrho^{**}\left(\sum z_n \, i_X(x_n)\right) ,$$

therefore $\varphi - \sum z_n \, i_X(x_n) \in \ker \varrho^{**}$.

(ii) claim: Any $\vartheta \in \Gamma H_X$ with $\vartheta(\ker \varrho^{**}) = \{0\}$ can be extended to a character on $\mathscr{L}_c\mathscr{C}_c(X)$.

proof: First we show that $\vartheta \circ i_X$ is compatible with $\sim$:

Let be $x, y \in X$ such that $\varrho(x) = \varrho(y)$, then

$$\varrho^{**}(i_X(x) - i_X(y)) = i_Y(\varrho(x)) - i_Y(\varrho(y)) = 0$$

and so $i_X(x) - i_X(y) \in \ker \varrho^{**}$. By assumption $\vartheta(i_X(x) - i_X(y)) = 0$ and so

$$\vartheta \circ i_X(x) = \vartheta \circ i_X(y) \quad \text{for all} \quad x, y \in X \quad \text{with} \quad \varrho(x) = \varrho(y) .$$

Since Y is a quotient of X we get together with lemma 2 a function $g \in \mathscr{C}(Y)$ with

$$v \circ g \circ \varrho = \vartheta \circ i_X .$$

Setting $f = g \circ \varrho$ it is easy to show with the help of claim (i) that $\varkappa_{\mathscr{C}_c(X)} \circ T \mid H_X = \vartheta$.

(iii) proof of the theorem: Denote by $E = \ker \varrho^{**} \subset \mathscr{L}_c \mathscr{C}_c(X)$ and take any $\zeta \in \Gamma H_X$. Then $\zeta \mid E \in \Gamma E$ and by Satz 1 in [5] there is a $\zeta_0 \in \mathscr{L} E$ with $v \circ \zeta_0 = \zeta \mid E$, but lemma III.5 in [7] shows that ζ_0 can be extended to a continuous linear functional $\zeta_1 \in \mathscr{L}\mathscr{L}_c \mathscr{C}_c(X)$ and for $\tilde{\zeta}_1 = v \circ \zeta_1 \in \Gamma \mathscr{L}_c \mathscr{C}_c(X)$ we have evidently

$$\zeta - \tilde{\zeta}_1 \mid E = 0 \Rightarrow (\zeta - \tilde{\zeta}_1 \mid H_X)(\ker \varrho^{**}) = \{0\} ,$$

and by claim (ii) there is a $\tilde{\zeta}_2 \in \Gamma \mathscr{L}_c \mathscr{C}_c(X)$ with

$$\zeta - \tilde{\zeta}_1 \mid H_X = \tilde{\zeta}_2 \mid H_X \Rightarrow \zeta = (\tilde{\zeta}_1 + \tilde{\zeta}_2) \mid H_X .$$

Corollary 1. *For any compact topological space X the convergence groups $\mathscr{C}_c(X)/\ker v_X$ and $v_X(\mathscr{C}_c(X))$ are P_c-reflexive.*

Proof. By lemma 1 both convergence groups are isomorphic. Since

$$j'_X \circ \varkappa_{\mathscr{C}_c(X,S)} = \mathrm{id}_{\mathscr{C}_c(X,S)}$$

for all convergence spaces X, we know that $\varkappa_{\mathscr{C}_c(X,S)}$ is an embedding und therefore the same holds for $\varkappa_{v_X(\mathscr{C}_c(X))}$ and $\varkappa_{\mathscr{C}_c(X)/G_X}$. So all that is left to prove is the surjectivity of $\varkappa_{\mathscr{C}_c(X)/G_X}$:
Take any $\zeta \in \Gamma\Gamma_c(\mathscr{C}_c(X)/G_X)$, then $\eta = \zeta \circ (T^{-1} \circ \pi')^{-1} \in \Gamma H_X$ and by theorem 1 there is a character $\tilde{\eta} \in \Gamma \mathscr{L}_c \mathscr{C}_c(X)$ with $\tilde{\eta} \mid H_X = \eta$. Since $\mathscr{C}_c(X)$ is P_c-reflexive by [5], there is an $f \in \mathscr{C}(X)$ with $\varkappa_{\mathscr{C}_c(X)}(f) = \tilde{\eta} \circ T^{-1}$ and it is easy to verify that $\varkappa_{\mathscr{C}_c(X)/G_X}(\pi(f)) = \zeta$.

Remark. If X is also connected then G_X contains just the constant functions and under this additional hypothesis the statement of Corollary 1 can easily be proven (cf. theorem 6 in [3]). So the complications in the proof are indeed due to the lack of connectedness of X.

