D. M. Shaydayeva

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ON BASIC CONCEPTS OF NON-COMMUTATIVE TOPOLOGY

D. M. SHAYDAYEVA, Moscow

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A non-commutative generalization of locally compact Hausdorff spaces was independently offered by Ch. A. Akemann [1–3], R. Giles and H. Kummer [8; 9]. They used the description of such topological spaces in terms of the bounded continuous functions algebra as the matter to be extended to the non-abelian situation. In that way a non-commutative or "quantum" topology associated with a C^*-algebra was defined as a certain family of projections in its atomic W^*-envelope, and the C^*-algebra was interpreted as an algebra of "continuous" elements in accordance with this topology.

In the present paper we give an intrinsic axiomatic definition of a general non-commutative topology in terms of the lattice of all projections in an arbitrary atomic W^*-algebra B. The system of axioms connects the properties of non-commutative topology with order, Jordan and C^*-structures on B. For all that two key ideas are pursued: firstly, to generalize to non-abelian case the description of any topology by means of the bounded lower semicontinuous functions cone and, secondly, to provide that the set of "continuous" elements in B be a C^*-algebra. Moreover, we give an effective characterization of compactness and show that a non-commutative topology is locally compact iff it is the Akemann-Giles topology associated with a certain C^*-algebra.

1. Preliminaries. Let us consider, together with any set X, the commutative W^*-algebra $B(X)$ of all complex-valued bounded functions on X. The field of all subsets of X may be naturally identified with the atomic boolean algebra $\mathrm{Pr}B(X)$ of all projections in $B(X)$ (each subset of X is assigned its characteristic function); the points of X are in one to one correspondence with the atoms in the lattice $\mathrm{Pr}B(X)$. Now, if there is a topology τ on X, it may be regarded as some family of projections in $B(X)$. Namely, $\tau = \mathrm{Pr}B(X) \cap L(X)$, where $L(X)$ is the convex cone of all lower semi-continuous bounded functions on X. In particular, if τ is completely regular, let $C(\tau)$ denote the C^*-algebra of all continuous bounded functions on X and let $C(\tau)^m$ denote the set of suprema in $B(X)$ (i.e. pointwise suprema) of all bounded increasing nets of real-valued elements in $C(\tau)$, then $\tau = \mathrm{Pr}B \cap C(\tau)^m$.

It easily follows from the spectral theory that an arbitrary commutative atomic

W^*-algebra M is isomorphic to the W^*-algebra $B(X)$, where X is the set of all minimal projections in M (i.e. atoms in the atomic boolean algebra PrM). In the light of these facts the lattice PrB of all projections in an arbitrary non-commutative atomic W^*-algebra B can be regarded as a non-commutative analogue of the concept of set; minimal projections in B play the role of points, any projection in B, being the supremum of minimal ones, plays the role of a subset. Finally, a "non-commutative topology" arises as the appropriate family of projections in a non-commutative atomic W^*-algebra.

This outlook at a non-commutative generalization of topology was the basis for the Akemann-Giles analogue of a locally compact Hausdorff space. In what follows we construct a non-commutative analogue of a general topological space and show that the Akemann-Giles construction coincides with ours in the locally compact case.

For the general theory of C^*- and W^*-algebras we shall make systematic use of books [6] and [12].

1.1. ***q*-sets.** An atomic W^*-algebra B together with the set of all minimal projections therein will be called a *q-set*, the elements of PrB *q-subsets* and the atoms in PrB *q-points*. Union and intersection of q-sets are to be taken in the lattice PrB. Two q-sets e and f will be called *disjoint* iff $ef = 0$. (The terminology is lifted from [8]).

Any C^*-algebra A is associated with a q-set as follows. The second conjugate space A^{**} is a W^*-algebra; let Z_A be the supremum of all minimal projections in A^{**}, then Z_A belongs to the center of the algebra A^{**} [1; p. 278]. Set $B_A = Z_A A^{**}$, then B_A is the atomic W^*-algebra and we can consider A as the weakly dense sub-algebra of B_A, since $A \subset A^{**}$ and $A \to Z_A A$ is an isomorphism (see [3], p. I). If A is abelian with the spectrum X, then the points of X are in one to one correspondence with the minimal non-zero projections of A^{**} so that $B_A \approx B(X)$ and the Gelfand isomorphism give $A \approx C_0(X) \subset B(X)$.

1.2. Notation. For any subset $E \subset B$ put $E^+ = \{a \in E \mid a \geqq 0\}$, $E^s = \{a \in E \mid a = a^*\}$, $E_1 = \{a \in E \mid \|a\| \leqq 1\}$. Recall that the self-adjoint part B^s of a W^*-algebra B is an ordered space, in which every norm bounded increasing net has a supremum. Let E^m denote the set of suprema in B^s of all norm-bounded increasing nets of elements of E^s; put $E_m = -(-E)^m$.

1.3. The Akeman-Giles *Q*-topology. For any C^*-algebra A consider $A \subset B_A$ as above. The family of q-sets $\tau_A = \mathrm{Pr}B_A \cap (A^+)^m$ will be called the *Akemann-Giles q-topology* on B_A. The elements of τ_A are called *q-open* q-sets, the elements of $1 - \tau_A = \{1 - e \mid e \in \tau_A\}$ are called *q-closed*. The pair (B_A, τ_A) will be called the *q-spectrum* of the C^*-algebra A and denote as q spec A. If A is commutative, then τ_A is the usual topology on the spectrum of A and therefore is Hausdorff and locally compact. In the non-abelian case τ_A has similar properties. Namely, a q-topology τ_A

is *Hausdorff*: i.e., given disjoint q-points x and y there exist disjoint open q-sets e and f with $x \leqq e$ and $y \leqq f$ [8; III.6]. After Akemann a q-set $p \in \mathrm{Pr}B_A$ is called *q-compact* if p is closed and there exists $a \in A_1^+$ with $p \leqq a$ [3; II.I and II.5]. The q-topology τ_A is *locally compact*: i.e., for any q-point x there exist an open q-set e and a compact q-set p with $x \leqq e \leqq p$ (this follows from [3; III.I]). In the abelian case, when $A \approx C_0(X) \subset B(X)$, these concepts of q-compactness are equivalent to the usual definitions.

