Czechoslovak Mathematical Journal

Jaromír Kuben

Asymptotic equivalence of second order differential equations

Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 2, 189–202

Persistent URL: http://dml.cz/dmlcz/101943

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

ASYMPTOTIC EQUIVALENCE OF SECOND ORDER DIFFERENTIAL EQUATIONS

JAROMÍR KUBEN, Brno

(Received July 19, 1982)

1. INTRODUCTION

In this paper the asymptotic properties of the linear differential equation

$$[p(t)\,x']' + q(t)\,x = 0 \tag{1}$$

and the perturbed differential equation

$$[p(t)\,y']' + q(t)\,y = f(t, y, y') \tag{2}$$

are compared. We shall suppose that $p, q \in C^0(j)$, p is positive on j, $f \in C^0(j \times R^2)$, $j = \langle t_0, \infty)$. Here R is the set of real numbers and $C^0(A)$ is the set of real continuous functions defined on A.

Notation 1. Let M_1 be the set of all noncontinuable solutions of the equation (1) and M_2 the set of all noncontinuable sollutions of the equation (2) that exist for all large t. Suppose that $M_2 \neq \emptyset$.

Let $\mu \in C^0(j)$. The symbols O and o have the usual meaning; i.e., $z(t) = O[\mu(t)]$ denotes that there exists $k > 0$ such that $|z(t)| \leqq |k\,\mu(t)|$ for large t, and $z(t) = o[\mu(t)]$ denotes that there exists $h(t)$ such that $z(t) = \mu(t)\,.\,h(t)$ and $\lim_{t\to\infty} h(t) = 0$.

Definition 1. We shall say that the equations (1) and (2) are *μ_0-asymptotically equivalent* if for each $x \in M_1$ there exists $y \in M_2$ such that

$$x(t) - y(t) = o[\mu_0(t)]\,, \tag{3}$$

and if for each $y \in M_2$ there exists $x \in M_1$ such that (3) holds. We shall say that the equations (1) and (2) are *weakly μ_1-asymptotically equivalent* if for each $x \in M_1$ there exists $y \in M_2$ such that

$$x'(t) - y'(t) = o[\mu_1(t)]\,, \tag{3'}$$

and conversely.

The equations (1) and (2) will be called *strongly (μ_0, μ_1)-asymptotically equivalent* if for appropriate $x(t)$ and $y(t)$, (3) and (3') hold.

If M_2 is the set of all noncontinuable solutions of the equation (2), we shall speak about the *complete asymptotic equivalence* of some type.

The asymptotic equivalence was studied by many authors, e.g. [1]–[20]. Our method is similar to that of [17] but is applied to the perturbed linear differential equation with nonconstant coefficients. In [17], the perturbed linear system with constant coefficients is considered. Some of our results are comparable with Theorems 2–4 in [19], but the types of perturbations considered below are more general.

Notation 2. In the sequel, let $u(t)$ and $v(t)$ form an appropriate fundamental system of the equation (1). Put

$$c = p(t)\left[u(t)\, v'(t) - u'(t)\, v(t)\right]$$

and denote

$$w_i(t) = \left|u^{(i)}(t)\right| + \left|v^{(i)}(t)\right|, \quad i = 0, 1 .$$

Here $u^{(i)}(t)$ denotes the i-th derivative, $i = 0, 1$; i.e. $u^{(0)}(t) = u(t)$.

2. EQUIVALENCE OF NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Let in (2) $f(t, y, y') = a(t)$, where $a \in C^0(j)$. Then (2) has the form

$$(4) \qquad [p(t)\, y']' + q(t)\, y = a(t) .$$

Let $t_0 \leqq \xi \leqq \infty$, $t_0 \leqq \eta \leqq \infty$. The method of variation of constants gives for each solution $y(t)$ of the equation (4) the relation

$$(5) \quad y(t) = c_1\, u(t) + c_2\, v(t) - c^{-1}\, u(t) \int_\xi^t v(s)\, a(s)\, \mathrm{d}s + c^{-1}\, v(t) \int_\eta^t u(s)\, a(s)\, \mathrm{d}s ,$$

where c_1, c_2 are arbitrary constants. The choice $\xi = \infty$ or $\eta = \infty$ is possible iff the corresponding integrals are convergent. Differentiating (5) we get an analogous relation for $y'(t)$:

$$(5') \quad y'(t) = c_1\, u'(t) + c_2\, v'(t) - c^{-1}\, u'(t) \int_\xi^t v(s)\, a(s)\, \mathrm{d}s + c^{-1}\, v'(t) \int_\eta^t u(s)\, a(s)\, \mathrm{d}s .$$

Theorem 1. *The equations* (1) *and* (4) *are completely* μ_0*-asymptotically equivalent (completely weakly* μ_1*-asymptotically equivalent, completely strongly* (μ_0, μ_1)*-asymptotically equivalent) iff there exists a solution* $y_0(t)$ *of the equation* (4) *such that* $y_0(t) = o[\mu_0(t)]$ ($y_0'(t) = o[\mu_1(t)]$, $y_0^{(i)}(t) = o[\mu_i(t)]$, $i = 0, 1$).

Proof. Evidently, M_2 is the set of all noncontinuable solutions of the equation (4) so that the equivalence is complete. Each solution of this equation can be expressed in the form

$$y(t) = x(t) + y_0(t) ,$$

where $x(t)$ is an arbitrary solution of the equation (1). This implies the assertion of the theorem.

