Czechoslovak Mathematical Journal

Luděk Zajíček

Differentiability of the distance function and points of multi-valuedness of the metric projection in Banach space

Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 2, 292–308

Persistent URL: http://dml.cz/dmlcz/101878

Czechoslovak Mathematical Journal, 33 (108) 1983, Praha

DIFFERENTIABILITY OF THE DISTANCE FUNCTION AND POINTS OF MULTI-VALUEDNESS OF THE METRIC PROJECTION IN BANACH SPACE

LUDĚK ZAJÍČEK, Praha

(Received March 1, 1982)

Introduction. In the whole article suppose that X is a real Banach space with the norm $q(x) = \|x\|$ and that F is a closed subset of X. For $x \in X$ denote by $d_F(x)$ the distance from the point x to the set F. The metric projection is defined as the (possibly) multivalued operator $P_F(x) = \{y \in F;\ \|x - y\| = d_F(x)\}$. Of course, there are connections between the singlevaluedness of P_F and the differentiability of d_F at a point $x \notin F$. In Euclidean spaces we have the Mises theorem [16] (see also [6], [10]), which can be stated in the following form.

Mises theorem. *If $X = R^n$ and $x \notin F$ then all one-sided directional derivatives $D_v\, d_F(x)$ exist and $D_v\, d_F(x) = \inf\{(v, \|x - y\|^{-1}(x - y));\ y \in P_F(x)\}$.*

In the present article we investigate the set $R(F)$ of points x for which $P_F(x)$ contains at least two points and the set $N(F)$ of points $x \notin F$ at which d_F is not Gateaux differentiable.

If $X = R^n$ then the Mises theorem implies $R(F) = N(F)$. The results of Section 4 imply that $R(F) = N(F)$ for any F if and only if X is a finite-dimensional strictly convex smooth space.

Large classes of spaces in which any $R(F)$ is a first category set are known, while the problem if the largest class of these spaces coincides with the class of strictly convex spaces remains open (see [13]). From Aronszajn's theorem [3] on differentiability of Lipschitz functions it follows that in separable spaces, $N(F)$ is small in a measure sense. In some smooth spaces $N(F)$ is of the first category (see ([9], Corollary 3.7) and Section 3 of the present article).

Our main effort is to determine the strictest sense in which all $R(F)$ (or $N(F)$) are small in special separable spaces. We obtain some partial results in this direction.

In Section 1 we discuss some classes of small sets in separable spaces, some of which are used in the subsequent sections.

Singlevaluedness of the metric projection in Euclidean spaces was investigated by Erdös [8]. He proved that $R(F)$ is of a σ-finite $(n - 1)$-dimensional measure. Konjagin [12] proved the Erdös theorem in finite-dimensional strictly convex spaces using an easy new lemma concerning differentiability of Lipschitz functions. In [25] the Erdös theorem is in a certain sense generalized to the case of a separable strictly convex X ($R(F)$ can be covered by countably many Lipschitz hypersurfaces).

In Section 2 we give simplified proofs of results from [25] based on Konjagin's method. The main idea of the new proofs is the same as that of the old ones. However, the method of the differentiation of the distance function is not explicitly used in [25]. This method together with Lemma 2 which is essentially Konjagin's lemma mentioned above considerably clarifies the proofs.

Section 3 includes some simple remarks concerning differentiability of d_F (also at $x \in F$) in general Banach spaces.

In Section 4 we characterize the Banach spaces, in which any distance function d_F has at each point $x \notin F$ all one-sided directional derivatives, as spaces which have a uniformly Gateaux differentiable norm. An analogue of the Mises theorem in these spaces is proved. In separable spaces which have a uniformly Gateaux differentiable norm we prove with help of Lemma 2 that $N(F)$ can be covered by countably many Lipschitz hypersurfaces (in fact we establish a slightly sharper result).

In Section 5 we prove that in Hilbert spaces and in finite-dimensional spaces the norm of which has a derivative which is Lipschitz on the unit sphere, the distance function d_F is locally δ-convex on $X - F$. (The function is said to be δ-convex if it is the difference of two continuous functions.) Consequently, d_F is as smooth on $X - F$ as a continuous convex function, in particular, $N(F)$ is very small (it can be covered by countably many δ-convex hypersurfaces). The main results of Section 5 are stated in [27]. The main method of Section 5 which generalizes an Asplund's argument from [4] was independently used by Abatzoglou in [1].

In Section 6 we obtain analogous results on $R(F)$ as easy consequences of the results of Section 5. Some additional propositions concerning the sharpness of these results in Hilbert spaces are given.

1. CLASSES OF EXCEPTIONAL SETS IN SEPARABLE SPACES

Let X be a separable real Banach space. We shall discuss six classes of exceptional sets.

(a) **The class of Gaussian null sets.** (See [17].) A Borel subset of X is called a *Gaussian null set* if $\mu B = 0$ for every nondegenerate Gaussian measure μ on X.

(b) **Aronszajn's system** $\mathscr{A}$. (See [3].) Let (a_n) be a sequence of nonzero elements complete in X (i.e., (a_n) has dense linear span in X). Define $\mathscr{A}(a_n)$ to be the family of all Borel sets of the form $\bigcup_{n=1}^{\infty} A_n$, where each A_n is a Borel set with the property that for each $x \in X$ the set $(A_n + x) \cap Ra_n$ has Lebesgue measure zero in the line Ra_n. Finally, let $\mathscr{A}$ be the intersection of the families $\mathscr{A}(a_n)$ over all such sequences (a_n) possible.

(c) **Aronszajn's system** $\mathscr{A}^0$. (See [3].) This system is defined similarly as $\mathscr{A}$. The only difference is that the condition "$(A_n + x) \cap Ra_n$ has Lebesgue measure zero" is replaced by the condition "$(A_n + x) \cap Ra_n$ is countable".

(d) **The system of sparse sets.** Let $o \neq v \in X$. We shall say that $A \subset X$ is a Lipschitz hypersurface associated with v if there exists a topological complement Z of Rv and a Lipschitz function $f: Z \to R$ such that $A = \{z + f(z)\, v,\ z \in Z\}$. A subset of X will be called *sparse* if it can be covered by countably many Lipschitz hypersurfaces. Sparse sets were defined in a different but equivalent way in the case $X = R^2$ by W. H. Young [21] under the name "ensemble ridée". Sparse sets were frequently used in the real analysis (see e.g. H. Blumberg [5], where the name "sparse" is used). For infinite-dimensional applications see [24] and [25]. The equivalence of Young's definition and ours in the case $X = R^2$ immediately follows from [22], Proposition 1.

(e) **The system of strongly sparse sets.** A set $M \subset X$ will be called *strongly sparse* if for any sequence of nonzero elements (a_n) complete in X we can write $M \subset \subset \bigcup_{n,m=1}^{\infty} L_{n,m}$, where each $L_{n,m}$ is a Lipschitz hypersurface associated with x_n.