1.4. Gelfand-Akemann-Giles theorem. An element $a \in B_A^s$ is called *τ_A-continuous* if each spectral projection of a which corresponds to an open subset of the real numbers is also an open q-set. A τ_A-continuous element a is called *vanishing at* ∞ if each spectral projection of a corresponding to a closed subset of the real numbers which do not contain 0 is q-compact [3; I.I and III.3].

Denote by $C(\tau_A)^s$ the set of all τ_A-continuous elements of B_A^s and by $C_0(\tau_A)^s$ the set of all elements vanishing at ∞. Set $C(\tau_A) = C(\tau_A)^s + iC(\tau_A)^s$ and $C_0(\tau)_A = C_0(\tau_A)^s + iC_0(\tau_A)^s$. The elements of $C(\tau_A)$ and $C_0(\tau_A)$ will also be called τ_A-continuous and vanishing at ∞, respectively.

Theorem [3; 5; 8]. *A C^*-algebra A is exactly the algebra of all τ_A-continuous elements of B_A vanishing at ∞, i.e. $A = C_0(\tau_A)$; the set of all continuous elements of B_A coincides with the C^*-algebra of all multipliers of A*, i.e. $C(\tau_A) = M(A) \equiv \{b \in B_A \mid Ab + bA \subset A\}$.

Throughout the whole paper B will always be used to denote an arbitrary atomic W^*-algebra, and $\tau \subset \mathrm{Pr}B$ a family of projections (i.e. q-sets) in B. We start to discuss the individual axioms which will connect the properties of τ with various structures on B.

2. Order axioms. It is easy to see that any Akemann-Giles q-topology contains 0 and 1 and that a union of open q-sets is also open. Akemann has shown that in contradistinction to the usual topology, the intersection of two open q-sets is not necessarily open (see a counterexample in [I]). The first three axioms describe the corresponding properties of the general q-topology.

AXIOM A1. $0, 1 \in \tau$;

AXIOM A2. $(e_\alpha) \subset \tau \Rightarrow \bigvee_\alpha e_\alpha \in \tau$, i.e. "union of open q-sets is open".

AXIOM A3. $e, f \in \tau$, $[e, f] = 0 \Rightarrow e \wedge f \in \tau$, i.e. "intersection of two commuting open q-sets is open".

Remark that actually axiom A3 is an order condition since $[e, f] = 0 \Leftrightarrow e = (e \wedge f) \vee (e \wedge (1 - f))$. Any Akemann-Giles q-topology satisfies all these axioms [I].

We have noticed in § 1 that in the commutative case a topology of a topological space X may be described algebraically by the equality $\tau = \mathrm{Pr}B(X) \cap L(X)$. Our fourth axiom will be a non-commutative version of this description, therefore we need an appropriate definition of the class of lower semicontinuous (LSC) elements in B.

Definition 2.1. The set $E \subset B^s$ is *lower monotone closed* (LMC) if $E = E^m$; E is *upper monotone closed* (UMC) if $E = E_m$. The minimal LMC set in B^s containing E is called the *lower monotone closure* of E and is denoted by $L(E)$. The *upper monotone closure* of E is similarly defined and denoted by $U(E)$.

Lemma 2.2. *If E is a convex cone in B^s, then $L(E)$ is a convex cone, too. If, besides, $E \supset \mathbb{R} \, . \, 1$, then the convex cone $L(E)$ is norm-closed in B^s.*

Proof. Take $\lambda, \mu \in R^+$ and $a \in E$, then

$$(*) \qquad E \subset M(a) \equiv \{b \in L(E) \mid \lambda a + \mu b \in L(E)\} \subset L(E)$$

and the set $M(a)$ is LMC. So $M(a) = L(E)$ and for each $a \in E$ and $b \in L(E)$ we have $\lambda a + \mu b \in L(E)$. This implies that for each $a \in L(E)$ $(*)$ is correct and we get similarly $M(a) = L(E)$. The last equality shows that $L(E)$ is a convex cone. Let $E \supset \mathbb{R} \, . \, 1$. If a sequence $(a_n) \subset L(E)$ converges to an element $b \in B^s$ we may suppose that $\|a_n - b\| \leqq 2^{-n}$. Then the increasing sequence $\tilde{a}_n = a_n - 2^{-n+1} \, . \, 1$ is contained in $L(E)$ and also converges to b. Hence $b = \bigvee_n \tilde{a}_n \in L(E)$ and $L(E)$ is norm-closed.

Put $A^+(\tau) = \{\sum_{i=1}^{n} \lambda_i e_i \mid \lambda_i \geqq 0, \, e_i \in \tau, \, n \in \mathbb{N}\}$ and consider $A(\tau) = A^+(\tau) + \mathbb{R} \, . \, 1$, the minimal convex cone containing τ and "constants". We now define the class of *lower semicontinuous* (LSC) *elements* in B^s as $L(\tau) = L(A(\tau))$. By Lemma 2.2, $L(\tau)$ is a norm-closed convex cone in B^s. If τ were the usual topology on a set X, $L(\tau)$ would be the class of all lower semicontinuous real-valued functions on X.