Theorem 2. *Let there exist* $\xi, \eta, t_0 \leqq \xi, \eta \leqq \infty$, *such that*

$$u(t) \int_\xi^t v(s)\, a(s)\, ds - v(t) \int_\eta^t u(s)\, a(s)\, ds = o[\mu_0(t)] \tag{6}$$

or

$$u'(t) \int_\xi^t v(s)\, a(s)\, ds - v'(t) \int_\eta^t u(s)\, a(s)\, d(s) = o[\mu_1(t)] \tag{6'}$$

or both (6) *and* (6′) *hold.*

Then the equation (4) *has a solution* $y_0(t)$ *with the property* $y_0(t) = o[\mu_0(t)]$ *or* $y_0'(t) = o[\mu_1(t)]$ *or* $y_0^{(i)}(t) = o[\mu_i(t)]$, $i = 0, 1$.

Proof. The assertion is an immediate consequence of the relations (5) and (5′).

Corollary 1. *If the hypotheses of Theorem 2 hold, then the equations* (1) *and* (4) *are completely* μ_0*-asymptotically equivalent or completely weakly* μ_1*-asymptotically equivalent or completely strongly* (μ_0, μ_1)*-asymptotically equivalent, respectively.*

3. EQUIVALENCE OF NONLINEAR DIFFERENTIAL EQUATIONS

In this chapter we shall give sufficient conditions for the types of asymptotic equivalence defined above. We shall suppose that the following hypotheses hold:

(7) (i) $f(t, r, s) \in C^0(j \times R^2)$;
(ii) *there exists a nonnegative function* $F(t, r, s)$, $F \in C^0(j \times R_+^2)$, *which is nondecreasing in* r *and* s *for each fixed* $t \in j$ *such that*

$$|f(t, r, s)| \leqq F(t, |r|, |s|)\,.$$

Here R_+ is the set of all nonnegative real numbers.

Notation 3. Let J_0, J_1 be positive functions, $J_i \in C^0(j)$, $i = 0, 1$, such that

$$u^{(i)}(t) = O[J_i(t)]\,, \quad v^{(i)}(t) = O[J_i(t)]\,, \quad i = 0, 1\,. \tag{8$^{(i)}$}$$

For example, we can take $J_i = w_i$, $i = 0, 1$.

Theorem 3. *Suppose that* (7) *holds and let for any* $k \geqq 0$,

$$\int_{t_0}^\infty |u(s)|\, F[s, k\, J_0(s), k\, J_1(s)]\, ds < \infty$$

and

$$u^{(i)}(t) \int_{t_0}^t |v(s)|\, F[s, k\, J_0(s), k\, J_1(s)]\, ds = o[J_i(t)]\,, \quad i = 0, 1\,. \tag{9$^{(i)}$}$$

Let for each solution $y \in M_2$,

$$(10^{(i)}) \qquad y^{(i)}(t) = O[J_i(t)]\,, \quad i = 0, 1\,.$$

Then the equations (1) *and* (2) *are strongly* (μ_0, μ_1)*-asymptotically equivalent for each pair of functions* μ_0, μ_1, *such that for any* $k \geqq 0$,

$$(11^{(i)}) \qquad \left|u^{(i)}(t)\right| \int_{t_0}^{t} \left|v(s)\right| F[s, k\, J_0(s), k\, J_1(s)]\, ds + $$
$$+ \left|v^{(i)}(t)\right| \int_{t}^{\infty} \left|u(s)\right| F[s, k\, J_0(s), k\, J_1(s)]\, ds = o[\mu_i(t)]\,, \quad i = 0, 1\,.$$

If F *does not depend on* r *or* s, *the assumptions* (9) *and* (10) *or* (9′) *and* (10′) *can be omitted. In this case, the equations* (1) *and* (2) *are weakly* μ_1*-asymptotically equivalent or* μ_0*-asymptotically equivalent for each function* μ_1 *or* μ_0 *satisfying* (11′) *or* (11).

Proof. I. Let $y \in M_2$. Consider a nonhomogeneous linear differential equation

$$(pz')' + qz = f[t, y(t), y'(t)]$$

that possesses the solution $y(t)$. For appropriate $k > 0$ and $t_1 \geqq t_0$ we have

$$\left|u^{(i)}(t) \int_{t_1}^{t} v(s) f[s, y(s), y'(s)]\, ds + v^{(i)}(t) \int_{t}^{\infty} u(s) f[s, y(s), y'(s)]\, ds\right| \leqq$$
$$\leqq \left|u^{(i)}(t)\right| \int_{t_1}^{t} \left|v(s)\right| F[s, k\, J_0(s), k\, J_1(s)]\, ds +$$
$$+ \left|v^{(i)}(t)\right| \int_{t}^{\infty} \left|u(s)\right| F[s, k\, J_0(s), k\, J_1(s)]\, ds = o[\mu_i(t)]\,, \quad i = 0, 1\,.$$

Theorem 2 guarantees the existence of a solution $z(t)$ such that $z^{(i)}(t) = o[\mu_i(t)]$, $i = 0, 1$. Then, $x(t) = y(t) - z(t)$ is the desired solution of the equation (1).