(f) **The system of d.c.-sparse sets.** A set $M \in X$ will be called a *δ-convex hypersurface* (or d.c.-hypersurface) associated with $v \neq 0$ if there exists a topological complement Z of Rv and convex Lipschitz functions f_1, f_2 on Z such that $M = = \{z + (f_1(z) - f_2(z))\, v,\ z \in Z\}$. Note that d.c.-hypersurfaces are called $(c - c)$-*hypersurfaces* in [26]. A subset of X will be called *d.c.-sparse* if it can be covered by countably many d.c.-hypersurfaces.

From results of [26] the following useful proposition easily follows.

Proposition 1. *Let (a_n) be a sequence of nonzero elements of X complete in X. Let $M \subset X$ be d.c.-sparse. Then $M \subset \bigcup_{n,m=1}^{\infty} H_{n,m}$, where $H_{n,m}$ is a d.c.-hypersurface associated with a_n.*

Proof. By ([26], Proposition 3) there exists a continuous convex function f on X such that f is not Gateuax differentiable at any point $x \in M$. Consequently for any $x \in M$ there exists n such that the two-sided directional derivative of f at x in the direction a_n does not exist. Using ([26], Lemma 2) we easily obtain the conclusion of our proposition.

For references concerning δ-convex functions in Euclidean spaces see [19].

Now we shall briefly discuss the relationships between the classes defined above.

Any set belonging to $\mathscr{A}$ is a Gaussian null set and it is not known whether there exists a Gaussian null set which does not belong to $\mathscr{A}$ [17].

The inclusion $\mathscr{A}^0 \subset \mathscr{A}$ is obvious. For a measure proof of the inequality $\mathscr{A}^0 \neq \mathscr{A}$ see [3]. We can also use a category argument which works in any X. In fact, using the Kuratowski-Ulam category analogue of the Fubini theorem, we obtain that any set from $\mathscr{A}^0$ is a first category set. On the other hand, in any X there exists a residual set $M \in \mathscr{A}$. It is sufficient to choose a residual set $T \subset R$ of Lebesgue measure zero and to put $M = \{x; x^*(x) \in T\}$ for a nonzero $x^* \in X$.

Any sparse set obviously is of the first category. Therefore there exists a set $M \in \mathscr{A}$ which is not sparse.

From the theory of Gaussian measures it easily follows that any Borel sparse set is a Gaussian null set ([25], p. 521).

Now we shall prove the (possibly stronger) proposition that any Borel sparse set M belongs to $\mathscr{A}$. Let A be a Lipschitz hypersurface, and let v, Z, f be as in the definition of a Lipschitz hypersurface. For $x = z + pv$, $p \in R$, $z \in Z$, put $g(x) = \max(p - f(v), 0)$. It is easy to prove that g is Lipschitz on X and Gateaux differentiable at no point $x \in A$. Therefore $A \in \mathscr{A}$ by Aronszajn's theorem. Now it is easy to complete the proof.

We are not able to prove any relationship between the class $\mathscr{A}^0$ and the class of all Borel sparse sets.

Any strongly sparse set is clearly sparse and any Borel strongly sparse set belongs to $\mathscr{A}^0$.

We have no example of a set from $\mathscr{A}^0$ which would not be strongly sparse.

Now we shall prove that there exist sparse sets which are not strongly sparse. Let $f : R \to R$ be a 1-Lipschitz function (see e.g. [15]) for which there exists a dense set $D \subset R$ such that Dini's derivates of f at any point $x \in D$ satisfy the following relations:

$$f^+(x) = f^-(x) = 1\,, \quad f_+(x) = f_-(x) = 0\,. \tag{1}$$

Let $M \subset R^2$ be the graph of f. Put $a_1 = (2,1)$, $a_2 = (-2,1)$ and suppose that $M \subset \bigcup_{n=1}^{\infty} A_{1,n} \cup \bigcup_{n=1}^{\infty} A_{2,n}$, where $A_{1,n}(A_{2,n})$ are Lipschitz hypersurfaces associated with a_1 (a_2, respectively). Using the Baire theorem we obtain that there exists a set $A_{i,n}$, $i = 1, 2$, containing a nonempty relatively open subset of M. But this is a contradiction with (1). The argument can be easily generalized to the case of an arbitrary space X.

Now we shall prove that in any X there exists a strongly sparse set M which is not d.c.-sparse. Let $f : R \to R$ be a differentiable function whose derivative has unbounded variation on each interval. Let x, y be linearly independent elements of X and let C be a topological complement of Span $\{x, y\}$. Define $M = \{f(t)\, y + tx + c;\ t \in R, c \in C\}$. It is not difficult to prove that M is strongly sparse. We can use e.g. ([25], Lemma 1) and the easy fact that contg (M, x) is a subspace of X of codimension 1 for any $x \in M$. Suppose that M is d.c.-sparse. Let (v_n) be a sequence complete in X such that $v_1 = x$, $v_2 = y$. By Proposition 1 and the Baire theorem we obtain that a nonempty relatively open subset of M is contained in a d.c.-hypersurface associated with a v_n. Now it is easy to obtain a contradiction, using the property of f.

2. POINTS OF MULTIVALUEDNESS OF THE METRIC PROJECTION IN A SEPARABLE STRICTLY CONVEX BANACH SPACE

If f is a function in a Banach space X we define the one-sided directional derivative of f at $x \in X$ in the direction $v \in X$ as $D_v f(x) = \lim_{h \to 0_+} (f(x + hv) - f(x)) h^{-1}$. We also define "directional Dini derivatives" $\bar{D}_v f(x) = \limsup_{h \to 0_+} (f(x + hv) - f(x)) h^{-1}$, $\underline{D}_v f(x) = \liminf_{h \to 0_+} (f(x + hv) - f(x)) h^{-1}$. The usual two-sided directional derivative will be denote by $\partial_v f(x)$. Clearly $\partial_v f(x)$ exists if and only if $D_v f(x) = -D_{-v} f(x)$.

The following proposition is due to S. V. Konjagin [12].

Proposition 2. *Let X be a strictly convex space and $x \in R(F)$. Then there exists $v \in X$ such that*

$$\bar{D}_v d_F(x) + \bar{D}_{-v} d_F(x) < 0 \tag{2}$$

and consequently, $\partial_v d_F(x)$ does not exist.

Proof. Let $y_1 \in P_F(x)$, $y_2 \in P_F(x)$, $y_1 \neq y_2$. Put $v = y_2 - y_1$. Since X is strictly convex we have

$$D_v q(x - y_2) + D_{-v} q(x - y_1) < 0 . \tag{3}$$

Since $(d_F(x + hv) - d_F(x)) h^{-1} \leqq (q(x + hv - y_2) - q(x - y_2)) h^{-1}$ for $h > 0$, we obtain $\bar{D}_v d_F(x) \leqq D_v q(x - y_2)$. Similarly $\bar{D}_{-v} d_F(x) \leqq D_{-v} q(x - y_1)$ and therefore (3) yields (2).