Similarly we may define the class $U(\tau)$ of *upper semicontinuous* (USC) *elements* in B^s by setting

$$A^+(1 - \tau) = \{\sum_{i=1}^{n} \lambda_i f_i \mid \lambda_i \geqq 0, \, f_i \in 1 - \tau\},$$

$$A(1 - \tau) = A^+(1 - \tau) + \mathbb{R} \, . \, 1 \quad \text{and} \quad U(\tau) = U(A(1 - \tau)).$$

For any subset $E \subset B^s$ we shall denote by $\bar{E}$ the norm-closure of E in B^s.

Lemma 2.3. *If A is a C^*-algebra and τ_A is the Akemann-Giles topology associated with A, then*

$$L(\tau_A) = \overline{\tilde{A}^m},$$

where $\tilde{A} = A + \mathbb{C} \, . \, 1 \subset B_A$ is the C^*-algebra obtained by adjoining the unit 1 of B_A to A.

Proof. We have $A(\tau_A) \subset \tilde{A}^m \subset \overline{\tilde{A}^m}$ since $\tau_A \subset (A^+)^m$. By [4; 3.3] the convex cone $\overline{\tilde{A}^m}$ is LMC, therefore we get $L(\tau_A) \subset \overline{\tilde{A}^m}$. To show the converse inclusion notice that by virtue of the Gelfand-Akemann-Giles theorem (1.4) we have $\tilde{A} \subset C(\tau_A)$ and so by the spectral theory $\tilde{A}^s \subset A(\tau_A)^m$. This implies $\tilde{A}^m \subset L(\tau_A)$, which means $\overline{\tilde{A}^m} \subset L(\tau_A)$ as desired.

The proof of the last lemma was based on the assertion in [4], which was stated there in terms of A^{**}, but applying theorems [9; 3.8 and 4; 2.6] we get the result in terms of B_A.

From the papers [9; 4] by Pedersen and Akemann we know that there is an isometric map of $\overline{\tilde{A}^m}$ onto the set $L^a(S(A))$ of all bounded LSC affine functions on the state space $S_A = \{\varphi \in A^{*+} \mid \|\varphi\| = 1\}$ provided with the weak* topology of A^*. Through Lemma 2.3 we may therefore identify the convex cone $L(\tau_A)$ with the cone $L^a(S(A))$ (see also § 5 below).

The next definition coincides in essence with that in 1.1, and in the commutative case it is the usual definition of continuous functions.

Definition 2.4. An element $a \in B^s$ is called *q-continuous*, if for each open set $I \subset \mathbb{R}$ the spectral projection $E_a(I)$ belongs to τ, $a = \int \lambda \, dE_a$ being the spectral representation of a. Let $C(\tau)^s$ denote the set of all q-continuous elements of B^s and $C(\tau) = C(\tau)^s + iC(\tau)^s$. The elements of $C(\tau)$ will also be called q-continuous.

Proposition 2.5. *If a family* $\tau \subset \mathrm{Pr}B$ *satisfies axioms* A1 – A3, *then*

$$C(\tau)^s = \bigcup\{C^s \mid C \subset C(\tau),\ C \text{ is commutative } C^*\text{-subalgebra}\}\,.$$

Proof. Let $a \in C(\tau)^s$ and let $C^*(1, a)$ be the commutative C^*-subalgebra of B, generated by a and the unit 1. By the Gelfand representation theorem $C^*(1, a)$ is isomorphic to the C^*-algebra $C(\mathrm{Sp}a)$ of all continuous functions on the spectrum of a, and any $b \in C^*(1, a)^s$ is associated with the continuous real-valued function f_b on $\mathrm{Sp}a$ with $E_b(I) = E_a(f_b^{-1}(I))$ for each $I \subset \mathbb{R}$. This shows that for any $b \in C^*(1, a)^s$ and any open set $I \subset \mathbb{R}$ the q-set $E_b(I) \in \tau$, which means that $C^*(1, a)^s \subset C(\tau)^s$ and completes the proof.

It follows from the spectral theory that $C(\tau)^s \subset L(\tau) \cap U(\tau)$, i.e. that any continuous element in B^s is both lower and upper semicontinuous. Set

$$Q(\tau)^s = L(\tau) \cap U(\tau) \quad \text{and} \quad Q(\tau) = Q(\tau)^s + iQ(\tau)^s\,.$$

The elements of $Q(\tau)$ will be called *q-quasicontinuous*. By Lemma 2.2, $Q(\tau)$ is a norm-closed linear subspace of B. In the commutative case, when τ is the usual topology, $Q(\tau)$ certainly coincides with $C(\tau)$. But it is not so for Akemann-Giles topologies.

Proposition 2.6. *Let A be a C^*-algebra and* $(B_A, \tau_A) = q$ spec A. *Then*

$$Q(\tau_A) = \{x \in B_A \mid a \times b \in A \ \forall a,\ b \in A\} \equiv Q(A)\,,$$

i.e. $Q(\tau_A)$ is the space of all quasimultiples of A.

Proof. It follows from Lemma 2.3 that $Q(\tau_A) = \overline{\tilde{A}^m} \cap \overline{\tilde{A}_m}$; by [4; 4.1] this intersection is exactly the space $Q(A)$ of all quasimultiples of A in B_A.

By the Gelfand-Akemann-Giles theorem, $C(\tau_A) = M(A)$ where $M(A)$ is the C^*-algebra of all multiples of A in B_A; in the paper [4] an example is given of a C^*-algebra A for which $Q(A) \neq M(A)$ and $Q(A)$ is not a Jordan algebra.

AXIOM A4. $\tau = \mathrm{Pr}B \cap L(\tau)$.