II. Take $x \in M_1$ and consider the integral equations

$$(12) \qquad y(t) = x(t) - c^{-1}\, u(t) \int_{t_1}^{t} v(s) f[s, y(s), z(s)]\, ds -$$
$$- c^{-1}\, v(t) \int_{t}^{\infty} u(s) f[s, y(s), z(s)]\, ds\,,$$
$$z(t) = x'(t) - c^{-1}\, u'(t) \int_{t_1}^{t} v(s) f[s, y(s), z(s)]\, ds -$$
$$- c^{-1}\, v'(t) \int_{t}^{\infty} u(s) f[s, y(s), z(s)]\, ds\,, \quad t \geqq t_1\,,$$

where $t_1 \geqq t_0$ will be chosen later.

Let $C_2^0\langle t_1, \infty)$ be the set of all pairs of continuous functions defined on $\langle t_1, \infty)$. For $g \in C_2^0\langle t_1, \infty)$, let

$$p_n(g) = \max_{t \in \langle t_1, t_1 + n\rangle} \|g(t)\|\,, \quad n \in N\,;$$

here $\|\cdot\|$ is some convenient norm in R^2. Then p_n is a pseudo-norm and $C_2^0\langle t_1, \infty)$ with the topology induced by the family of pseudo-norms $\{p_n\}_{n=1}^{\infty}$ is a Fréchet space. Denote

$$\mathscr{B}_\varrho(\tau) = \{\varphi = [\varphi^0, \varphi^1] \in C_2^0\langle \tau, \infty): |\varphi^i(t)| \leqq \varrho\, J_i(t),\ i = 0, 1\}\,,$$

$\tau \geqq t_0$. There exists $k > 0$ such that $[x, x'], [u, u'], [v, v'] \in \mathscr{B}_k(t_0)$. Let $\varrho \geqq 2k$ and choose t_1 so that

$$\int_{t_1}^{\infty} |u(s)|\, F[s, \varrho\, J_0(s), \varrho\, J_1(s)]\, \mathrm{d}s \leqq \frac{|c|}{2}$$

and

$$|u^{(i)}(t)| \int_{t_1}^{t} |v(s)|\, F[s, \varrho\, J_0(s), \varrho\, J_1(s)]\, \mathrm{d}s \leqq \tfrac{1}{2}\, k|c|\, J_i(t)\,, \quad t \geqq t_1\,, \quad i = 0, 1\,.$$

Let $T\colon \mathscr{B}_\varrho(t_1) \to \mathscr{B}_\varrho(t_1)$ be an operator, $T\varphi = [T_0\varphi, T_1\varphi]$, $\varphi = [\varphi^0, \varphi^1]$ and

$$(T_i\varphi)(t) = x^{(i)}(t) - c^{-1}\, u^{(i)}(t) \int_{t_1}^{t} v(s) f[s, \varphi^0(s), \varphi^1(s)]\, \mathrm{d}s -$$

$$- c^{-1}\, v^{(i)}(t) \int_{t}^{\infty} u(s) f[s, \varphi^0(s), \varphi^1(s)]\, \mathrm{d}s\,, \quad i = 0, 1\,.$$

The convergence in $C_2^0\langle t_1, \infty)$ is the uniform convergence on each compact subinterval of $\langle t_1, \infty)$. If $\varphi \in \mathscr{B}_\varrho(t_1)$, then

$$|(T_i\varphi)(t)| \leqq k\, J_i(t) + |c|^{-1}\, \tfrac{1}{2}\, k|c|\, J_i(t) + |c|^{-1}\, k\, J_i(t) \frac{|c|}{2} = 2k\, J_i(t) \leqq$$

$$\leqq \varrho\, J_i(t)\,, \quad t \geqq t_1\,, \quad i = 0, 1\,.$$

Therefore, $T\mathscr{B}_\varrho(t_1) \subset \mathscr{B}_\varrho(t_1)$.

Let $\{\varphi_n\}_{n=0}^{\infty} \subset \mathscr{B}_\varrho(t_1)$ and $\varphi_n \to \varphi_0$ in the Fréchet space $C_2^0\langle t_1, \infty)$. Let $t_2 > t_1$ and $\varepsilon > 0$. Denote $m = \max w_0(t)$, $t \in \langle t_1, t_2\rangle$. Choose $t_3 > t_2$ such that

$$\int_{t_3}^{\infty} |u(s)|\, F[s, \varrho\, J_0(s), \varrho\, J_1(s)]\, \mathrm{d}s < \frac{|c|\, \varepsilon}{8m}\,.$$

Put

$$\vartheta = \min\left\{\frac{\varepsilon|c|}{4m \int_{t_1}^{t_3} |u(s)|\, \mathrm{d}s}\,, \frac{\varepsilon|c|}{2m \int_{t_1}^{t_2} |v(s)|\, \mathrm{d}s}\right\}.$$

As $\varphi_n \to \varphi_0$ uniformly on $\langle t_1, t_3\rangle$, we have for large n, e.g. $N \leqq n$, and $t \in \langle t_1, t_3\rangle$,

$$|f[t, \varphi_n^0(t), \varphi_n^1(t)] - f[t, \varphi_0^0(t), \varphi_0^1(t)]| < \vartheta\,.$$