We shall use also the following version of Proposition 2.

Proposition 3. *Let X be a separable strictly convex smooth space and let (a_n) be a complete sequence in X. Let $x \in R(F)$. Then there exists $v \in \{a_1, a_2, \dots\}$ such (2) holds.*

Proof. Let $y_1 \in P_F(x)$, $y_2 \in P_F(x)$, $y_1 \neq y_2$. Suppose that the conclusion of our proposition does not hold. Then the proof of Proposition 2 easily yields that $D_v q(x - y_2) + D_{-v} q(x - y_1) \geqq 0$ for any $v \in \{a_1, a_2, \dots\}$. Since X is smooth we have $-D_{-v} q(x - y_2) = D_v q(x - y_1)$ and therefore $D_v q(x - y_2) \geqq D_v q(x - y_1)$. Since the role of y_1, y_2 is symmetric we obtain $D_v q(x - y_2) = D_v q(x - y_1)$ for any $v \in \{a_1, a_2, \dots\}$. Consequently, $D_{y_2 - y_1} q(x - y_2) = D_{y_2 - y_1} q(x - y_1)$ and this is a contradiction with the strict convexity of X.

Lemma 1. *Let f be a 1-Lipschitz function on X, $x \in X$. Then the function $g(v) = \bar{D}_v f(x)$ is 1-Lipschitz on X.*

Proof. The function $g_t(v) = (f(x + tv) - f(x)) t^{-1}$ is 1-Lipschitz for any $t > 0$ (see e.g. [3], Chap. II, Proposition 6). Therefore $g(v) = \limsup_{h \to 0_+} \{g_t(v);\ 0 < t < h\}$ is 1-Lipschitz.

The following lemma is essentially due to S. V.Konjagin [12].

Lemma 2. *Let f be a Lipschitz function on X, $0 \neq v \in X$. Let $\bar{D}_v f(x) + \bar{D}_{-v} f(x) < < 0$ for any $x \in M \subset X$. Then M can be covered by countably many Lipschitz hypersurfaces associated with v.*

Proof. For $x \in M$ there exist rational numbers $p < q$ such that $\bar{D}_v f(x) < p < < q < -\bar{D}_{-v} f(x)$ and we can find a natural number n such that

$$(4) \qquad (f(x + tv) - f(x))\, t^{-1} < p < q < (f(x) - f(x - tv))\, t^{-1}$$

whenever $0 < t \leqq n^{-1}$.

If we denote by $M(p, q, n)$ the set of all $x \in M$ for which (4) holds, obviously $M = \bigcup M(p, q, n)$. Choose a triple p, q, n, a topological complement W of $V := := \text{Span}\,\{v\}$ and a countable covering (B_k) of X by sets of diameters less than $\|v\|/n\|\pi_V\|$, where π_V is the projector on V parallel to W. To complete the proof it is sufficient to show that for any k the set $M(p, q, n) \cap B_k$ is a subset of a Lipschitz hypersurface associated with v. For this purpose fix k and choose points $a \neq b$ from $M(p, q, n) \cap B_k$. Let $a = a_V + a_W$, $b = b_V + b_W$, where $a_V, b_V \in V$ and $a_W, b_W \in W$. We can suppose that $b_V - a_V = tv$, $t > 0$. By the definition of B_k we have $\|a_V - b_V\| \leqq \|\pi_V\|\,\|b - a\| \leqq \|v\|/n$ and consequently,

$$(5) \qquad 0 < t \leqq 1/n\,.$$

Put

$$D_1 = f(b) - f(b_W + a_V)\,, \qquad D_3 = f(b) - f(a_W + b_V)\,,$$

$$D_2 = f(b_W + a_V) - f(a)\,, \qquad D_4 = f(a_W + b_V) - f(a)\,.$$

Obviously

$$(6) \qquad D_1 + D_2 = D_3 + D_4 = f(b) - f(a)$$

and

$$(7) \qquad |D_2| \leqq K\|b_W - a_W\|\,, \quad |D_3| \leqq K\|b_W - a_W\|\,,$$

where K is a Lipschitz constant of f. By the definition of $M_{p,q,n}$ and (5) we have

$$(8) \qquad D_1 = f(b) - f(b - tv) > qt \quad \text{and} \quad D_4 = f(a + tv) - f(a) < pt\,.$$

On account of (6), (7) and (8) we obtain

$$(q - p)\,\|V\|^{-1}\,\|b_V - a_V\| = (q - p)\,t < D_1 - D_4 = D_3 - D_2 \leqq$$
$$\leqq 2K\|b_W - a_W\|\,.$$

Since for every Lipschitz function defined on a subset of a metric space there exists a Lipschitz extension on the the whole space (see e.g. [14]), there exists a Lipschitz function g on W such that $M(p, q, n) \cap B_k \subset \{g(w)\, v + w;\ w \in W\}$ and therefore

$M(p, q, n) \cap B_k$ is a subset of a Lipschitz hypersurface associated with v. The proof is complete.

Theorem 1. *Let X be a separable strictly convex space. Then $R(F)$ is a sparse set.*

Proof. Let C be a countable dense subset of X and $x \in R(F)$. By Proposition 2 and Lemma 1 there exists $v \in C$ such that (2) holds. Using Lemma 2 we obtain that $R(F)$ is a sparse set.

Theorem 2. *Let X be a separable strictly convex smooth space. Then $R(F)$ is strongly sparse.*

Proof. It is sufficient to use Proposition 3 and Lemma 2.

3. SOME NOTES ON DIFFERENTIABILITY OF THE DISTANCE FUNCTION IN GENERAL BANACH SPACES

Any distance function is a l-Lipschitz function. Consequently, by the well-known Rademacher's theorem $N(F)$ is a set of Lebesgue measure zero if $\dim X < \infty$. Aronszajn's theorem on the differentiation of Lipschitz function yields $N(F) \in \mathscr{A}$ if X is separable [3]. If $X = R^n$ then by the Mises theorem and Theorem 2 we obtain that $N(F)$ is strongly sparse (and, consequently, a first category set). This result will be improved in Section 5. In Section 4 it will be proved in the case of a separable X which has a uniformly Gateaux differentiable norm. S. Fitzpatrick ([9], Corollary 3.7) has proved that in reflexive locally uniformly convex smooth (or Frechet smooth) spaces d_F is Gateaux (Frechet, respectively) differentiable at all $x \notin F$ except a set of the first category. Note that by ([9], Theorem 3.1) and ([13], Corollary 5) in strongly convex (for the definition see [13]) smooth (or Frechet smooth) spaces the same theorem holds. It is an interesting problem to characterize the spaces which have this property. We sketch the proof of the following simple proposition concerning the problem mentioned above.

