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ON WEAK DIRECT PRODUCT DECOMPOSITIONS OF LATTICES AND GRAPHS

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All lattices dealt with in this paper are assumed to be of locally finite lengths. All graphs considered here are undirected.

We denote by C_0 the class of all lattices which are determined up to isomorphisms by their graphs. (Cf. [1], Problem 8.) Some results on lattices belonging to C_0 were established in [7]. In the present paper further results on C_0 will be deduced; we shall apply the notion of the weak direct product as the main tool in this investigation. It will be shown that the class C_0 is closed with respect to weak direct products. It will be proved that a lattice $\mathscr{L}$ belongs to C_0 if and only if all directly indecomposable direct factors of $\mathscr{L}$ belong to C_0. Also it will be shown that each lattice can be embedded into a lattice belonging to C_0.

For a lattice $\mathscr{L}$ we denote by $T(\mathscr{L})$ the set of all nonisomorphic types of lattices whose graphs are isomorphic to the graph of $\mathscr{L}$. By using weak direct product decompositions it will be proved that for each cardinal α there exists a lattice $\mathscr{L}$ with card $T(\mathscr{L}) \geqq \alpha$.

Some questions on the relations between lattices and their graphs were investigated by G. Birkhoff [2], M. Kolibiar and the author [8], and the author (cf. [7] and the papers quoted there).

1. PRELIMINARIES

For the sake of completeness, let us recall some notions concerning graphs of partially ordered sets.

Let $\mathscr{L} = (L; \leqq)$ be a partially ordered set. $\mathscr{L}$ is said to be of *locally finite length* if each bounded chain in $\mathscr{L}$ is finite.

If $a, b \in L$ and a is covered by b (i.e., the interval $[a, b]$ of $\mathscr{L}$ is prime), then we write $a \prec b$ or $b \succ a$.

By the graph $\mathscr{G}(\mathscr{L})$ we mean the graph whose set of vertices is L and whose edges are those pairs (a, b) which satisfy either $a \prec b$ or $a \succ b$. The notion of isomorphism of graphs is defined in the usual way. If $\mathscr{L}_1 = (L_1; \leqq_1)$ is a lattice and h is an isomorphism of $\mathscr{G}(\mathscr{L})$ onto $\mathscr{G}(\mathscr{L}_1)$, then h is called a *graph isomorphism of the lattice $\mathscr{L}$ onto $\mathscr{L}_1$*.

Let C_0 be the class of all lattices which are determined up to isomorphisms by their graphs (i.e., a lattice $\mathscr{L}$ belongs to C_0 iff, whenever for some lattice $\mathscr{L}_1$ the graph $\mathscr{G}(\mathscr{L})$ is isomorphic to $\mathscr{G}(\mathscr{L}_1)$, then $\mathscr{L}$ isomorphic to $\mathscr{L}_1$). Further, let C_1 be the class of all lattices $\mathscr{L}$ having the property that whenever h is an isomorphism of the graph of $\mathscr{L}$ onto the graph of a lattice $\mathscr{L}_1$, then h is either an isomorphism or a dual isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1$.

The following results were established in [7]:

(A) *Each lattice can be embedded into a lattice belonging to* C_1.

(B) *Each bounded lattice can be embedded into a lattice belonging to* $C_0 \cap C_1$.

(C) *Each bounded modular (distributive) lattice can be embedded as a convex sublattice into a bounded modular (distributive) lattice belonging to* $C_0 \cap C_1$.

Also, in [7] it was shown by examples that $C_0 \setminus C_1 \neq \emptyset$ and $C_1 \setminus C_0 \neq \emptyset$.

Again, let $\mathscr{L} = (L; \leqq)$ be a partially ordered set. Each nonempty subset of L is considered partially ordered (by means of the partial order inherited from $\mathscr{L}$). Let x_1, x_2, x_3, x_4 be distinct elements of L such that (x_1, x_2), (x_2, x_3), (x_3, x_4) and (x_4, x_1) are edges of $\mathscr{G}(\mathscr{L})$. Then $Q = (x_1, x_2, x_3, x_4)$ is said to be an elementary quadruple in $\mathscr{L}$.

1.1. Lemma. *Let $Q = (x_1, x_2, x_3, x_4)$ be an elementary quadruple in $\mathscr{L}$. Then Q is isomorphic either to the partially ordered set in Fig.* 1a *or to the partially ordered set in Fig.* 1b.