Thus

$$|(T_0\varphi_n)(t) - (T_0\varphi_0)(t)| \leqq$$

$$\leqq |c|^{-1} |u(t)| \int_{t_1}^{t} |v(s)| |f[s, \varphi_n^0(s), \varphi_n^1(s)] - f[s, \varphi_0^0(s), \varphi_0^1(s)]| \, ds +$$

$$+ |c|^{-1} |v(t)| \int_{t}^{\infty} |u(s)| |f[s, \varphi_n^0(s), \varphi_n^1(s)] - f[s, \varphi_0^0(s), \varphi_0^1(s)]| \, ds <$$

$$< |c|^{-1} m\vartheta \int_{t_1}^{t_2} |v(s)| \, ds +$$

$$+ |c|^{-1} m \int_{t_1}^{t_3} |u(s)| |f[s, \varphi_n^0(s), \varphi_n^1(s)] - f[s, \varphi_0^0(s), \varphi_0^1(s)]| \, ds +$$

$$+ 2|c|^{-1} m \int_{t_3}^{\infty} |u(s)| F[s, \varrho J_0(s), \varrho J_1(s)] \, ds < \frac{\varepsilon}{2} + |c|^{-1} m\vartheta \int_{t_1}^{t_3} |u(s)| \, ds + \frac{\varepsilon}{4} \leqq \varepsilon$$

for $n \geqq N$ and $t \in \langle t_1, t_2 \rangle$. This estimate implies that T_0 is continuous. The same is true for T_1 and, therefore, T is continuous. As the functions of $T\mathscr{B}_\varrho(t_1)$ are uniformly bounded together with their derivatives, they are equicontinuous at each t, $t \geqq t_1$. By Ascoli's theorem $T\mathscr{B}_\varrho(t_1)$ is relatively compact in $C_2^0\langle t_1, \infty)$. Therefore, as $\mathscr{B}_\varrho(t_1)$ is convex and closed in $C_2^0\langle t_1, \infty)$, T has a fixed point in $\mathscr{B}_\varrho(t_1)$. This assertion is due to Tychonoff's fixed point theorem – see e.g. [24], p. 45.

At the same time, we have proved that the system (12) has a solution. Evidently, $y'(t) = z(t)$ and $y(t)$ is a solution of the equation (2). Moreover, $[y, y'] \in \mathscr{B}_\varrho(t_1)$ so that we have found a solution for which $y^{(i)}(t) = O[J_i(t)]$, $i = 0, 1$, without using (10) and (10'). The relations (11), (11') and (12) imply that (3) and (3') hold.

If e.g. F does not depend on s, then part I is the same as above. In part II define

$$\mathscr{B}_\varrho(\tau) = \{\varphi = [\varphi^0, \varphi^1] \in C_2^0\langle \tau, \infty): |\varphi^0(t)| \leqq \varrho J_0(t), \ |\varphi^1(t)| \leqq \psi(t)\},$$

where

$$\psi(t) = |x'(t)| + |c|^{-1} |u'(t)| \int_{t_0}^{t} |v(s)| F[s, \varrho J_0(s)] \, ds +$$

$$+ |c|^{-1} |v'(t)| \int_{t}^{\infty} |u(s)| F[s, \varrho J_0(s)] \, ds .$$

The rest of the proof requires no essential changes.

Theorem 4. *Suppose that* (7) *holds and let for any* $k \geqq 0$,

$$\int_{t_0}^{\infty} (|u(s)| + |v(s)|) F[s, k J_0(s), k J_1(s)] \, ds < \infty .$$

For each $y \in M_2$ *let* (10) *and* (10') *hold.*

If F does not depend on r or s, the assumption (10) *or* (10') *can be omitted.*

Then the equations (1) *and* (2) *are strongly* (μ_0, μ_1)*-asymptotically equivalent for each pair of functions* μ_0, μ_1 *such that for any* $k \geqq 0$,

$$\int_t^\infty (|u^{(i)}(t)\, v(s)| + |u(s)\, v^{(i)}(t)|)\, F[s, k\, J_0(s), k\, J_1(s)]\, ds = o[\mu_i(t)]\,, \quad i = 0, 1\,.$$

The proof is similar to that of Theorem 3.

Remark 1. We can always put e.g. $\mu_i = J_i$, $i = 0, 1$, in Theorems 3 and 4.

4. SPECIAL CASES OF PERTURBATIONS

Suppose that

$$(13) \qquad |f(t, r, s)| \leqq h(t)\, |r|$$

or

$$(13') \qquad |f(t, r, s)| \leqq k(t)\, |s|\,,$$

where $h, k \in C^0(j)$ are nonnegative.

Lemma 1. *If* (8), (8') *and* (13) *hold, then each solution* $y(t)$ *of the equation* (2) *exists on the whole interval* j *and*

$$y^{(i)}(t) = O\left[J_i(t) \exp \int_{t_0}^t l_1\, J_0^2(s)\, h(s)\, ds\right], \quad i = 0, 1\,,$$

where l_1 *is a positive constant.*

Proof. Let $\langle t_1, T)$, $t_1 \geqq t_0$, be the right-hand maximal interval of existence of $y(t)$. As

$$(14) \qquad y(t) = c_1\, u(t) + c_2\, v(t) - c^{-1}\, u(t) \int_{t_1}^t v(s) f[s, y(s), y'(s)]\, ds + \\ + c^{-1}\, v(t) \int_{t_1}^t u(s) f[s, y(s), y'(s)]\, ds$$

with appropriate constants c_1, c_2, we have for $t \in \langle t_1, T)$

$$|y(t)| \leqq k_1\, J_0(t) + l_1\, J_0(t) \int_{t_1}^t h(s)\, J_0(s)\, |y(s)|\, ds\,,$$

where k_1, l_1 are positive constants and l_1 does not depend on $y(t)$. Using a generalized Gronwall's inequality – see [22] – we obtain an estimate

$$|y(t)| \leqq k_1 l_1\, J_0(t) \exp \int_{t_1}^t l_1\, J_0^2(s)\, h(s)\, ds\,.$$

Differentiating (14) and substituting the preceding result we obtain an analogous estimate for $y'(t)$. This implies that $T = \infty$. If $t_1 > t_0$, we put $t = -s$ in the equation (2) and we show in the same way that $y(-s)$ exists on $\langle -t_1, -t_0 \rangle$. Thus, $y(t)$ exists on j.