Proposition 4. *If X is not smooth then there exists a closed set $F \subset X$ for which $N(F)$ is a residual set.*

Proof. Choose an $a \in X$, $\|a\| = 1$, at which the norm q is not Gateaux differentiable. It is easy to prove that there exists a support hyperplane H of the closed unit ball at the point a and $v \in H$, $\|v\| = 1$, for which $D_v q(a) > 0$ and $D_{-v} q(a) > 0$. Let T be a topological complement to Span $\{v\}$ in H. Let g be a l-Lipschitz function on R which is not differentiable on a residual set $M \in R$ (see e.g. [15]). For any $c > 0$ define the Lipschitz hypersurface $H_c = \{c f(p)\, a + pv + t;\ p \in R,\ t \in T\}$. It is easy to prove the following geometrically obvious fact: For all sufficiently small c the distance from a point of the form $xa + pv + t$ to H_c equals $a - c f(p)$. Consequently, $\partial_v d_{H_c}(y)$ exists for no point y from the residual set $\{xa + pv + t;\ p \in M,\ t \in T,\ x \in R\}$.

We can also investigate the set $N_g^*(F)$ (or $N_f^*(F)$) of the points $x \in F$ at which d_F is not Gateaux (Frechet, respectively) differentiable. The following notes generalize an Erdös [8] observation.

Obviously $N_g^*(F) \subset N_f^*(F) \subset Bd\, F$. If $x \in Bd\, F$ then it is easy to see that $x \notin N_g^*(F)$ if and only if $D_v d_F(x) = 0$ for any $v \in X$. It is not difficult to characterize the magnitude of sets $N_g^*(F)$, $N_f^*(F)$.

Definition. We shall say that $A \subset X$ is *directionally porous* (*or porous*) if for any $a \in A$ there exist $0 \neq v \in X$ ((v_n), $\|v_n\| = 1$, respectively), $p > 0$, $t_n \searrow 0$ and $r_n \searrow 0$ such that $B(a + t_n v, pr_n) \subset B(a, r_n) - A$ (resp. $B(a + t_n v_n, pr_n) \subset B(a, r_n) - A$). ($B(s, r)$ denotes the open ball with a centre s and a radius r.)

It is easy to see that if $\dim X < \infty$ then the both notions coincide. The notion of a porous (and σ-porous) set was introduced by E. P. Dolženko [7] (see also [23]).

Proposition 5. *$A \subset X$ is a subset of some $N_g^*(F)$ if and only if A is directionally porous.*

Proof. If $x \in N_g^*(F)$ then $\bar{D}_v d_F(x) > 0$ for a $v \in X$. This implies that $N_g^*(F)$ is directionally porous. If A is directionally porous then obviously $A \subset N_g^*(\bar{A})$.

Similarly we obtain the following easy proposition.

Proposition 6. *$A \subset X$ is a subset of some $N_f^*(F)$ if and only if A is porous.*

4. DIFFERENTIABILITY OF THE DISTANCE FUNCTION IN SPACES WITH A UNIFORMLY GATEAUX DIFFERENTIABLE NORM

X is said to have *uniformly Gateuax* (UG) *differentiable norm in the direction v* if the limit $\lim_{t \to 0} (\|x + tv\| - \|x\|)\, t^{-1} = \partial_v q(x)$ is uniform on $\{x;\ \|x\| = 1\}$. X is said to have (UG) *differentiable norm* if it is (UG) differentiable in any $v \in X$.

The following simple facts are essentially known but we have not been able to reach a reference for them.

Proposition 7. *The following conditions are equivalent*:

(i) *q is (UG) differentiable in the direction v.*

(ii) $\lim_{t \to 0+} (\|x + tv\| - \|x\|)\, t^{-1} = D_v q(x)$ *is uniform on* $\{x;\ \|x\| = 1\}$.

(iii) *The function $g(v) = D_v q(x)$ is uniformly continuous on* $\{x;\ \|x\| = 1\}$.

(iv) *For any $r > 0$, g is uniformly continuous on* $\{x;\ \|x\| > r\}$.

(v) *For any $r > 0$,* $\lim_{t \to 0} (\|x + tv\| - \|x\|)\, t^{-1} = \partial_v q(x)$ *is uniform on* $\{x;\ \|x\| > r\}$.

Proof. The implications (i) ⇒ (ii), (v) ⇒ (i) are obvious. Since the functions $g_t(x) = (\|x + tv\| - \|x\|)\, t^{-1}$, $t > 0$, are uniformly continuous on X, we obtain that the implications (ii) ⇒ (iii) and (v) ⇒ (iv) are valid. The implication (iii) ⇒ (iv) follows from the fact that $D_v q(x) = D_v q(x/\|x\|)$ and the mapping $G(x) = x/\|x\|$ is $2/r$ – Lipschitz on $\{x;\ \|x\| > r\}$. Now we shall prove the implication (iv) $\Rightarrow$ (v). Let $r > 0$ and $\varepsilon > 0$ be fixed. By (iv) we can choose $\delta > 0$ such that $D_v f(x) - \varepsilon \leqq \leqq D_v f(x + vt) \leqq D_v f(x) + \varepsilon$ whenever $\|x\| > 2r$ and $-\delta < t < \delta$. From the classical Dini's theorem (see e.g. [20], p. 204) easily follows that

$$D_v f(x) - \varepsilon \leqq (f(x + vt) - f(x))\, t^{-1} \leqq D_v f(x) + \varepsilon$$

whenever $\|x\| > 2r$ and $-\delta < t < \delta$. Therefore (v) holds. The proof is complete.

Let $x \notin F$ and $v \in X$ be given. Denote by $\mathscr{F}$ the filter with the basis $\{\{y;\ \|y - x\| < < d_F(x) + \varepsilon\};\ \varepsilon > 0\}$ and put $L(F, x, v) = \liminf_{y,\mathscr{F}} D_v q(x - y)$.

The following lemma is obvious.

Lemma 3. $L(F, x, v) = \lim_{\varepsilon \to 0+} (\inf \{D_v q(x - y);\quad y \in F,\quad \|y - x\| < d_F(x) + \varepsilon\}) = = \min \{\liminf_{n \to \infty} D_v q(x - y_n);\ (y_n) \subset F,\ \|y_n - x\| \to d_F(x)\} = \min \{\lim_{n \to \infty} D_v q(x - y_n);\ (y_n) \subset F,\ \|y_n - x\| \to d_F(x)\}$.

The following lemma is an easy consequence of Lemma 3 and Lemma 1.

Lemma 4. $|L(F, x, v)| \leqq 1$. *The function* $L(v) := L(F, x, v)$ *is* 1-*Lipschitz. If* X *is smooth, then* $L(v)$ *is a concave function.*

Theorem 3. *Let* $v \in X$. *Then the following conditions are equivalent*:

(i) *X has a* (UG) *differentiable norm in the direction* v.