Definition 2.7. A family $\tau \subset \mathrm{Pr}B$, which obeys the axioms A1 – A4, is called a *q-topology*. The elements of τ are *q-open q*-sets the elements of $1 - \tau$ are *q-closed q*-sets. Given any q-set $e \in \mathrm{Pr}B$, its *q-closure* is $\bar{e} = \bigwedge\{f \mid f$ is q-closed and $e \leqq f\}$; similarly its *q-interior* is $^\circ e = \bigvee\{g \mid g$ is open and $e \geqq g\}$. The pair (B, τ) is called the *q-topological space.*

It follows from Lemma 2.3 and [4; 3.6] that any Akemann-Giles q-topology satisfies axiom A4. In the commutative case that axiom follows from A1 – A3, but if B is non-abelian, A4 is independent of them. Axioms A1 – A3 imply neither that $C(\tau)$ is a linear subspace of B nor that it is norm-closed.

Theorem 2.8. *If τ is a q-topology, then*

$$C(\tau)^s = \{a \in B^s \mid a^n \in Q(\tau)^s \text{ for all } n \in \mathbb{N}\}\,.$$

Proof. If $a \in C(\tau)^s$ then by Proposition 2.5 $a^n \in C(\tau)^s$ for all $n \in \mathbb{N}$ and so $C(\tau)^s \subset \subset \{a \in B^s \mid a^n \in Q(\tau)^s$ for all $n \in \mathbb{N}\}$. Conversely, let $b \in B^s$, $b^n \in Q(\tau)^s$ for all $n \in \mathbb{N}$ and denote by $C^*(1, b)$ the C^*-subalgebra of B generated by b and the unit 1, then through the Stone-Weierstrass theorem we get $C^*(1, b)^s \subset Q(\tau)^s$. Since for any open $I \subset \mathbb{R}$ such that $E_b(I) \neq 0$ there exists an increasing sequence $(b_n) \subset C^*(1, b)_1^+$ with $E_b(I) = \bigvee_n b_n$, it implies that $E_b(I) \in (Q(\tau))^m \subset L(\tau)$. Thus in virtue of axiom A4 $E_b(I)$ is open, whence $b \in C(\tau)^s$.

Corollary 2.9. *If τ is a q-topology then $C(\tau)^s$ is a norm-closed subset of B^s.*

Recall that B^s is a real Jordan algebra and B is a Jordan C^*-algebra (JC^*-algebra) with multiplication $a \circ b = \frac{1}{2}(ab + ba)$.

Proposition 2.10. *Let τ be a q-topology. If $C(\tau)^+$ is a convex cone in B^s then $C(\tau)^s$ is a norm-closed real Jordan subalgebra of B^s and $C(\tau)$ is a JC^*-subalgebra of B.*

Proof. By Proposition 2.5, given $a, b \in C(\tau)^s$ we have $a + \|a\| . 1, b + \|b\| . 1 \in C(\tau)^+$ so $(a + b) + (\|a\| + \|b\|) . 1 \in C(\tau)^+$ whence $a + b \in C(\tau)^s$. Moreover we get again by Proposition 2.5 $a^2 \in C(\tau)^s$ whenever $a \in C(\tau)^s$ and thus $a \circ b = \frac{1}{4}((a + b)^2 - (a - b)^2) \in \in C(\tau)^s$ whenever $a, b \in C(\tau)^s$. Together with Corollary 2.9 this proves the first assertion. Take a sequence $(a_n) \subset C(\tau)$ which uniformly converges to $b \in B$. Setting $\mathrm{Re}\, x = = \frac{1}{2}(x + x^*)$, $\mathrm{Im}\, x = \frac{1}{2}(x - x^*) \in B^s$ for any $x \in B$ we see that $\|\mathrm{Re}\, a_n - \mathrm{Re}\, b\| \to 0$ and $\|\mathrm{Im}\, a_n - \mathrm{Im}\, b\| \to 0$. Since $C(\tau)^s$ is closed, we obtain that $\mathrm{Re}\, b$, $\mathrm{Im}\, b \in C(\tau)^s$ which implies $b \in C(\tau)$.

Lemma 2.11. *Let τ be a q-topology. Suppose $X \subset Q(\tau)^+$ is a convex cone satisfying* (i) $X \supset \mathbb{R}^+ . 1$; (ii) $a^{1/2} \in Q(\tau)^s$ for any $a \in X$.
Then $X \subset C(\tau)^+$.

Proof. Take $a \in X$. By Theorem 2.8 it suffices to prove that $a^n \in Q(\tau)^+$ whenever $n \in \mathbb{N}$. This assertion is trivial for $n = 1$ and we assume that it has been proved for all values of $n < m$. Consider $\alpha > \|a\|$; since by (i) $1 + \alpha^{-1}a \in X$ we see from (ii) that the element

$$(1+\alpha^{-1}a)^{1/2} = 1 + \tfrac{1}{2}\alpha^{-1}a + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\alpha^{-2}a^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\alpha^{-3}a^3 + \dots$$

belongs to $Q(\tau)^+$. So a^m is the uniform limit

$$a^m = \lim_{\alpha\to+\infty} \alpha^m \binom{\frac{1}{2}}{m}^{-1}\left[(1+\alpha^{-1}a)^{1/2} - \sum_{k=0}^{m-1}\binom{\frac{1}{2}}{k}\alpha^{-k}a^k\right]$$

of elements in $Q(\tau)^s$. Because $Q(\tau)^s$ is closed, we get the lemma.

3. Algebraic regularity of q-topology. Next we introduce a condition connecting τ with the Jordan algebra structure on B.

AXIOM A5. If $a \in A^+(\tau)$, then $a^{1/2} \in L(\tau)$; if $a \in A^+(1-\tau)$, then $a^{1/2} \in U(\tau)$.