Lemma 2. *If* (8), (8′) *and* (13′) *hold, then each solution* $y(t)$ *of the equation* (2) *exists on the whole interval* j *and*

$$y^{(i)}(t) = O\left[J_i(t) \exp \int_{t_0}^{t} l_2 \, J_0(s) \, J_1(s) \, k(s) \, ds\right], \quad i = 0, 1,$$

where l_2 *is a positive constant.*

The proof is analogous to that of Lemma 1.

Theorem 5. *Suppose that* (13) *holds and* $\int_{t_0}^{\infty} w_0^2(s) \, h(s) \, ds < \infty$ *or that* (13′) *holds and* $\int_{t_0}^{\infty} w_0(s) \, w_1(s) \, k(s) \, ds < \infty$.

Then the following implications are true:

(i) *If all solutions of* (1) *are bounded, then the equations* (1) *and* (2) *are completely* w_0*-asymptotically equivalent.*

(ii) *If all solutions of* (1) *have bounded first derivatives, then the equations* (1) *and* (2) *are completely weakly* w_1*-asymptotically equivalent.*

(iii) *If all solutions of* (1) *are bounded together with their first derivatives, then the equations* (1) *and* (2) *are completely strongly* (w_0, w_1)*-asymptotically equivalent.*

Proof. The assertions are immediate consequences of Theorem 4 and Lemmas 1 and 2.

Further we shall suppose that

$$|f(t, r, s)| \leqq a(t) \, g(|r|) \tag{15}$$

or

$$|f(t, r, s)| \leqq b(t) \, h(|s|), \tag{15'}$$

where $a, b \in C^0(j)$ are nonnegative, $g, h \in C^0(R_+)$ are nonnegative and nondecreasing and $g(r) > 0$ for $r \geqq M \geqq 0$, $h(s) > 0$ for $s \geqq N \geqq 0$.

Lemma 3. *If* (15) *holds and* $\int_M^{\infty} dr/g(r) = \infty$, *then each solution of the equation* (2) *exists for large* t. *If, moreover, all solutions of the equation* (1) *are bounded and* $\int_{t_0}^{\infty} w_0(s) \, a(s) \, ds < \infty$, *then all solutions of* (2) *are bounded for large* t.

Proof. Let $y(t)$ and $\langle t_1, T)$ have the same meaning as in the proof of Lemma 1. Suppose that $T < \infty$. Denote $\varphi_i(t) = \max w_i(s)$, $s \in \langle t_1, t \rangle$, $t \geqq t_1$, $i = 0, 1$. Clearly, φ_i is nondecreasing. As $y(t)$ fulfils (14), we get

$$|y(t)| \leqq K \, \varphi_0(T) + |c|^{-1} \, \varphi_0(T) \int_{t_1}^{t} w_0(s) \, a(s) \, g[|y(s)|] \, ds,$$

where $K = \max\{|c_1|, |c_2|\}$. Bihari's inequality – see [23] – gives an estimate

$$|y(t)| \leqq G^{-1}\left\{G[K\,\varphi_0(T)] + |c|^{-1}\,\varphi_0(T)\int_{t_1}^{t} w_0(s)\,a(s)\,\mathrm{d}s\right\},$$

where $G(u) = \int_M^u \mathrm{d}r/g(r)$ and G^{-1} is the inverse function. Differentiating (14) we further get

$$|y'(t)| \leqq K\,\varphi_1(T) + |c|^{-1}\,\varphi_1(T)\int_{t_1}^{t} w_0(s)\,a(s)\,g[|y(s)|]\,\mathrm{d}s\,.$$

Thus, $y(t)$ and $y'(t)$ are bounded but this contradicts the assumption $T < \infty$. The remainder of the lemma is now evident.

Lemma 4. *If (15') holds and $\int_N^\infty \mathrm{d}s/h(s) = \infty$, then each solution of the equation (2) exists for large t. If, moreover, all solutions of the equation (1) have bounded their first derivatives and $\int_{t_0}^\infty w_0(s)\,b(s)\,\mathrm{d}s < \infty$, then all solutions of (2) have bounded their first derivatives for large t.*

The proof is the same as that of Lemma 3.

Using Theorem 4 and Lemmas 3 and 4 we obtain

Theorem 6. *Suppose that (15) holds, w_0 is bounded and $\int_{t_0}^\infty w_0(s)\,a(s)\,\mathrm{d}s < \infty$, $\int_M^\infty \mathrm{d}r/g(r) = \infty$. Then the equations (1) and (2) are completely w_0-asymptotically equivalent.*

Theorem 7. *Suppose that (15') holds, w_1 is bounded and $\int_{t_0}^\infty w_0(s)\,b(s)\,\mathrm{d}s < \infty$, $\int_N^\infty \mathrm{d}s/h(s) = \infty$. Then the equations (1) and (2) are completely weakly w_1-asymptotically equivalent.*

Lemma 5. *Let (7) hold and*

$$\int_{t_0}^\infty w_0(s)\,F[s, \lambda\,w_0(s), \lambda\,w_1(s)]\,\mathrm{d}s < \infty$$

for any $\lambda \geqq 0$. Let there exist $\lambda_0 > 0$ such that

$$\sup_{\lambda\in\langle\lambda_0,\infty)} \frac{1}{\lambda}\int_{t_1}^\infty w_0(s)\,F[s, \lambda\,w_0(s), \lambda\,w_1(s)]\,\mathrm{d}s = S < |c| \tag{16}$$

for an appropriate $t_1 \geqq t_0$.