Theorem 2. *For any compact topological space X there is a discrete topological subgroup D_X of $\mathscr{C}_c(X, S)$ such that*

$$\mathscr{C}_c(X, S) = v_X(\mathscr{C}_c(X)) \oplus D_X .$$

Proof. $L_X := v_X(\mathscr{C}_c(X))$ is a divisibe subgroup of $\mathscr{C}_c(X, S)$ and so has by a theorem of Baer (see e.g. theorem 18.1 in [6]) an algebraic direct complement D_X. Denoting by $\varrho : \mathscr{C}_c(X, S) \to D_X$ the projection we have to show that ϱ is continuous. But there is a group isomorphism $\tilde{\rho} : \mathscr{C}_c(X, S)/L_X \to D_X$ so that the following diagramme commutes:

$$\begin{array}{ccc} \mathscr{C}_c(X, S) & \xrightarrow{\varrho} & D_X \\ \pi \downarrow & & \uparrow \tilde{\varrho} \\ \mathscr{C}_c(X,S)/L_X & \longrightarrow & \end{array}$$

Since ν_X is by lemma 1 open, L_X is open in $\mathscr{C}_c(X, S)$ and therefore $\mathscr{C}_c(X, S)/L_X$ is discrete, showing the continuity of ϱ. Since D_X is a direct summand of $\mathscr{C}_c(X, S)$ it is indeed isomorphic to the discrete topological group $\mathscr{C}_c(X, S)/L_X$ and therefore itself discrete.

Corollary 2. *For any compact topological space X the topological group $\mathscr{C}_c(X, S)$ is P_c-reflexive.*

Proof. $\nu_X(\mathscr{C}_c(X))$ is P_c-reflexive by corollary 1 and D_X is P_c-reflexive by the classical Pontryagin-duality-theorem. Since the direct sum of P_c-reflexive convergence groups has this property again, the corollary follows from theorem 2.

Remark. For a compact, connected C^∞-manifold X and $0 \leqq k \leqq \infty$ define $\nu_X^k : \mathscr{C}^k(X) \to C(X, S)$ in the usual way. Now the same argument as in the proof of theorem 2 shows that $\nu_X^k(\mathscr{C}(X))$ is a direct summand of $\mathscr{C}^k(X, S)$ having a discrete direct complement. The obvious modifications of the proof of corollary 2 then give a quick proof of the main result of [3].

The heart of the reduction of the locally compact case to the compact one is given in proposition 5. Before we state it we notice the following elementary but often useful fact:

Lemma 3. *For every convergence space X the convergence group $\mathscr{C}_c(X, S)$ is P_c-reflexive if and only if for all $\zeta \in \Gamma\Gamma_c(\mathscr{C}_c(X, S))$ we have*

$$\varkappa_{\mathscr{C}_c(X,S)}(\zeta \circ j_X) = \zeta .$$

Proof. Since $j'_X \circ \varkappa_{\mathscr{C}_c(X,S)} = \mathrm{id}_{\mathscr{C}_c(X,S)}$ the P_c-reflexivity of $\mathscr{C}_c(X, S)$ implies

$$\varkappa_{\mathscr{C}_c(X,S)} \circ j'_X = \mathrm{id}_{\mathscr{C}_c(X,S)} , \quad \text{i.e.}$$

$$\zeta = \varkappa_{\mathscr{C}_c(X,S)} \circ j'_X(\zeta) = \varkappa_{\mathscr{C}_c(X,S)}(\zeta \circ j_x) \quad \text{for all} \quad \zeta .$$

The other implication is clear.

Proposition 5. *For any locally compact, c-embedded convergence space X and any compact set $K \subset X$ the embedding $e_K : K \to X$ induces an open mapping*

$$e'_K : \mathscr{C}_c(X, S) \to \mathscr{C}_c(K, S) .$$

Furthermore we have

$$\Gamma\mathscr{C}_c(X, S) = \bigcup\{e''_K(\Gamma\mathscr{C}_c(K, S)) : K \subset X \ \textit{compact}\} .$$

Proof. For $A \subset X$ and $\varepsilon > 0$ we set

$$T'_\varepsilon(A) = \{f \in \mathscr{C}(X, \boldsymbol{S}) : f(A) \subset v([-\varepsilon, \varepsilon])\}\,.$$

