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ON PSEUDONORMABILITY OF SOME PARTICULAR CLASSES OF SPACES

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It is known that the concept of locally convex spaces is equivalent to the concept of pseudonormed spaces (*espaces à pseudonormes* or *espaces à seminormes*). In the first case, topological methods are widely used. The aim of this note is to prove, by purely analytic methods, a few facts on pseudonormed spaces which can be considered as known when transformulated into the topological language. In particular, we determine families of pseudonorms which generate convergences in inductive limits of increasing sequences of normed and pseudonormed spaces.

1. A family P of pseudonorms on a linear space E generates a convergence in the following way

$$x_i \to^P x \quad \textit{iff} \quad p(x_i - x) \quad \textit{tends to zero for every} \quad p \in P\,.$$

We say that a convergence G on a linear space E is a *pseudonorm convergence* if it is equal to the convergence generated by the family of all continuous pseudonorms on E. Such a convergence is also called the locally convex convergence ([1]). In this note, by *continuity* we always mean the sequential continuity. We call two convergences *identical* or *equal*, if each sequence convergent in the sense of any of these convergences converges to the same element in the sense of the other convergence.

Clearly, every convergence generated by any family of pseudonorms, not necessarily all pseudonorms, is a pseudonorm convergence.

2. Let $(E, \|\cdot\|)$ be a linear normed space and let F be a linear subspace of E. It is easy to show that *the distance of a point* $x \in E$ *from the subspace* F,

$$D(x) = \inf\{\|x - \xi\|;\ \xi \in F\} \tag{1}$$

is a pseudonorm on the space E.

Let $E_1, E_2, \ldots$ be an increasing sequence of closed linear subspaces of a linear normed space $(E, \|\cdot\|)$ such that $E = \bigcup E_n$. By D_i we denote the distance from the subspace E_i. (In (1) we then put E_i instead of F.) By D_0 we mean the distance from the origin, so that $D_0(x) = \|x\|$.

Theorem 1. *The convergence*

(H) $x_i \to^H x$ *iff* $\|x_i - x\|$ *tends to zero and there exists an index* n *such that* $x_i \in E_n$ *for all* i,

is equal to the convergence generated by the family of all pseudonorms of the form

$$p_a(x) = \sum_{i=0}^{\infty} \alpha_i D_i(x), \tag{2}$$

where $a = (\alpha_0, \alpha_1, \ldots)$ *is any increasing sequence of positive integers.*

Proof. Since the space E is the union of all subspaces E_n, the sum in (2) is always finite. The functionals p_a are pseudonorms, because so are the D_i's.

Now we shall prove the identity of both convergences. For every pseudonorm p_a we have

$$\|x\| \leqq p_a(x), \tag{3}$$

because $\alpha_0 \geqq 1$ and $D_0(x) = \|x\|$. If $x \in E_n$ then

$$p_a(x) \leqq \left(\sum_{i=0}^{n-1} \alpha_i\right) \|x\|, \tag{4}$$

as $D_0(x) \geqq D_1(x) \geqq \ldots \geqq D_{n-1}(x)$ and $D_k(x) = 0$ for $k \geqq n$. From (3) it follows that the pseudonorms p_a are norms.

Since the inequalities (3) and (4) hold for all n, the norms $\|\cdot\|$ and p_a are equivalent on each subspace E_n. Let, for a sequence x_i, there exist an index n such that $x_i \in E_n$ for all i. Then x_i is convergent in the norm $\|\cdot\|$, iff x_i is convergent in the norm p_a. This is true for each norm p_a.

Assume that a sequence x_i is not G-convergent. If the sequence x_i is contained in E_n for some index n, then, by the above remark, x_i is not convergent in any norm p_a. It remains to consider the case when there is no index n such that $x_i \in E_n$ for all i. We may assume, for simplicity, that $x_i \in E_{i+1} \setminus E_i$ for $i = 1, 2, \ldots$. Since the subspaces E_n are closed, the distance $D_i(x_i)$ si always positive. Let $\alpha_0, \alpha_1, \ldots$ be an increasing sequence of positive integers such that

$$\alpha_i \geqq \frac{i}{D_i(x_i)}$$

for $i = 1, 2, \ldots$. Then

$$p_a(x_i) = \sum_{k=0}^{\infty} \alpha_k D_k(x_i) \geqq i .$$

Thus, there exists a pseudonorm p_a such that the sequence $p_a(x_i)$ is unbounded.