Then each solution y of the equation (2), $y(t_1) = y_1$, $y'(t_1) = y_1'$, exists for $t \geqq t_1$ and $y^{(i)}(t) = O[w_i(t)]$, $i = 0, 1$.

Proof. Let $\langle t_1, T)$ and K have the same meaning as in the proof of Lemma 3. Assume that $T < \infty$. From (14) we obtain

$$|y^{(i)}(t)| \leqq K\,w_i(t) + |c|^{-1}\,w_i(t)\int_{t_1}^{t} w_0(s)\,F[s, |y(s)|, |y'(s)|]\,\mathrm{d}s\,,$$

$t \in \langle t_1, T)$, $i = 0, 1$. Denote

$$(17) \qquad \varphi(\tau) = K|c| + \int_{t_1}^{\tau} w_0(s)\, F[s, |y(s)|, |y'(s)|]\, ds\,,$$

$\tau \in \langle t_1, T)$. Then

$$(18) \qquad |y^{(i)}(t)| \leqq |c|^{-1}\, w_i(t)\, \varphi(\tau)\,, \quad t \in \langle t_1, \tau\rangle\,, \quad i = 0, 1\,.$$

If $\varphi(\tau) < |c|\, \lambda_0$ for each $\tau \in \langle t_1, T)$, then

$$(19) \qquad |y^{(i)}(t)| \leqq \lambda_0\, w_i(t)\,, \quad t \in \langle t_1, T)\,, \quad i = 0, 1\,.$$

If there exists $\tau_0 \in \langle t_1, T)$ such that $\varphi(\tau_0) \geqq |c|\, \lambda_0$, then $\varphi(\tau) \geqq |c|\, \lambda_0$ for $\tau \in \langle \tau_0, T)$. Relation (16) gives

$$\sup_{\lambda \in \langle \lambda_0, \infty)} \frac{1}{\lambda} \int_{t_1}^{T} w_0(s)\, F[s, \lambda\, w_0(s), \lambda\, w_1(s)]\, ds = S_1 \leqq S < |c|\,.$$

Put $\lambda = |c|^{-1}\, \varphi(\tau)$, $\tau \in \langle \tau_0, T)$. Thus

$$\int_{t_1}^{T} w_0(s)\, F[s, |c|^{-1}\, \varphi(\tau)\, w_0(s), |c|^{-1}\, \varphi(\tau)\, w_1(s)]\, ds \leqq |c|^{-1}\, S\, \varphi(\tau)\,.$$

We obtain from (17) and (18) that

$$\varphi(\tau) \leqq K|c| + |c|^{-1}\, S\, \varphi(\tau)\,, \quad \tau \in \langle \tau_0, T)\,.$$

Therefore,

$$\varphi(\tau) \leqq \frac{|c|\, K}{1 - |c|^{-1}\, S}\,,$$

since $|c|^{-1}\, S < 1$. Relation (18) implies

$$(20) \qquad |y^{(i)}(t)| \leqq \frac{K}{1 - |c|^{-1}\, S}\, w_i(t)\,, \quad t \in \langle t_1, \tau)\,, \quad \tau \in \langle \tau_0, T)\,, \quad i = 0, 1\,.$$

But this estimate does not depend on τ, thus, (20) holds for each $t \in \langle t_1, T)$. As (19) or (20) holds, we get that $y^{(i)}$, $i = 0, 1$, are bounded on $\langle t_1, T)$. This is a contradiction and hence necessarily $T = \infty$. At the same time we have obtained that $y^{(i)} = O[w_i(t)]$, $i = 0, 1$.

Theorem 8. *Let the assumptions of Lemma 5 hold. Then the equations* (1) *and* (2) *are strongly* (w_0, w_1)*-asymptotically equivalent.*

Proof. The assertion is an immediate consequence of Theorem 4 and Lemma 5.

In this chapter we shall use some special properties of nonoscillatory second order linear differential equations. For these properties and the concept of the principal solution see [25], pp. 350–370.

Theorem 9. *Suppose*

i) $p, q, q_0 \in C^0\langle t_0, \infty)$, $p > 0$, $q_0 \leqq 0$;

ii) u_0, u_1 *are respectively a principal and a nonprincipal solution of* $[p(t)\, u']' + q_0(t)\, u = 0$;

iii) $\int_{t_0}^{\infty} |u_0(t)\, u_1(t)|\, |q(t) - q_0(t)|\, \mathrm{d}t < \infty$.

Then

i) *it can be supposed that*

$$u_0 > 0\,, \quad u_0' \leqq 0\,, \quad u_1 > 0\,, \quad u_1' > 0\,, \quad t \in \langle t_0, \infty)\,; \tag{21}$$

ii) *the equation* (1) *is nonoscillatory*;

iii) *each principal or nonprincipal solution* x_0, x_1 *respectively of the equation* (1) *satisfies* $x_i \sim \varkappa_i u_i$, $i = 0, 1$, *where* $\varkappa_i$ *are appropriate constants.*

Proof. Theorem 9 is a consequence of Corollary 11.6.4 and Theorem 11.9.1 of [25].