An arbitrary Akemann-Giles q-topology satisfies this axiom. Indeed, if $\tau = \tau_A$ for a C^*-algebra A, then $\tau \subset (A^+)^m$ whence $A^+(\tau) \subset (A^+)^m$ and $A^+(\tau)^{1/2} \subset \subset ((A^+)^m)^{1/2}$. Since $(A^+)^{1/2} \subset A^+$ and $t^{1/2}$ is an operator monotone function on B^+ [10] we have $((A^+)^m)^{1/2} \subset (A^+)^m$; aplying Lemma 2.3 we see that $(A^+)^m \subset L(\tau)$. From all that we obtain $(A^+(\tau))^{1/2} \subset L(\tau)$. Similarly we can conclude that $(A^+(1-\tau))^{1/2} \subset U(\tau)$.

Theorem 3.1. *If a q-topology τ satisfies axiom* A5, *then $C(\tau)^s$ is a norm-closed real Jordan subalgebra of B^s and $C(\tau)$ is a JC^*-subalgebra of B.*

Proof. Let X denote the convex cone $\overline{A^+(\tau)} \cap \overline{A^+(1-\tau)}$. In view of Proposition 2.10 it suffices to show that $C(\tau)^+ = X$. The inclusion $C(\tau)^+ \subset X$ follows from the spectral theory. The inverse inclusion is valid since the cone X satisfies all the conditions of Lemma 2.11 (we have $a^{1/2} \in Q(\tau)^s$ whenever $a \in X$ in virtue of axiom A5 and norm-continuity of the operator function $t^{1/2}$).

Axiom A5 may be weakened to get a necessary and sufficient condition for the set $C(\tau)$ be a JC^*-subalgebra of B. Such a weak variant, being equivalent to the original axiom A5 for completely regular q-topologies (see Definition 3.4 below), will concern only the part of q-topology τ, which can be reproduced by means of $C(\tau)$.

Let us define the *regularization* τ^{reg} of a q-topology τ by setting $\tau^{\mathrm{reg}} = \mathrm{Pr}B \cap \cap (C(\tau)^+)^m$ (in general the projections family τ^{reg} need not be a q-topology).

Lemma 3.2. *If $\tau \subset \mathrm{Pr}B$ satisfies axioms* A1–A3, *then $C(\tau) = C(\tau^{\mathrm{reg}})$.*

Proof. Clearly, $C(\tau^{\mathrm{reg}}) \subset C(\tau)$. Conversely, by Proposition 2.5, given $a \in C(\tau)^s$ the C^*-subalgebra $C^*(1, a) \subset C(\tau)$, so for any open $I \subset \mathbb{R}$ the spectral projection $E_a(I)$, being a supremum of an increasing sequence of elements in $C^*(1, a)$, belongs to τ^{reg}. Thus we get $C(\tau)^s \subset C(\tau^{\mathrm{reg}})^s$ and, consequently, $C(\tau) \subset C(\tau^{\mathrm{reg}})$.

AXIOM A5°. If $a \in A^+(\tau^{\mathrm{reg}})$, then $a^{1/2} \in L(\tau)$, if $a \in A^+(1-\tau^{\mathrm{reg}})$ then $a^{1/2} \in U(\tau)$.

Theorem 3.3. *A q-topology τ satisfies axiom* A5° *iff $C(\tau)$ is a JC^*-subalgebra of B.*

Proof. Necessity follows from Proposition 2.9 and the equality

$$C(\tau)^+ = \overline{\Lambda^+(\tau^{\text{reg}})} \cap \overline{\Lambda^+(1 - \tau^{\text{reg}})},$$

which can be easily deduced from Lemma 3.2 and axiom A5° in a manner similar to the proof of Theorem 3.1.

Sufficiency. If $C(\tau)$ is a Jordan algebra, then $C(\tau)^+$ is a convex cone and $\Lambda^+(\tau^{\text{reg}}) \subset (C(\tau)^+)^m$. Indeed, whenever $a = \sum_{i=1}^{n} \lambda_i e_i$, $\lambda_i \geqq 0$, $e_i \in \tau^{\text{reg}}$, there exist increasing sequences $(b_{ik})_{k=1}^{\infty} \subset C(\tau)^+$, $i = 1, 2, \ldots, n$, with $e_i = \bigvee_k b_{ik}$ and $a = \bigvee_k c_k$, where $c_k = \sum_{i=1}^{n} \lambda_i b_{ik}$ is an increasing sequence in $C(\tau)^+$. Now by Proposition 2.5 we have $(C(\tau)^+)^{1/2} \subset C(\tau)^+$ and the operator monotonicity of the function $t^{1/2}$ gives $((C(\tau)^+)^m)^{1/2} \subset (C(\tau)^+)^m \subset L(\tau)$. Similarly we can conclude that $\Lambda^+(1 - \tau^{\text{reg}})^{1/2} \subset U(\tau)$. Hence axiom A5° holds.

Definition 3.4. A q-topology τ is called *completely regular*, if for any q-point x disjoint from a closed q-set f there exists an element $a \in C(\tau)_1^+$ with $ax = x$ and $af = 0$ (this means that a takes the value 1 at x and the value 0 on f).

Any Akemann-Giles q-topology is completely regular [8; 4.7].

Theorem 3.5. *If a completely regular q-topology τ satisfies axiom* A5°, *then* $\tau = \tau^{\text{reg}}$.