Since X is locally compact, $\mathscr{C}_c(X, \boldsymbol{S})$ is topological and a zero-neighborhood base is given by

$$\{T'_\varepsilon(L) : L \subset X \text{ compact, } \varepsilon > 0\}\,.$$

Fix a compact set $K \subset X$, then we show at first for all compact sets $L \subset X$ and all $\varepsilon < \frac{1}{2}$

$$e'_K(T'_\varepsilon(L)) \supset e'_K(T'_\varepsilon(X) \supset \{g \in \mathscr{C}(K, \boldsymbol{S}) : g(K) \subset v([-\varepsilon, \varepsilon])\}\,,$$

which implies that e'_K is open:

Given any g from the right side there is an $f \in \mathscr{C}(K)$ with $v \circ f = g$ and $f(K) \subset \subset [-\varepsilon, \varepsilon]$. Since X is c-embedded, there is an $\tilde{f} \in \mathscr{C}(X)$ with $\tilde{f} \,|\, K = f$ and $\tilde{f}(X) \subset \subset [-\varepsilon, \varepsilon]$. Evidently $v \circ \tilde{f} \in T'_\varepsilon(X)$ and $e'_K(v \circ f) = g$.

To prove the second statement, take any $\varphi \in \Gamma\, \mathscr{C}_c(X, \boldsymbol{S})$. Then there are $\varepsilon > 0$ and a compact set $K \subset X$ such that

$$\varphi(T'_\varepsilon(K)) \subset v([-\tfrac{1}{4}, \tfrac{1}{4}])\,.$$

Since $I'(K) := \{f \in \mathscr{C}(X, \boldsymbol{S}) : f(K) = \{0\}\} \subset T'_\varepsilon(K)$ is a subgroup the same holds for its image under φ. Since $v([-\frac{1}{4}, \frac{1}{4}])$ contains only the trivial subgroup we deduce:

$$\{0\} = \varphi(I'(K)) = \varphi(\ker e'_K)\,.$$

Since e'_K is open we get therefore a character $\psi_0 \in \Gamma(e'_K(\mathscr{C}(X, \boldsymbol{S}))$ with $\psi_0 \circ e'_K = \varphi$. But the image of e'_K is open and so ψ_0 can be extended to a character $\psi \in \Gamma \mathscr{C}_c(X, \boldsymbol{S})$ for which evidently $e''_K(\psi) = \varphi$.

We are now ready to prove the main result of this paper:

Theorem 3. *For any locally compact convergence space X the topological group $\mathscr{C}_c(X, \boldsymbol{S})$ is P_c-reflexive.*

Proof. It is well-known that to any convergence space X there is a c-embedded one X' such that $\mathscr{C}_c(X, \boldsymbol{S})$ and $\mathscr{C}_c(X', \boldsymbol{S})$ are in a natural way isomorphic. Moreover, X' is locally compact if X has this property. Therefore we assume w.l.o.g. that X is c-embedded. Take any $\zeta \in \Gamma\Gamma_c\, \mathscr{C}_c(X, \boldsymbol{S})$ and regard for any compact set $K \subset X$ the following commutative diagramme:

$$\begin{array}{ccc} K & \xrightarrow{\;e_K\;} & X \\ \big\downarrow{\scriptstyle j_K} & & \big\downarrow{\scriptstyle j_X} \\ \Gamma_c\, \mathscr{C}_c(K, \boldsymbol{S}) & \xrightarrow{\;e''_K\;} & \Gamma_c\, \mathscr{C}_c(X, \boldsymbol{S}) \end{array}$$

As a compact subset of a c-embedded convergence space, K is by proposition 27 in [1] topological and therefore $\mathscr{C}_c(K, \boldsymbol{S})$ is by theorem 2 P_c-reflexive, implying with lemma 3

$$\varkappa_{\mathscr{C}_c(K,S)}(\zeta \circ e''_K \circ j_K) = \zeta \circ e''_K .$$

This gives for all $\varphi \in \Gamma \mathscr{C}_c(K, S)$:

$$\varkappa_{\mathscr{C}_c(X,S)}(\zeta \circ j_X)(e''_K(\varphi)) = \varphi(\zeta \circ j_X \circ e''_K) = \varphi(\zeta \circ e''_K \circ j_K) =$$
$$= \varkappa_{\mathscr{C}_c(K,S)}(\zeta \circ e''_K \circ j_K)(\varphi) = (\zeta \circ e''_K)(\varphi) = \zeta(e''_K(\varphi)) .$$

Since $K \subset X$ was an arbitrary compact subset of X we conclude with proposition 5

$$\varkappa_{\mathscr{C}_c(X,S)}(\zeta \circ j_X) = \zeta \quad \text{for all} \quad \zeta \in \Gamma\Gamma_c \mathscr{C}_c(X, S) .$$

Applying now lemma 3 we get the statement of the theorem.

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