(ii) *For any closed* $\emptyset \neq F \subset X$ *and* $x \notin F$, $D_v d_F(x)$ *exists.*

If these conditions hold, then $D_v d_F(x) = L(F, x, v)$.

Proof. a) Suppose that (i) holds and F, $x \notin F$ are fixed. It is sufficient to prove

(9) $$\underline{D}_v d_F(x) \geqq L(F, x, v)$$

and

(10) $$\overline{D}_v d_F(x) \leqq L(F, x, v)\,.$$

Choose $t_n \searrow 0$ such that $\lim_{n \to \infty} (d_F(x + t_n v) - d_F(x))\, t_n^{-1} = \underline{D}_v d_F(x)$. Choose further $(y_n) \subset F$ such that

(11) $$\lim_{n \to \infty} (\|x + t_n v - y_n\| - d_F(x))\, t_n^{-1} = \underline{D}_v d_F(x)\,.$$

Since $d_F(x) \leqq \|x - y_n\|$ we obtain

$$\liminf_{n \to \infty} (\|x + t_n v - y_n\| - \|x - y_n\|)\, t_n^{-1} \leqq \underline{D}_v d_F(x)$$

and using the convexity of the norm we infer

(12) $$\liminf_{n \to \infty} D_v q(x - y_n) \leqq \underline{D}_v d_F(x)\,.$$

The inequality $\|x - y_n\| \leqq \|x - y_n + t_n v\| + \|t_n v\|$ together with (11) easily yields $\|x - y_n\| \to d_F(x)$. This fact and (12) yield (9).

By Lemma 3 we can choose $(y_n) \subset F$ such that $\|x - y_n\| \to d_F(x)$ and $\lim_{n\to\infty} D_v(x - - y_n) = L(F, x, v)$. We can clearly suppose that $\|x - y_n\| > r$ for some $r > 0$ and any n. By Proposition 7, (v), for any $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$|D_v q(x - y_n) - (\|x - y_n + tv\| - \|x - y_n\|)\, t^{-1}| < \varepsilon$$

for any $0 < t < t_0$ and any index n. Let $t \in (0, t_0)$ be fixed and let n be an arbitrary index. Then $(d_F(x + tv) - d_F(x))\, t^{-1} \leqq (\|x - y_n + tv\| - \|x - y_n\|)\, t^{-1} + (\|x - y_n\| - d_F(x))\, t^{-1} \leqq D_v q(x - y_n) + \varepsilon + (\|x - y_n\| - d_F(x))\, t^{-1}$ and consequently, $(d_F(x + tv) - d_F(x))\, t^{-1} \leqq L(F, x, v) + \varepsilon$. From this inequality (10) easily follows.

b) Suppose that (i) does not hold. Then by Proposition 7, (ii), we can choose $\varepsilon > 0$, $(x_n) \subset X$, $(t_n) \subset R$ such that $\|x_n\| = 1$, $0 < t_n < 1/n$ and $(\|x_n + t_n v\| - - 1)\, t_n^{-1} - D_v q(x_n) > 3\varepsilon$. Using the convexity of the norm we obtain

(13) $$(\|x_n + tv\| - 1)\, t^{-1} - D_v q(x_n) > 3\varepsilon \quad \text{for} \quad 1/n \leqq t\,.$$

We can suppose that $\lim_{n\to\infty} D_v q(x_n) := a$ exists and $|D_v q(x_n) - a| < \varepsilon$ for any n. Put $n_1 = 1$ and choose $0 < d_1 < 1$ such that $(q(x_{n_1} + d_1 v) - 1)\, d_1^{-1} < a + \varepsilon$. Further, choose $0 < \lambda_1 < 1$ for which

(14) $$(q(x_{n_1} + d_1 v) - 1 + \lambda_1)\, d_1^{-1} < a + \varepsilon\,,$$

and $0 < u_1 < d_1$ such that $(q((1 + \lambda_1)\, x_{n_1} + uv) - 1)\, u^{-1} > a + 2\varepsilon$ for any $0 < < u \leqq u_1$. Let n_2 be an index such that $1/n_2 < u_1/2$ and choose $0 < d_2 < u_1$, $d_2 < 1/2$, $0 < \lambda_2 < 1/2$ for which $(q(x_{n_2} + d_2 v) - 1 + \lambda_2)\, d_2^{-1} < a + \varepsilon$. Proceeding in this way we obtain sequences $(n_i), (\lambda_i), (d_i), (u_i)$ such that $d_1 > u_1 > d_2 > \ldots > 0$, $d_i \to 0$, $\lambda_i \to 0$, $1 > \lambda_i > 0$,

(15) $$1/n_{i+1} < u_i/2\,,$$

(16) $$(q(x_{n_i} + d_i v) - 1 + \lambda_i)\, d_i^{-1} < a + \varepsilon$$

and

(17) $$(q((1 + \lambda_i)\, x_{n_i} + u_j v) - 1)\, u_i^{-1} > a + 2\varepsilon \quad \text{for} \quad i \leqq j\,.$$

Put $y_i = -(1 + \lambda_i)\, x_{n_i}$ and $F = \overline{\{y_1, y_2, \ldots\}}$. Obviously $d_F(0) = 1$. On account of (15), $(d_F(d_i v) - 1)\, d_i^{-1} \leqq (q(d_i v - y_i) - 1)\, d_i^{-1} = (q(d_i v + x_{n_i} + \lambda_i x_{n_i}) - 1)\,.\, d_i^{-1} \leqq (q(x_{n_i} + d_i v) - 1 + \lambda_i)\, d_i^{-1} < a + \varepsilon$ and consequently, $\underline{D}_v d_F(0) \leqq a + \varepsilon$. On the other hand, for $i > j$ we have by (15) the inequality $1/n_i < u_j/(1 + \lambda_i)$ and therefore by (13),

$$(1 + \lambda_i)\, u_j^{-1}(q(x_{n_i} + (1 + \lambda_i)^{-1}\, u_j v) - 1) > a + 2\varepsilon\,.$$

Consequently,

$$\left(q\left((1 + \lambda_i) x_{n_i} + u_j v\right) - 1\right) u_j^{-1} =$$
$$= \left((1 + \lambda_i)\left(q\left(x_{n_i} + (1 + \lambda_i)^{-1} u_j v\right) - 1\right) + \lambda_i\right) u_j^{-1} > a + 2\varepsilon\,.$$

By (17) the same inequality also holds for $i \leqq j$ and consequently, $(d_F(u_j v) - 1)u_j^{-1} \geqq a + 2\varepsilon$ for any j. Therefore $\bar{D}_v d_F(0) \geqq a + 2\varepsilon$ and $D_v d_F(0)$ does not exist.