Proof. Given any open q-set $e \in \tau$ put

$$I(e) = \{(1 + a)^{-1} . a \mid a \in C(\tau)^+,\ a \leqq \lambda e \text{ for some } \lambda > 0\}.$$

Whenever $(1 + a_i)^{-1} . a_i \in I(e)$, $i = 1, 2$, we have $\tilde{a} = (1 + (a_1 + a_2))^{-1} . (a_1 + a_2) \in I(e)$ and $(1 + a_i)^{-1} . a_i \leqq \tilde{a}$ for $i = 1, 2$ (since by [6; 16.8] the function $(1 + t)^{-1}$ is antimonotone and so the function $(1 + t)^{-1} . t = 1 - (1 + t)^{-1}$ is operator monotone). This means that $I(e)$ is a directed set and there is a supremum e_1 of $I(e)$ in B^s. To complete the proof we shall show that $e = e_1$. The implication $a \leqq \lambda e \Rightarrow (1 + a)^{-1} . a \leqq e$ gives $e_1 \leqq e$. Inasmuch τ is completely regular, for any q-point $x \leqq e$ there exists $a_x \in C(\tau)^+$ with $a_x x = x$ and $a_x \leqq e$. So for each natural n we have $(1 + na_x)^{-1}\, na_x \in I(e)$ whence $e_1 \geqq (1 + na_x)^{-1}\, na_x \geqq (1 + nx)^{-1} . nx$ and $e_1 \geqq x$. Finally, we have $e_1 \geqq \bigvee_{x \leqq e} x = e$ and $e_1 = e$.

Corollary 3.6. *A complete regular q-topology τ satisfies axiom* A5 *iff the set $C(\tau)$ is a JC^*-subalgebra of B.*

4. Symmetry: the sixth axiom. Every unitary $u \in B$ (i.e. such that $uu^* = 1 = u^*u$) yields an * automorphism $\varphi_u : a \to u^*au$ of the W^*-algebra B, which induces an automorphism of the lattice PrB onto itself. If τ is a q-topology in B and a unitary element u is τ-continuous, it is very natural to require φ_u to be a "homeomorphism" of τ. Such requirement seems to be independent of axioms A1 – A5 and so it becomes our last axiom.

AXIOM A6. $u^*\tau u \subset \tau$ for any unitary $u \in C(\tau)$.

As a matter of fact we need only the weakened variant of A6 like that in § 3.

AXIOM A6°. *$u^*\tau^{\mathrm{reg}}u \subset \tau$ for any unitary $u \in C(\tau)$.*

Theorem 4.1. *Let τ be a q-topology. Then $C(\tau)$ is a C^*-subalgebra of B iff τ satisfies axioms* A5° *and* A6°.

Proof. Necessity. In view of Theorem 3.3 we need to check axiom A6° only. Take a unitary $u \in C(\tau)$, then $u^*\, C(\tau)^+\, u = C(\tau)^+$ since φ_u is an * automorphism *and $C(\tau)$ is a C^*-algebra. Axiom A6° holds because*

$$u^*\tau^{\mathrm{reg}}u = u^*((C(\tau)^+)^m \cap \mathrm{Pr}B)\, u =$$

$$= u^*(C(\tau)^+)^m\, u \cap \mathrm{Pr}B = (C(\tau)^+)^m \cap \mathrm{Pr}B = \tau^{\mathrm{reg}}\,.$$

Sufficiency. By Theorem 3.3, $C(\tau)$ is a JC^*-subalgebra of B and we have to prove that $\mathrm{i}[a, b] = \mathrm{i}(ab - ba) \in C(\tau)^s$ whenever $a, b \in C(\tau)^s$ for this implies that $ab = a \circ b + \frac{1}{2}[a, b] \in C(\tau)$. For any $t > 0$ consider the unitary element.

$$u_t = \exp(\mathrm{i}tb) \equiv \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{n!}\, b^n \in C(\tau)^s\,.$$

Then $(1/t)\,(u_t^* a u_t - a) = a_t$ uniformly converges to $\mathrm{i}[a, b]$ as t tends to 0. Since for any unitary $u \in C(\tau)$ we have by the spectral theory, Lemma 3.2 and by virtue of axiom A6°

$$u^*\, C(\tau)^s\, u = u^*\, C(\tau^{\mathrm{reg}})^s\, u \subset C(u^*\tau^{\mathrm{reg}}u)^s \subset C(\tau)^s\,,$$

the elements a_t belong to $C(\tau)^s$ and thus $\mathrm{i}[a, b] \in C(\tau)^s$.

Definition 4.2. A q-topology τ is a *T_1 q-topology*, if for any two disjoint q-points x and y there exists an open q-set $e \in \tau$ with $e \geqq x$ and $ey = 0$. It also means that any q-point is τ-closed.

Proposition 4.3. *Let a T_1 q-topology τ satisfy axioms* A5° *and* A6°, *then τ is completely regular iff $\tau = \tau^{\mathrm{reg}}$.*

Proof. By virtue of Theorem 3.5 it is enough to prove sufficiency. Since τ is T_1, elements of $C(\tau)$ distinguish normal pure states of B and so the C^*-algebra $C(\tau)$ is weakly dense in B. For any $e \in \tau$ consider the C^*-subalgebra $A(e) = \{a \in C(\tau) \mid eae = a\}$. By [8; 4.2 and 4.5] $A(e)$ is weakly dense in the W^*-algebra eBe, so by the transitivity theorem [8; 2.7] for any q-point $x \leqq e$ there exists $a \in A(e)_1^+$ (i.e. $a \leqq e$) with $ax = x$.

We now define a *C^*-topology* as a q-topology which obeys axioms A5 and A6. Any Akemann-Giles q-topology is a C^*-topology.