Lemma 6. *Suppose that*

i) $g \geqq 0$ *on* $\langle t_0, \infty)$ *and* $\int_{t_0}^{\infty} g(t)\, \mathrm{d}s < \infty$;

ii) $c \in C^1\langle t_0, \infty)$, $c > 0$ *is nondecreasing,* $\lim_{t \to \infty} c(t) = \infty$ *and* $c'(t)$ *is nondecreasing.*

Then

$$c^{-1}(t) \int_{t_0}^{t} c(s)\, g(s)\, \mathrm{d}s = o(1)\,.$$

For the proof see [16] and [21].

Theorem 10. *Suppose that*

i) *the assumptions of Theorem 9 are satisfied and* u_0, u_1 *are chosen so that* (21) *holds*;

ii) $p(t)\, u_0^2(t)$ *is nonincreasing*;

iii) $|f(t, r, s)| \leqq F(t, |r|)$, *where* $F \in C^0(j \times R_+)$, $F(t, r)$ *is nondecreasing in* r *for each fixed* $t \in j$;

iv) $\int_{t_0}^{\infty} u_0(s)\, F[s, k\, u_1(s)]\, \mathrm{d}s < \infty$ *for any* $k \geqq 0$;

v) *for each* $y \in M_2$ *the identity* $y(t) = O[u_1(t)]$ *holds.*

Then the equations (1) *and* (2) *are* u_1*-asymptotically equivalent.*

Proof. We shall show that the hypotheses of Theorem 3 are satisfied. Theorem 9 allows us to choose a fundamental system x_0, x_1 of the equation (1) so that $x_i \sim u_i$,

$i = 0, 1$. Put $J_0 = u_1$. Then

$$\int_{t_0}^{\infty} |x_0 s)| \, F[s, k\, u_1(s)] \, ds \leqq m_1 \int_{t_0}^{\infty} u_0(s) \, F[s, k\, u_1(s)] \, ds < \infty ,$$

where $m_1 > 0$ is a constant. Further, $u_1/u_0 \to \infty$ monotonously since $(u_1/u_0)' = = c/pu_0^2 > 0$, and $(u_1/u_0)'$ is nondecreasing. Then, using Lemma 6, we obtain

$$|x_0(t)| \int_{t_0}^{t} |x_1(s)| \, F[s, k\, u_1(s)] \, ds \leqq m_2\, u_0(t) \int_{t_0}^{t} u_1(s) \, F[s, k\, u_1(s)] \, ds =$$

$$= u_1(t)\, m_2 \frac{u_0(t)}{u_1(t)} \int_{t_0}^{t} \frac{u_1(s)}{u_0(s)} u_0(s) \, F[s, k\, u_1(s)] \, ds = o[u_1(t)] ,$$

where $m_2 > 0$ is a constant. The proof is complete.

Theorem 11. *Let the assumptions* i) *and* ii) *of Theorem* 10 *hold. Let* $f(t, r, s)$ *satisfy condition* (13) *and*

$$\int_{t_0}^{\infty} u_0(s)\, u_1(s)\, h(s) \exp \int_{t_0}^{s} l_1\, u_1^2(\sigma)\, h(\sigma) \, d\sigma \, ds < \infty ,$$

where $l_1 > 0$ *is the constant from Lemma* 1.

Then the equations (1) *and* (2) *are completely* u_1*-asymptotically equivalent.*

The proof is similar to that of Theorem 10. Lemma 1 must be used.

6. EXAMPLES

Example 1. a) $(t^4 y')' + 2t^2 y = h(t)\, y,\ t \geqq 1$.Put $u(t) = t^{-1}$, $v(t) = t^{-2}$. If $h \in \in C^0\langle 1, \infty)$ and $\int_1^\infty t^{-2}|h(t)| \, dt < \infty$, then the complete strong (t^{-1}, t^{-2})-asymptotic equivalence holds by Theorem 5.

b) $(t^4 y')' + 2t^2 y = k(t)\, y',\ t \geqq 1$.

Analogously, if $k \in C^0\langle 1, \infty)$ and $\int_1^\infty t^{-3}|k(t)| \, dt < \infty$, then the same assertion as in a) holds.

Example 2. $(t^{\alpha+1} y')' + \beta t^{\alpha-1} y = f(t, y, y'),\ t \geqq 1,\ \alpha^2 - 4\beta < 0$. Put

$$u(t) = t^{-\alpha/2} \cos \frac{\sqrt{(4\beta - \alpha^2)}}{2} \ln t , \quad v(t) = t^{-\alpha/2} \sin \frac{\sqrt{(4\beta - \alpha^2)}}{2} \ln t .$$

We can take $J_0(t) = t^{-\alpha/2}$, $J_1(t) = t^{-\alpha/2-1}$.

a) If (15) holds and $\alpha \geqq 0$, $\int^\infty (1/g) = \infty$, $\int_1^\infty s^{-\alpha/2}\, a(s) \, ds < \infty$, then the complete $t^{-\alpha/2}$-asymptotic equivalence holds by Theorem 6.

b) If (15′) holds and $\alpha \geqq -2$, $\int^\infty (1/h) = \infty$, $\int_1^\infty s^{-\alpha/2}\, b(s) < \infty$, then the complete weak $t^{-\alpha/2-1}$-asymptotic equivalence holds by Theorem 7.