From Theorem 1 we can derive the following

Corollary. *Let $(E_n, \|\cdot\|_n)$ be a sequence of linear normed spaces such that the space E_n is a closed linear subspace of $(E_{n+1}, \|\cdot\|_{n+1})$ for $n = 1, 2, \ldots$, and that the restriction of the norm $\|\cdot\|_{n+1}$ to E_n is equivalent to the norm $\|\cdot\|_n$ for $n = 1, 2, \ldots$. Let $E = \bigcup E_n$. Then the convergence*

(I) *$x_i \to^I x$ iff there exists an index n such that $x_i, x \in E_n$ and $\|x_i - x\|_n$ tends to zero,*

is a pseudonorm convergence in E.

To prove Corollary we shall use

Lemma. *Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be a pair of linear normed spaces such that E_1 is a subspace of the space $(E_2, \|\cdot\|_2)$. If the restriction of the norm $\|\cdot\|_2$ to E_1 is equivalent to the norm $\|\cdot\|_1$ then there exists a norm $\|\cdot\|_3$ on E_2, equivalent to the norm $\|\cdot\|_2$, such that $\|x\|_1 = \|x\|_3$ for $x \in E_1$.*

Proof of Lemma. We may assume that $\|x\|_1 \leqq \|x\|_2$ for $x \in E_1$. If it is not true, we replace the norm $\|\cdot\|_2$ by the norm $M\|\cdot\|_2$, where M is a positive number such that $\|x\|_1 \leqq M\|x\|_2$ for $x \in E_1$. Such a number M exists because the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on E_1.

The functional

$$\|x\|_3 = \inf\{\|\xi_1\|_1 + \|\xi_2\|_2;\ x = \xi_1 + \xi_2,\ \xi_1 \in E_1\}$$

is a norm on E_2 and possesses the required properties.

In fact, since the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on E_1 there exists a positive number $N > 1$ such that $\|x\|_2 \leqq N\|x\|_1$ for $x \in E_1$. Let $\xi_1 \in E_1$, $\xi_2, x \in E_2$ and $x = \xi_1 + \xi_2$. The following inequalities hold:

$$\|x\|_2 \leqq \|\xi_1\|_2 + \|\xi_2\|_2 \leqq N\|\xi_1\|_1 + \|\xi_2\|_2 \leqq N\|\xi_1\|_1 + N\|\xi_2\|_2\,.$$

We hence have

$$\|x\|_2 \leqq \inf\{N(\|\xi_1\|_1 + \|\xi_2\|_2);\ x = \xi_1 + \xi_2,\ \xi_1 \in E_1\} = N\|x\|_3\,. \tag{5}$$

It follows from (5) that, if $\|x\|_3 = 0$, then $x = 0$. The homogeneity of the functional $\|\cdot\|_3$ follows directly from homogeneity of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Let

$$x = \xi_1 + \xi_2\,, \quad y = \eta_1 + \eta_2 \quad \text{and} \quad \xi_1, \eta_1 \in E_1\,. \tag{6}$$

Then

$$\|x + y\|_3 \leqq \|\xi_1 + \eta_1\|_1 + \|\xi_2 + \eta_2\|_2 \leqq (\|\xi_1\|_1 + \|\xi_2\|_2) + (\|\eta_1\|_1 + \|\eta_2\|_2)\,.$$

Since these inequalities hold for all $\xi_1, \xi_2, \eta_1, \eta_2$ satisfying (6), the following inequality also holds

$$\|x + y\|_3 \leqq \inf\{\|\xi_1\|_1 + \|\xi_2\|_2;\ x = \xi_1 + \xi_2,\ \xi_1 \in E_1\} + \\ + \inf\{\|\eta_1\|_1 + \|\eta_2\|_2;\ y = \eta_1 + \eta_2,\ \eta_1 \in E_1\}\,.$$

This is the triangle inequality for the functional $\|\cdot\|_3$. Hence the functional $\|\cdot\|_3$ is a norm on the space E_2.