5. Compactness. Ch. Akemann introduced the notion of q-compactness in terms of the C^*-algebra A (see 1.3), but an intrinsic q-topological description of q-compact q-sets has not been given. Nevertheless, Akemann showed in [3] that the intersection condition for a q-set p (for any decreasing net (q_α) of τ_A-closed q-sets $\forall p \wedge (\bigwedge_{i=1}^{n} q_{\alpha_i}) \neq 0$ implies $p \wedge (\bigwedge_\alpha q_\alpha) \neq 0$) and the regularity after Effros [7] follow from the q-compactness of p, it being unknown whether these conditions are sufficient. We make use of a multiplicative version of the intersection condition.

Let us define a *q-topological space* to be a pair (B, τ) where τ is a q-topology. A q-set $p \in \mathrm{Pr}B$ is called *regular* if for any open q-set $e \in \tau$ $\|pe\| = \|\bar{p}e\|$ ($\bar{p}$ is τ-closure of p).

Definition 5.1. Let (B, τ) be a q-topological space. A q-set $p \in \mathrm{Pr}B$ is called *quasicompact* if for any decreasing net $(b_\alpha) \subset U(\tau)^+$ with $b = \bigwedge_\alpha b_\alpha \in U(\tau)^+$, $\inf \|pb_\alpha p\| = \|pbp\|$. If (B, τ) is a completely regular T_1 q-topological space, then a q-set $p \in \mathrm{Pr}B$ is called *compact* if p is quasicompact and regular. (B, τ) is called a *compact q-space* if the unit 1 is a compact q-set.

Proposition 5.2. *If a completely regular T_1 q-topology satisfies axiom* A5, *then any compact q-set p is closed.*

Proof. Suppose, on the contrary, that $p \neq \bar{p}$ and consider a q-point x with $x \leqq \bar{p} - p$. By Theorem 3.5 there exists an increasing net $(a_\alpha) \subset C(\tau)_1^+$ with $1 - x = \bigvee_\alpha a_\alpha$ Put $b = 1 - a_\alpha$, then $x = \bigwedge_\alpha b_\alpha$, $(b_\alpha) \subset C(\tau)^+ \subset U(\tau)^+$. For all natural n and all α we have $x \leqq E_{b_\alpha}((1 - 1/n, \infty)) \equiv e_{\alpha n} \in \tau$ and $b_\alpha \geqq ((n-1)/n) \,.\, e_{\alpha n}$. Since p is regular we see that

$$1 \geqq \|pb_\alpha p\| \geqq \frac{n-1}{n} \|pe_{\alpha n}p\| = \frac{n-1}{n} \|\bar{p}e_{\alpha n}\bar{p}\| \geqq \frac{n-1}{n} \|\bar{p}x\bar{p}\| =$$

$$= \frac{n-1}{n} \|x\| = \frac{n-1}{n}$$

and finally $\|pb_\alpha p\| = 1$. Since p is compact, this implies that $\|pxp\| = 1$ which contradicts $xp = 0$.

Any Akemann-Giles q-topology is T_1 complete regular [3; III.1 and 8; 3.9] so Definition 5.1 of compact q-sets is applicable.

Theorem 5.3. *Let (B_A, τ_A) be the q-spectrum of a C^*-algebra A. A q-set $p \in \mathrm{Pr}B_A$ is q-compact after Akemann* (see 1.3) *iff p is compact in the sense of Definition* 5.1.

Proof. In § 2 we have mentioned an isomorphism of $\overline{\tilde{A}_m}$ on the cone $U^a(S_A)$. With any q-closed q-set $p \in 1 - \tau_A$ this isomorphism correlates the USC affine function $\hat{p}$ on S_A and the closed face $F(p) = \{\varphi \in S_A \mid \hat{p}(\varphi) = 1\}$. The map $p \mapsto F(p)$ is induced by the Effros-Akemann correspondence between the q-closed q-sets and the $\sigma(A^*, A)$-closed order ideals of A^* ([7], [2]). So that map is a bijection of $1 - \tau_A$ on the set of all closed faces of S_A and for each $p \in 1 - \tau_A$ and $b \in \overline{\tilde{A}_m}$ we have $\|pbp\| = \max_{\varphi \in F(p)} |\hat{b}(\varphi)|$.

Notice that a q-closed p is q-compact after Akemann iff $F(p)$ is a compact subset of S_A. It may be proved in the same way as Akemann-Urysohn's lemma [3; III.1].

Necessity. It is enough to prove that p is quasicompact. Consider a decreasing net $(b_\alpha) \subset U(\tau)^+$ with $b = \bigwedge_\alpha b_\alpha$ and put $\lambda = \inf_\alpha \|pb_\alpha p\| \geqq \|pbp\|$. For each α, $F_\alpha = \{\varphi \in F(p) \mid \hat{b}_\alpha(\varphi) \geqq \lambda\} \neq \emptyset$ is a closed subset of the compact set $F(p)$ so $F_0 = \bigcap_\alpha F_\alpha \neq \emptyset$. Take $\tilde{\varphi} \in F_0$, then $\hat{b}(\tilde{\varphi}) = \inf_\alpha \hat{b}_\alpha(\tilde{\varphi}) \geqq \lambda$ whence $\|pbp\| = \max_{\varphi \in F(p)} \hat{b}(\varphi) \geqq \lambda$.