Example 3. $y'' - \mu t^{-2} y = f(t, y, y')$, $t \geqq 1$, $\mu > \frac{1}{4}$. Put $u(t) = t^{1/2} \cos (\mu - \frac{1}{4})^{1/2}$ . . $\ln t$, $v(t) = t^{1/2} \sin (\mu - \frac{1}{4})^{1/2} \ln t$. We can take $J_0(t) = t^{1/2}$, $J_1(t) = t^{-1/2}$.

a) Let $|f(t, r, s)| \leqq h(t) |r|^\alpha |s|^\beta$, $\alpha, \beta \geqq 0$, $\alpha + \beta \leqq 1$, $h \in C^0\langle 1, \infty)$. If $\int_1^\infty s^{(\alpha - \beta + 1)/2} h(s) \, ds < \infty$, then the strong $(t^{1/2}, t^{-1/2})$-asymptotic equivalence holds by Theorem 8.

b) Let $|f(t, r, s)| \leqq k(t) (|r|^\alpha + |s|^\beta)$, $0 \leqq \alpha$, $\beta \leqq 1$, $k \in C^0\langle 1, \infty)$. If $\int_1^\infty s^{(\alpha+1)/2}$ . . $k(s) \, ds < \infty$, then the same assertion as in a) holds.

Example 4. $y'' + q(t) y = f(t, y, y')$, $t \geqq t_0$. Let $|f(t, r, s)| \leqq h(t) |r|$, $h \in C^0\langle t_0, \infty)$ and

$$\int_{t_0}^{\infty} s |q(s)| \, ds < \infty , \quad \int_{t_0}^{\infty} s^2 h(s) \, ds < \infty .$$

Put $q_0 = 0$, $u_0(t) = 1$, $u_1(t) = t$ in Theorem 11. Then the complete t-asymptotic equivalence holds.

References

[1] *Wintner A.:* Small perturbations, Am. J. Math. *67* (1945), 417—430.

[2] *Levinson N.:* The asymptotic behavior of a system of linear differential equations, Am. J. Math. *68* (1946), 1—6.

[3] *Weyl H.:* Comment on the preceding paper, Am. J. Math. *68* (1946), 7—12.

[4] *Wintner A.:* Linear variation of constants, Am. J. Math. *68* (1946), 185—213.

[5] *Jakubovič V. A.:* On asymptotic behaviour of solutions of a system of differential equations (Russian), Mat. Sbornik, T *28* (*70*), 1951, 217—240.

[6] *Brauer F.:* Asymptotic equivalence and asymptotic behavior of linear systems, Michigan J. Math. *9* (1962), 33—43.

[7] *Onuchic N.:* Relationship among the solutions of two systems of ordinary differential equations, Michigan J. Math. *10* (1963), 129—139.

[8] *Onuchic N.:* Nonlinear perturbation of a linear system of ordinary differential equations, Michigan J. Math. *11* (1964), 237—242.

[9] *Kato J.:* The asymptotic relations of two systems of ordinary differential equations, Contr. Diff. Eqs. *3* (1964), 141—161.

[10] *Brauer F.:* Nonlinear differential equations with forcing terms, Proc. Am. Math. Soc. *15* (1964), 758—765.

[11] *Onuchic N.:* Asymptotic relationships at infinity between the solutions of two systems of ordinary differential equations, J. Diff. Eq. *3* (1967), 47—58.

[12] *Brauer F., Wong:* On the asymptotic relationship between solutions of two systems of ordinary differential equations, J. Diff. Eq. *6* (1969), 527—543.

[13] *Brauer F., Wong:* On asymptotic behavior of perturbed linear systems, J. Diff. Eq. *6* (1969), 142—153.

[14] *Hallam T. G.:* On asymptotic equivalence of the bounded solutions off two systems of differential equations, Michigan J. Math. *16* (1969), 353—363.

[15] *Hallam T. G., Onuchic N.:* Asymptotic relations between perturbed linear systems of ordinary differential equations, Pac. J. Math. *45* (1973), 187—199.

[16] *Hallam T. G.:* Asymptotic relationships between the solutions of two second order differential equations, Ann. Polon. Math. *24* (1971), 295—300.

[17] *Švec M.:* Asymptototic relationship between solutions of two systems of differential equations, Czech. Math. J. *24* (99), Nr. 1 (1974), 44—58.
[18] *Ráb M.:* Asymptotic relationships between the solutions of two systems of differential equations, Ann. Polon. Math. *30* (1974), 119—124.
[19] *Kitamura Y.:* Remarks on the asymptotic relationships between solutions of two systems of ordinary differential equations, Hiroshima Math. J. *6*, No. 2 (1976), 403—420.
[20] *Marlin, Struble:* Asymptotic equivalence of nonlinear systems, J. Diff. Eq. *6* (1969), 578—596.
[21] *Hallam T. G.:* Asymptotic expansions in certain second order non-homogeneous differential equations, Mathematika *15* (1968), 30—38.
[22] *Charlamov P. V.:* On estimates of solutions of a system of differential equations (Russian), Ukrainskij mat. žurnal *7* (1955), 471—473.
[23] *Bihari I.:* A generalisation of a lemma of Bellman and its applications to uniqueness problems of differential equations, Acta Math. Acad. Sci. Hungar *7* (1956), 71—94.
[24] *Lakshmikantham V., Leela S.:* Differential and Integral Inequalities, Volume I. New York and London, Academic Press 1969.
[25] *Hartman P.:* Ordinary Differential Equations. New York—London—Sydney, John Wiley & Sons, Inc. 1964.

Author's address: 625 00 Brno, Sevastopolská 11, ČSSR.