Proof. Q cannot be linearly ordered, hence there are $a, b \in Q$ such that a is incomparable with b. Let $x \in Q \setminus \{a, b\}$. Then we have either (i) $x \prec a$ and $x \prec b$, or (ii) $x \succ a$ and $x \succ b$. Similarly, if $y \in Q \setminus \{a, b, x\}$, then either ($\mathrm{i}_1$) $y \prec a$ and $y \prec b$, or (ii_1) $y \succ a$ and $y \succ b$.

Fig. 1a.
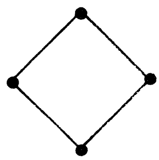
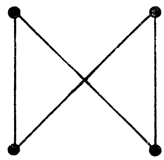
Fig. 1b.

If (i) and (i_1), or (ii) and (ii_1) are valid, then Q is isomorphic to Fig. 1b; if (i) and (ii_1), or (ii) and (i_1) hold, then Q is isomorphic to Fig. 1a.

An elementary quadruple is called regular (irregular) if it is isomorphic to Fig. 1a (or 1b, respectively).

The following two lemmas are obvious.

1.2. Lemma. *Let $Q = (x_1, x_2, x_3, x_4)$ be a regular quadruple in $\mathscr{L}$. Then we have either* (i) $x_1 \prec x_2$ *and* $x_4 \prec x_3$, *or* (ii) $x_1 \succ x_2$ *and* $x_4 \succ x_3$.

1.3. Lemma. *Let $\mathscr{L}$ be a lattice. Then $\mathscr{L}$ does not contain any irregular quadruple.*

2. DIRECT DECOMPOSITIONS

We recall the notion of the direct product of graphs.

Let I be a nonempty set and for each $i \in I$ let $\mathcal{G}_i = (G_i, E_i)$ be a graph, where G_i and E_i are the sets of all vertices and all edges of G_i, respectively. Let G be the cartesian product of the sets $G_i (i \in I)$ and let $\mathcal{G}$ be the graph whose set of vertices is G and whose set of edges consists of those pairs $(x, y) \in G \times G$ which satisfy the following condition: there is $i \in I$ such that $x_j = y_j$ for each $j \in I$ with $j \neq i$, and $(x_i, y_i) \in E_i$. Then $\mathcal{G}$ is said to be the *direct product of the graphs* $\mathcal{G}_i (i \in I)$ and we denote this fact by writing $\mathcal{G} = \prod_{i \in I} \mathcal{G}_i$.

The direct product of partially ordered sets is defined in the usual way (cf., e.g., [4]). If φ is an isomorphism of a partially ordered set $\mathcal{L}$ onto the direct product $\prod_{i \in I} \mathcal{L}_i$ of partially ordered sets $\mathcal{L}_i$, then φ is said to be a *direct product representation of* $\mathcal{L}$. An analogous terminology is adopted for graphs. The following lemma is obvious.

2.1. Lemma. *Let $\mathcal{L}_i = (L_i; \leqq)$ be partially ordered sets $(i \in I)$. Let $\mathcal{L} = \prod_{i \in I} \mathcal{L}_i = (L; \leqq)$ and $x, y \in L$. Then $x \prec y$ if and only if there exists $i \in I$ such that $x_j = y_j$ for each $j \in I$ with $j \neq i$, and $x_i \prec y_i$.*

From 2.1 we immediately obtain

2.2. Corollary. *Let L be as in* 2.1. *Suppose that I is finite. Then $\mathcal{G}(\mathcal{L}) = \prod_{i \in I} \mathcal{G}(\mathcal{L}_i)$.*

The following result was established in [3] (Thm. 4) by applying a theorem of Kolibiar [9]:

(D) *Let $\mathcal{L}$ be a lattice. Let $\mathcal{G}_i = (\mathcal{G}_i, \mathcal{E}_i)$ $(i = 1, 2)$ be graphs and let $\varphi: \mathcal{G}(\mathcal{L}) \to \mathcal{G}_1 \times \mathcal{G}_2$ be a direct product representation of $\mathcal{G}(\mathcal{L})$. Then there exist partial orders $\leqq_1$ and $\leqq_2$ on G_1 and G_2, respectively, such that $\varphi: \mathcal{L} \to \mathcal{L}_1 \times \mathcal{L}_2$ is a direct product representation of $\mathcal{L}$, where $\mathcal{L}_1 = (G_1; \leqq_1)$ and $\mathcal{L}_2 = (G_2; \leqq_2)$. Moreover, $\mathcal{G}(\mathcal{L}_1) = \mathcal{G}_1$ and $\mathcal{G}(\mathcal{L}_2) = \mathcal{G}_2$.*