Sufficiency. If $A \ni 1$, it follows from Proposition 5.2. Let $A \not\ni 1$ and $\tilde{A} = A + \mathbb{C}.1$ as in 2.3. Denote as $\overline{F(p)} \subset S_{\tilde{A}}$ the $\sigma(\tilde{A}^*, \tilde{A})$-closure of $F(p)$. Since $S_{\tilde{A}}$ is compact, it is enough to show $F(p) = \overline{F(p)}$. For any $\varphi \in \overline{F(p)}$ we have $\varphi = \varphi_0 + \lambda\varphi_\infty$, where $\varphi_0 \in A^{*+}$, $\lambda \geqq 0$, φ_∞ is the unique pure state of $\tilde{A}$ which vanishes on A and $\|\varphi\| = \|\varphi_0\| + |\lambda|$. We shall show $\|\varphi\| = \|\varphi_0\|$, whence $\lambda = 0$ and $\varphi \in F(p)$. Let $(u_\alpha) \subset A^+$ be an increasing approximate unit, then $\|pu_\alpha p - p\| \to 0$ since $(1 - u_\alpha) \subset C(\tau_A)^+$, $0 = \bigwedge_\alpha(1 - u_\alpha)$ in B^+ and p is compact. Let us choose $u_n = u_{\alpha_n}$ with $u_{n+1} \geqq u_n$ and $pu_n p \geqq p - 2^{-n}p$, n being natural. Consider the closed subsets $F_n = \{\theta \in \tilde{A}^{*+} \mid \hat{u}_n(\theta) \geqq 1 - 2^n\} \subset \tilde{A}^+$; then $F(p) \subset F_n$, hence $\overline{F(p)} \subset F_n$. This means that $\varphi(u_n) = \varphi_0(u_n) \geqq 1 - 2^{-n}$. Since n was arbitrary, $\|\varphi_0\| = 1 = \|\varphi\|$ as desired.

Definition 5.4. A completely regular T_1 q-topological space (B, τ) is called *locally compact* if for any q-point x there exists an open q-set $e \geqq x$ with the compact q-closure $\bar{e}$.

Theorem 5.5. *A q-topological space (B, τ) is the q-spectrum of a certain $\mathbb{C}^*$-algebra iff τ is a locally compact C^*-topology.*

Proof. Necessity. Let A be a C^*-algebra, then τ_A is the complete regular T_2 C^*-topology by the above. For any q-point $x \in B_A$ take $a \in A_1^+$ with $ax = x$, then $e = E_a(\frac{1}{2}, \infty)) = E_a((\frac{1}{2}, 1]) \geqq x$, $e \in \tau$ and $\bar{e} \leqq E_a([\frac{1}{2}, 1]) \leqq 2a$. So e is q-compact after Akemann, hence e is compact by Theorem 5.3.

Sufficiency. Let (B, τ) be a locally compact C^*-topological space. Then $C(\tau)$ is the weakly dense C^*-subalgebra of B and $\tau = \mathrm{Pr}B \cap (C^+(\tau))^m$ (see 4.3). Put $(\hat{B}, \hat{\tau}) = q \operatorname{spec} C(\tau)$. By [8; 3.4] there exists a central projection $z \in \mathrm{Pr}\hat{B}$ with $B \approx z\hat{B}$. Since the indentification $B = z\hat{B}$ agrees with the inclusions $C(\tau) \subset B$ and $C(\tau) \subset \hat{B}$, we have

$$\tau = \mathrm{Pr}B \cap (C(\tau)^+)^m = \mathrm{Pr}(z\hat{B}) \cap z(C(\tau)^+)^m = z(\mathrm{Pr}\hat{B} \cap (C(\tau)^+)^m) = z\hat{\tau}\,.$$

Let us show $z \in \hat{\tau}$. By hypothesis, if $x \leqq z$ is a q-point in $\hat{B}$ (hence in B) there exists $e \in \tau$ with $e \geqq x$ and $\bar{e}$ q-compact. Besides, there exists $a \in C(\tau)_1^+$ with $ax = x$ and $a \leqq e$. Let p be the support of a in $\hat{B}$. Then $p \in \hat{\tau}$ and $x \leqq p$. If we show $p \leqq z$, the assertion will follow for then $z = \bigvee\{p \in \hat{\tau} \mid x \leqq p \leqq z,\ x \text{ is a } q\text{-point}\}$ which is $\hat{\tau}$-open. To prove $p \leqq z$ we consider any q-point $y \leqq 1 - z$ and show $ay = 0$. Indeed, by [8; 3.9 and 4.2] there exists a decreasing net $(b_\alpha) \subset C(\tau)_1^+$, with $y = \bigwedge_\alpha b_\alpha$

in $\hat{B}$. Then $\|aya\| \leqq \inf_\alpha \|ab_\alpha a\|$ and

$$\|ab_\alpha a\| = \|(b_\alpha^{1/2}a) * (b_\alpha^{1/2}a)\| = \|b^{1/2}a^2b^{1/2}\| \leqq \|b^{1/2}\bar{e}b^{1/2}\| = \|\bar{e}b_\alpha\bar{e}\|\,.$$

Since $0 = \bigwedge_\alpha(zb_\alpha)$ we have $\|\bar{e}b_\alpha\bar{e}\| \to 0$, for $\bar{e}$ is a compact q-set. Thus $\|aya\| = 0$, i.e. $ay = 0$. It implies that $py = 0$, hence $p \leqq z$. So we have that z is $\hat{\tau}$-open. Now consider $A = \{a \in C(\tau) \mid az = a\}$. Then by [8; 5.9] q spec $A = (z\hat{B}, z\hat{\tau}) = (B, \tau)$, which completes the proof.

Corollary 5.6. *A C^*-topological space (B, τ) is compact iff $(B, \tau) = q$ spec $C(\tau)$.*

This last theorem shows that for an arbitrary locally compact C^*-topological space (B, τ) the q-space $(\hat{B}, \hat{\tau}) = q$ spec $C(\tau)$ may be described as "the Stone-Čech compactification of (B, τ)".

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Author's address: University of People's Friendship, Dept. of Math. Analysis, Ordzhonikidze 3, 117923 Moscow, USSR.