For $x \in E_1$ we have

$$\inf\{\|\xi_1\|_1 + \|\xi_2\|_2;\ x = \xi_1 + \xi_2,\ \xi_1 \in E_1\} \leqq \|x\|_1 + \|0\|_2 = \|x\|_1 .$$

Hence $\|x\|_3 \leqq \|x\|_1$ for $x \in E_1$. On the other hand, if $x, \xi_1 \in E_1$ then $\xi_2 = x - \xi_1 \in \in E_1$. Consequently

$$\|x\|_1 = \|\xi_1 + \xi_2\|_1 \leqq \|\xi_1\|_1 + \|\xi_2\|_1 \leqq \|\xi_1\|_1 + \|\xi_2\|_2 ,$$

and

$$\|x\|_1 \leqq \inf\{\|\xi_1\|_1 + \|\xi_2\|_2;\ x = \xi_1 + \xi_2,\ \xi_1 \in E_1\} = \|x\|_3 .$$

The norms $\|\cdot\|_1$ and $\|\cdot\|_3$ are identical on the space E_1.

For each $x \in E_2$, we have

$$\|x\|_3 = \inf\{\|\xi_1\|_1 + \|\xi_2\|_2,\ x = \xi_1 + \xi_2,\ \xi_1 \in E_1\} \leqq \|0\|_1 + \|x\|_2 = \|x\|_2 .$$

This, together with the inequality (5), implies that the norms $\|\cdot\|_2$ and $\|\cdot\|_3$ are equivalent and the proof of Lemma is complete.

Proof of Corollary. By using Lemma for a countable number of steps, we can replace the sequence of norms $\|\cdot\|_n$ by a single norm $\|\cdot\|$ on the space E such that for each n the norm $\|\cdot\|$ restricted to the subspace E_n is equivalent to the norm $\|\cdot\|_n$. Then the convergence I in the space E is equal to the convergence H.

2. In this part we present a theorem analogous to Theorem 1 for a linear space endowed with a family of pseudonorms.

Theorem 2. *Let a linear space E, endowed with a family of pseudonorms P, be the union of an increasing sequence of subspaces E_n such that*

(7) *if $x \notin E_n$ then there exists a pseudonorm $p \in P$ such that* $\inf\{p(x - \xi);\ \xi \in E_n\} > 0$.

The convergence

(J) *$x_i \to^J x$ iff $x_i \to^P x$ and there exists an index n such that $x_i \in E_n$ for all i,*

is a pseudonorm convergence.

Proof. It suffices to show that for every sequence x_i such that

(8) $$x_i \in E_{i+1} \setminus E_i$$

there exists a continuous (with respect to the convergence J) pseudonorm q such that the sequence $q(x_i)$ is unbounded. Let a sequence x_i satisfy (8). There exists, by (7) and (8), a sequence of pseudonorms $p_i \in P$ such that $\inf\{p_i(x_i - \xi);\ \xi \in E_i\} > > 0$ for $i = 1, 2, \ldots$. Let $r(x) = p_1(x)$ for $x \in E_2$. Arguing by induction, assume that an extension of the pseudonorm r is defined on the space E_n. Let

$$r(x) = \inf\{r(\xi) + \max\{p_1(\eta), \ldots, p_n(\eta)\};\ x = \xi + \eta,\ \xi \in E_n\}$$

for $x \in E_{n+1} \setminus E_n$. Let $R_i(x) = \inf\{r(x - \xi);\ \xi \in E_i\}$. We shall show that $R_i(x_i) > 0$ for $i = 1, 2, \ldots$. Assume, on the contrary, that $R_{i_0}(x_{i_0}) = 0$ for some index i_0. This means that $\inf\{r(x_{i_0} - \xi);\ \xi \in E_{i_0}\} = 0$. Consequently, there exists a sequence $x_k \in E_{i_0}$ such that $r(x_{i_0} - x_k)$ tends to zero, i.e.

$$\inf\{r(\xi) + \max\{p_1(\eta), \ldots, p_{i_0}(\eta)\};\ x_{i_0} - x_k = \xi + \eta,\ \xi \in E_{i_0}\}$$

converges to zero. Therefore, for some pair of sequences $\xi_k \in E_{i_0}$ and $\eta_k \in E_{i_0} + 1$ such that $\xi_k + \eta_k = x - x_k$, the sequence

$$r(\xi_k) + \max\{p_1(\eta_k), \ldots, p_{i_0}(\eta_k)\}$$

tends to zero. Hence $p_{i_0}(\eta_k)$ converges to zero. Since $\eta_k = x - (x_k + \xi_k)$ and $x_k + \xi_k \in E_{i_0}$, then $p_{i_0}(x - (x_k + \xi_k))$ tends to zero. But this contradicts the condition (8).