Corollary. *If X is finite dimensional and $\partial_v q(x)$ exists for any $x \neq 0$ then $D_v d_F(x)$ exists for any $x \notin F$ and $D_v d_F(x) = \inf\{D_v q(x - y);\ y \in P_F(x)\}$.*

From Theorem 3 and Lemma 4 we immediately obtain the following theorem.

Theorem 3*. *The following conditions are equivalent:*

(i) *X has a (UG) differentiable norm.*

(ii) *For any closed $\emptyset \neq F \subset X$ and for any $x \notin F$, $v \in X$, $D_v d_F(x)$ exists.*

If these conditions hold, then $D_v d_F(x) = L(F, x, v)$ and the function $g(v) = D_v d_F(x)$ is a 1-Lipschitz concave function.

Corollary. *If X is a smooth finite dimensional space, then $D_v d_F(x) = \inf\{D_v q(x - y);\ \|x - y\| = d_F(x)\}$ for any $x \notin F$, $v \in X$. The function $g(v) = D_v d_F(x)$ is 1-Lipschitz and concave.*

The special case of Corollary is the Mises theorem.

Theorem 4. *Let X be a separable space with a (UG) differentiable norm. Then $N(F)$ is strongly sparse.*

Proof. Let (a_n) be a sequence complete in X and denote by A_n the set of all $x \in F$ for which $\partial_{a_n} d_F(x)$ does not exist. By Theorem 3 and Lemma 2, A_n can be covered by countably many Lipschitz hypersurfaces associated with a_n. If $\partial_{a_n} d_F(x)$ exists for any n, then d_F is Gateaux differentiable at x, since by Theorem 3, $g(v)$ is a continuous concave function and $D_v d_F(x) = D_v g(0) = g(v)$ for any $v \in X$. Therefore $N(F) \subset \bigcup_{n=1}^{\infty} A_n$ and the proof is complete.

Corollary. *Let X be a smooth n-dimensional space. Then $N(F)$ has a σ-finite $(n - 1)$-dimensional (Hausdorff) measure.*

It is easy to see that Theorem 4 can be slightly generalized in the following way. The notion of strongly sparse subset of an affine manifold is defined in the obvious way.

Theorem 4*. *Let X be a Banach space and $S \subset X$ a separable subspace such that the norm of X is (UG) differentiable in any direction $v \in S$. Let A be an affine manifold of the form $A = a + S$. Then the set of points $x \in A - F$ at which the restriction d_F/A is not Gateaux differentiable is a strongly sparse subset of A.*

5. SMOOTHNESS OF THE DISTANCE FUNCTION IN HILBERT AND SOME FINITE DIMENSIONAL SPACES

The main results of the present section are stated without proofs in [27]. The main method which generalizes an Asplund's argument [4] was independently used in [1].

Lemma 5. *Let H be a real Hilbert space and $G \subset H$ an open convex set. Let g be a Frechet differentiable function on G and let the function $x \to g'(x)$ be K-Lipschitz on G. Then the function $f(x) := K\|x\|^2/2 - g(x)$ is convex on G.*

Proof. Let $a \in G$ and $\|v\| = 1$. Put $h(t) = f(a + tv)$. Let $t_2 > t_1$ and $a + t_1 v$, $a + t_2 v \in G$. Then $h'(t_2) - h'(t_1) = K(a + t_2 v, v) - K(a + t_1 v, v) - (g'(a + t_2 v), v) + (g'(a + t_1 v), v)) \geqq K(t_2 - t_1) - \|g'(a + t_2 v) - g'(a + t_1 v)\| \geqq 0$. Therefore h is convex on the interval $\{t;\ a + tv \in G\}$ and consequently f is convex on G.

A function on a convex open subset of X is said to be *δ-convex* if it is the difference of two continuous convex functions on G.

Theorem 5. *Let $(X, \|\cdot\|)$ be a Banach space such that the Frechet derivative of the norm $\|\cdot\|$ is C-Lipschitz on $\{x;\ \|x\| = 1\}$. Suppose that on X there exists an equivalent Hilbert norm $\|\cdot\|_h$. Then d_F is locally δ-convex on $X - F$.*

Proof. Choose $B > 0$ such that $\|x\|_h/B < \|x\| < B\|x\|_h$ for any $x \in X$. Choose $x_0 \in X - F$ and put $G = \{x;\ \|x - x_0\| < d_F(x_0)/2\}$. Since the mapping $x \to x/\|x\|$ is $4/d_F(x_0)$-Lipschitz on $M := \{x;\ \|x\| > d_F(x_0)/2\}$, we obtain that $\|\cdot\| : (X, \|\cdot\|) \to R$ has a $4C/d_F(x_0)$-Lipschitz derivative on M and consequently $\|\cdot\| : (X, \|\cdot\|_h) \to R$ has a $4B^2C/d_F(x_0)$-Lipschitz derivative on M. Therefore each function $g_y : (X, \|\cdot\|_h) \to R$, $g_y(x) = \|x - y\|$, $y \in F$, has a K-Lipschitz derivative on G, where $K = 4B^2C/d_F(x_0)$. By Lemma 5 each function $x \to K\|x\|_h^2/2 - \|x - y\|$, $y \in F$, is convex on G and therefore the function

$$V(x) := K\|x\|_h^2/2 - d_F(x) - \sup\{K\|x\|_h^2/2 - \|x - y\|;\ y \in F\}$$

is a continuous convex function on G. Thus the function $d_F(x) = K\|x\|_h^2/2 - V(x)$ is δ-convex on G.

Since any Hilbert space is an Asplund space we obtain the following proposition which is a special case of ([9], Corollary 3.7).

Corollary 1. *Let X be a Hilbert space. Then the set of points $x \notin F$ at which d_F is not Frechet differentiable is a first category set.*

If we define the notion of a d.c.-sparse subset of an affine manifold in the obvious way, then from Theorem 5 and ([26], Theorem 2) we obtain the following corollary.

Corollary 2. *Let X be a Hilbert space and let $S \subset X$ be a separable affine manifold. Then the set of points $x \in S - F$ at which d_F/S is not Gateaux differentiable is a d.c.-sparse subset of S.*

Using the Buseman-Feller-Aleksandrov theorem [2] on twice differentiability of convex functions we obtain the following corollary.

Corollary 3. *Let X be a Hilbert space and let $M \subset X$ be a finite dimensional affine manifold. Then d_F/M is twice differentiable (in any one of the two most natural generalized senses, cf. [9]) almost everywhere on $M - F$.*

Corollary 4. *Let X be a finite dimensional space such that the Frechet derivative of the norm $\|\cdot\|$ is Lipschitz on $\{x;\ \|x\| = 1\}$. Then $N(F)$ is d.c.-sparse and d_F is twice differentiable almost everywhere on $X - F$.*

Slightly modifying in the obvious way the proof of Theorem 5 we can obtain the following theorem.

Theorem 5*. *Let X be a Banach space such that the Frechet derivative of the norm $\|\cdot\|$ is Lipschitz on $\{x;\ \|x\| = 1\}$. Let $M \subset X$ be a finite dimensional affine manifold. Then d_F/M is locally δ-convex on $M - F$.*

The following proposition on "differentiability of the distance function on δ-convex curves" will be applied in the next section. Note that the same argument can be used to obtain an analogous result for "k-dimensional δ-convex surfaces in R^n".

Proposition 8. *Let $X = R^n$. Let $f_1, \ldots, f_n$ be locally δ-convex function on (a, b) such that $(f_1(x), \ldots, f_n(x)) \notin F$ for any $x \in (a, b)$. Then $d_F(f_1(x), \ldots, f_n(x))$ is differentiable on (a, b) except for a countable set.*

Proof. Using Theorem 5 and a P. Hartman's theorem ([11], II) we obtain that $d_F(f_1(x), \ldots, f_n(x))$ is locally δ-convex on (a, b).

6. POINTS OF MULTIVALUEDNESS OF THE METRIC PROJECTION IN SEPARABLE HILBERT SPACES AND SOME FINITE DIMENSIONAL SPACES

If X is strictly convex then $R(F) \subset N(F)$ (e.g. by Proposition 2). Therefore Corollary 2 and Corollary 4 of Theorem 5 immediately yield the following theorems.

Theorem 6. *Let X be a separable Hilbert space. Then $R(F)$ is a d.c.-sparse set.*

Theorem 7. *Let X be a strictly convex finite dimensional space such that the derivative of the norm $\|\cdot\|$ is Lipschitz on $\{x;\ \|x\| = 1\}$. Then $R(F)$ is a d.c.-sparse set.*

The following two propositions illustrate the sharpness of Theorem 6.

Proposition 4. *Let H be a Hilbert space and $C \subset H$ a closed convex set for which* Int $C \neq \emptyset$. *Then there exists a closed set $F \subset H$ such that* Bd$C \subset R(F)$.

Proof. Choose a point $a \in \text{Int}\, C$. For any $b \in \text{Bd}C$ choose a hyperplane H_b supporting C at the point b and denote by x_b the point symmetric to the point a with respect to the hyperplane H_b. Put $F = \overline{a \cup \bigcup_{b \in \mathrm{Bd}C} \{x_b\}}$. Now it is sufficient to prove that for any $b, c \in \mathrm{Bd}C$, $b \neq c$, the inequality $\|b - a\| = \|b - x_b\| \leqq \|b - x_c\|$ holds. But the inequality $\|b - a\| \leqq \|b - x_c\|$ is obvious, since a and x_c are symmetric with respect to H_c and b, a belong to the same closed half-space determined by the hyperplane H_c.

Proposition 10. *Let H be a Hilbert space. Let $M \subset H$ be a subspace of codimension 1 and let $0 \neq a \in M^{\perp}$. Let $f : M \to R$ be Frechet differentiable on M and let $f(x)$ be K-Lipschitz on M. Then there exists a closed set $F \subset H$ such that*

$$S := \{x + f(x)\, a;\ x \in M\} \subset R(F)\,.$$

Proof. For $x \in M$ denote by T_x the tangent hyperplane to S at the point $x + f(x)\, a$. Denote by n_x the unit vector orthogonal to T_x for which $(a, n_x) > 0$. Put $F = \overline{\bigcup\{x + f(x)\, a \pm K^{-1} n_x;\ x \in M\}}$. Let $g(x)$ be the "lower" function in M implicitly defined by the sphere with the centre $x + f(x) + K^{-1} n_x$ and the radius K^{-1}. Analogously as in Lemma 5 we can easily prove that $g(x) - f(x)$ is convex on the domain D_g of g and hence $g(x) \geqq f(x)$ for $x \in D_g$. Consequently, we obtain that $d_S(x + f(x)\, a + K^{-1} n_x) = K^{-1}$ for $x \in M$. Similarly we obtain that $d_S(x + f(x)\, a - K^{-1} n_x) = K^{-1}$ for $x \in M$. Consequently, $S \subset R(F)$.

We have not been able to solve the following problem.

Problem 1. *Let $A \subset R^2$ be a δ-convex hypersurface. Is A (locally) a subset of an $R(F)$?*

The following proposition (which can be in an obvious way generalized to the case of R^n, $n \geqq 3$) can be of interest in connection with the preceding problem.

Proposition 11. *Let $C \subset R^2$ be a δ-convex hypersurface and let $F \subset R^2$ be a closed set. Then for all $x \in C \cap R(F)$ except for a countable set we have $P_F(x) = \{P_1(x), P_2(x)\}$, where the points $P_1(x)$, $P_2(x)$ are symmetric with respect to the tangent to C at x.*

Proof. For all points $x \in C \cap R(F)$ except for a countable set the tangent to C exists. Let v_x be the direction of this tangent. It is not difficult to prove that Proposition 8 yields that $\partial_{v_x} d_F(x)$ exists for any $x \in C \cap R(F)$ except for a countable set. Consequently, the Mises theorem implies the conclusion of our proposition.

The following example shows that there exist closed sets F_n such that $\bigcup_{n=1}^{\infty} R(F_n)$ is a subset of no $R(F)$.

Example 1. Put $A = (-1, 1) \times \{0, \mp 1, \mp 1/2, \mp 1/3, \ldots\}$. Clearly $A \subset \bigcup_{n=1}^{\infty} R(F_n)$ for some F_n. Suppose that $A \subset R(F)$ for some F and choose $b \in P_F(0)$. Clearly no point from the segment $s = \overline{b, 0}$ belongs to $R(F) - \{0\}$. But $s \cap A$ is infinite and this is a contradiction.

Example 2. Put $S_n = (-1, 1) \times \{1/n\}$, $n = 1, 2, \ldots$, and $S_0 = (-1, 1) \times \{0\}$. Let $A_n \subset S_n$ be a set such that any relatively open subset of S_n contains uncountably many points from A_n. Then $A := \bigcup_{n=0}^{\infty} A_n$ is a subset of no $R(F)$. Suppose on the contrary $A \subset R(F)$. By Proposition 11 we can choose $a = (t, 0)$, $|t| < 1$, such that $P_F(a) = \{(x, y), (x, -y)\}$, $0 < x < 1$, $y > 0$. Choose n for which there exists a point $c \in S_n$ in the relative interior of the segment $\overline{a, (x, y)}$. By Proposition 11 there exists a sequence $(c_k) \subset S_n$, $c_k \to c$, such that $P_F = \{P_1(c_k), P_2(c_k)\}$, where $P_1(c_k)$, $P_2(c_k)$ are symmetric with respect to S_n. Since $P_F(c) = \{(x, y)\}$ and P_F is upper semicontinuous we obtain a contradiction.