Now, we define the pseudonorm q

$$q(x) = \sum_{i=0}^{\infty} \alpha_i R_i(x)$$

where α_i is a sequence of positive numbers such that

$$\alpha_i \geqq \frac{i}{R_i(x_i)}.$$

The pseudonorm q has all the required properties.

We are often concerned with the case when there is no single family of pseudonorms, but each of the subspaces E_n has its own family of pseudonorms P_n. Theorem 2 is useful when every continuous pseudonorm on the subspace E_n can be extended to a continuous pseudonorm on the subspace E_{n+1}. For example, this is satisfied when every family P_n is countable. This follows from

Lemma. *Let (E, P) be a linear space with a countable family of pseudonorms $(P = p_1, p_2, \ldots)$ and let F be a subspace of E. For every continuous pseudonorm q on the subspace F there exists a continuous pseudonorm p on the space E such that $p(x) = q(x)$ for $x \in F$.*

Proof. The convergence generated by the family P is equal to the convergence generated by the increasing sequence of pseudonorms:

$$q_n(x) = \max\{p_1(x), \ldots, p_n(x)\}.$$

Let q be a continuous pseudonorm on the subspace F. We shall show that there exists an index n_0 and a positive number M such that the inequality

$$q(x) \leqq M q_{n_0}(x) \tag{9}$$

holds for $x \in F$. Assume that it is not true. Consequently, there exists a sequence $x_n \in F$ such that $q(x_n) > nq_n(x_n)$. Let

$$y_n = \frac{x_n}{q(x_n)}.$$

The sequence y_n converges to zero, but $q(y_n) = 1$ for all n, a contradiction.

Let M and n_0 be a pair of positive integers such that the inequality (9) holds. The pseudonorm p defined by the following formula

$$p(x) = \inf\{q(\xi) + Mq_{n_0}(\eta);\ x = \xi + \eta,\ \xi \in F\}$$

has the required properties.

This lemma fails when we omit the assumptions that the family P is countable. We present an example due to J. Burzyk (communicated orally).

Let E be the space of all real valued functions on the segment $[0, 1]$ and let P be the family of all pseudonorms $p_t(x) = |x(t)|$, where $t \in [0, 1]$. For the subspace F we take the space of all functions of the following form

$$x = \lambda\, \chi([0, 1]) + \sum_{i=1}^{m} \lambda_i\, \chi(\{t_i\}),$$

where $\chi(A)$ is a characteristic function of the set A.

The functional

$$p(\lambda\, \chi([0, 1]) + \sum_{i=1}^{m} \lambda_i\, \chi(\{t_i\})) = |\lambda|$$

is a continuous pseudonorm on F.

Assume that there exists a continuous pseudonorm q on E such that $p(x) = q(x)$ for $x \in F$. Let $x_0 = \chi([0, 1])$. It is easy to show that there exists a decreasing sequence of segments $A_i \subset [0, 1]$ such that $\bigcap A_i = \{t_0\}$ for some point $t_0 \in [0, 1]$ and such that $q(\chi(A_i)) > 0$ for all i. The functions

$$x_i = \frac{\chi(A_i \setminus \{t_0\})}{q(\chi(A_i))}$$

form a sequence converging to $x \equiv 0$. But, since $p(\chi(\{t\})) = 0$ for each $t \in [0, 1]$, then $q(x_n) = 1$ for all n. This contradiction proves that there is no continuous extension of the pseudonorm p.

Reference

[1] *J. Mikusiński:* Convergence and Locally Convex Spaces, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., Vol. *XXIII*, No. 11, 1975, 1171—1173.

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