The examples indicate that the problem of characterization of the magnitude of sets $R(F)$ (i.e. the problem of characterization of the class $\{A; A \supset R(F)$ for some $F\}$) is very difficult.

We pose the following easier problem.

Problem 2. *Let H be a separable Hilbert space. What is the smallest σ-ideal I which contains all sets $R(F)$?*

Theorem 6 and Propositions 9, 10 suggest the following natural conjecture.

(i) I is the class of all d.c.-sparse sets.

(ii) I is the class of all sets which can be covered by countably many Rešetnjak's hypersurfaces. We say that $A \subset H$ is a Rešetnjak's hypersurface (cf. [18]) if A is a Lipschitz hypersurface and the function $f : Z \to R$ from the definition of a Lipschitz hypersurface is of the form $f(x) = K\|x\|^2 - c(x)$, where $K > 0$ and $c(x)$ is a continuous convex function on Z.

In [12] Konjagin clasified the points from $R(F)$ in the following way. Let k be a positive integer. Then we denote by $R_k(F)$ ($R^k(F)$) the set of points x for which the dimension (the codimension) of the closed affine manifold spanned by $P_F(x)$ is at least k (at most k, respectively).

Define further $R^k_*(F)$ as the set of points x for which the convex closure of $P_F(x)$ contains a ball in an affine manifold of codimension k. Clearly $R^k_*(F) \subset R^k(F)$.

If H is a Hilbert space then there exists [4] a continuous convex function f (namely, $f(x) = 1/2(\|x\|^2 - d_F^2(x))$ such that $P_F(x) \subset \partial f(x)$. Using Proposition 2 from [26] we easily obtain the following theorem.

Theorem 8. *If X is a separable Hilbert space then $R_k(F)$ can be covered by countably many d.c.-surfaces of codimension k.*

By a d.c.-surface of codimension k we mean an $(\infty - k)$-dimensional $(c - c)$-surface from [26].

In the case $X = R^n$, Theorem 8 improves result by Erdös (cf. Introduction).

Since the subdifferential $x \to \partial f(x)$ of a continuous convex function is a monotone operator and $\partial f(x) \subset X^*$ is a closed convex set for any x, we obtain by Theorem 3 from [24] the following theorem.

Theorem 9. *If X is a separable Hilbert space then $R_*^k(F)$ can be covered by countably many Lipschitz surfaces of dimension k.*

For the definition of a Lipschitz surface of dimension k see ([24], p. 181). We have no analogous result concerning $R^k(F)$.

References

[1] *T. Abatzoglou:* Finite-dimensional Banach spaces with a.e. differentiable metric projection, Proc. Amer. Math. Soc. *78* (1980), 492—496.

[2] *A. D. Aleksandrov:* Existence almost everywhere of the second differential of a convex function and some properties of convex surfaces connected with it (Russian), Učenye Zapiski Leningrad. Gos. Univ. Math. Ser. *6* (1939), 3—35.

[3] *N. Aronszajn:* Differentiability of Lipschitzian mappings between Banach spaces, Studia Math. *57* (1976), 147—190.

[4] *E. Asplund:* Differentiability of the metric projection in finite-dimensional Euclidean spaces, Proc. Amer. Math. Soc. *38* (1973), 218—219.

[5] *H. Blumberg:* Exceptional sets, Fund. Math. *32* (1939), 3—32.

[6] *G. Bouligand:* Sur la distance d'un point variable a un ensemble fixe, C.R. Acad. Sci. Paris *206* (1938), 552—554.

[7] *E. P. Dolženko:* Boundary properties of arbitrary functions (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. *31* (1967), 3—14.

[8] *P. Erdös:* On the Hausdorff dimension of some sets in Euclidean space, Bull. Amer. Math. Soc. *52* (1946), 107—109.

[9] *S. Fitzpatrick:* Metric projection and the differentiability of distance functions, Bull. Austral. Math. Soc. *22* (1980), 291—312.

[10] *S. Golab:* Sur la fonction representant la distance d'un point variable a un ensemble fixe, C. R. Acad. Sci. Paris *206* (1938), 406—408.

[11] *P. Hartman:* On Functions representable as a Difference of Convex Functions, Pacific J. Math. *9* (1959), 707—713.

[12] *S. V. Konjagin:* Approximation properties of arbitrary sets in Banach spaces, Dokl. Akad. Nauk SSSR *239* (1978), No. 2, 261—264, Soviet. Math. Dokl. *19* (1978), No. 2, 309—312.

[13] *S. V. Konjagin:* On approximation properties of closed sets in Banach spaces and the characterization of strongly convex spaces, (Russian), Dokl. Akad. Nauk SSSR *251* (1980), No. 2, 276—280, Soviet. Math. Dokl. *21* (1980), No. 2, 418—422.

[14] *E. J. Mc Shane:* Extension of range of functions, Bull. Amer. Math. Soc. *40* (1934), 837—842.

[15] *F. Mignot:* Controle dans les inéqualitions variationelles elliptiques, J. Functional Analysis *22* (1976), 130—185.

[16] *M. R. de Mises:* La base géométrique du théoreme de M. Mandelbrojt sur les points singuliers d'une fonction analytique, C. R. Acad. Sci. Paris *205* (1937), 1353—1355.

[17] *R. R. Phelps:* Gaussian null sets and differentiability of Lipschitz maps on Banach spaces, Pacific J. Math. *18* (1978), 523—531.

[18] *Ju. G. Rešetnjak:* On a generalization of convex surfaces (Russian), Mat. Sbornik *40 (82)* (1956), 381—398.

[19] *A. W. Roberts* and *D. E. Varberg:* Convex functions, New York, 1973.
[20] *S. Saks:* Theory of the Integral, New York, 1937.
[21] *W. H. Young:* La symétrie de structure des fonctions de variables réeles, Bull. Sci. Math. *52* (1928), 265—280.
[22] *L. Zajíček:* On cluster sets of arbitrary functions, Fund. Math. *83* (1974), 197—217.
[23] *L. Zajíček:* Sets of σ-porosity and sets of σ-porosity (q), Čas. pěst. mat. *101* (1976), 350—359.
[24] *L. Zajíček:* On the points of multiplicity of monotone operators, Comment. Math. Univ. Carolinae *19* (1978), 179—189.
[25] *L. Zajíček:* On the points of multivaluedness of metric projections in separable Banach spaces, Comment. Math. Univ. Carolinae *19* (1978), 513—523.
[26] *L. Zajíček:* On the differentiation of convex functions in finite and infinite dimensional spaces, Czechoslovak Math. J. *29* (*104*) (1979), 340—348.
[27] *L. Zajíček:* On metric projections and distance functions in Banach spaces, Abstracta of Eighth winter school on abstract analysis (1980), 207—208, Prague 1980.

Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).