Since direct products are associative, from (D) we obtain by induction:

2.3. Corollary. *Let $\mathcal{L}$ be a lattice. Let $\mathcal{G}_i = (G_i, E_i)$ $(i \in I = \{1, 2, \ldots, n\})$ be graphs and let $\varphi: \mathcal{G}(\mathcal{L}) \to \prod_{i \in I} \mathcal{G}_i$ be a direct product representation of $\mathcal{G}(\mathcal{L})$. Then there exist lattice orders $\leqq_i$ on $G_i (i \in I)$ such that $\varphi: \mathcal{L} \to \prod_{i \in I} \mathcal{L}_i$ is a direct product representation of $\mathcal{L}$, where $\mathcal{L}_i = (G_i; \leqq_i)$. Moreover, $\mathcal{G}(\mathcal{L}_i) = \mathcal{G}_i$ for each $i \in I$.*

3. THE CLASS C_0

In this section it will be proved that the class C_0 is closed with respect to finite direct products. Two types of embeddings are described which show that each lattice can be embedded into a lattice belonging to C_0.

3.1. Theorem. *Let $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n \in C_0$, $\mathscr{L} = \mathscr{L}_1 \times \mathscr{L}_2 \times \ldots \times \mathscr{L}_n$. Then $\mathscr{L}$ belongs to C_0.*

Proof. Let $\mathscr{L}_{(1)}$ be a lattice and let h be a graph isomorphism of the lattice $\mathscr{L}$ onto $\mathscr{L}_{(1)}$. In view of 2.2, there exists an isomorphism φ of $\mathscr{G}(\mathscr{L})$ onto $\prod_{i\in I} \mathscr{G}(\mathscr{L}_i)$ (where $I = \{1, 2, \ldots, n\}$). Thus $\varphi' = h^{-1} \circ \varphi$ is an isomorphism of $\mathscr{G}(\mathscr{L}_{(1)})$ onto $\prod_{i\in I} \mathscr{G}(\mathscr{L}_i)$. According to 2.3, φ' is at the same time a direct product representation $\varphi' : \mathscr{L}_{(1)} \to \prod_{i\in I} \mathscr{L}_i^1$ such that $\mathscr{G}(\mathscr{L}_i) = \mathscr{G}(\mathscr{L}_i^1)$ for each $i \in I$. Since $\mathscr{L}_i \in C_0$, we have $\mathscr{L}_i \cong \mathscr{L}_i^1$, whence $\mathscr{L} \cong \mathscr{L}_{(1)}$. Therefore $\mathscr{L} \in C_0$.

Let us remark that the class C_1 fails to be closed with respect to finite direct products. Example: Let $\mathscr{L}_1$ be as in Fig. 3.1. Then $\mathscr{L}_1 \in C_1$, but $\mathscr{L}_1 \times \mathscr{L}_1$ does not belong to C_1.

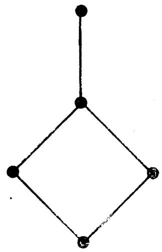

Fig. 3.1.

Let $\mathscr{L} = (L; \leqq)$ be a lattice. In [7] a lattice $\mathscr{L}' = (L'; \leqq) \in C_1$ was constructed such that $\mathscr{L}$ was a sublattice of $\mathscr{L}'$. The idea of constructing $\mathscr{L}'$ from $\mathscr{L}$ was as follows: for each triple of elements u, x, v of L such that $u \prec x \prec v$ was valid in $\mathscr{L}$ two new elements a_1 and a_2 were added to L such that $u \prec a_i \prec v$ $(i = 1, 2)$ was fulfilled in $\mathscr{L}'$. For details, cf. [7].

The construction of $\mathscr{L}'$ is intrinsic: if $\mathscr{A}$ and $\mathscr{B}$ are lattices such that $\mathscr{A} \cong \mathscr{B}$, then $\mathscr{A}' \cong \mathscr{B}'$. Hence if $\mathscr{L}$ is isomorphic to $\mathscr{L}^\sim$, then $\mathscr{L}' \cong (\mathscr{L}^\sim)'$. (For each partially ordered set $\mathscr{L}$ we denote by $\mathscr{L}^\sim$ the dual of $\mathscr{L}$.)

Now let us denote

$$\mathscr{L}^0 = (\mathscr{L} \times \mathscr{L}^\sim)' .$$

Let $\mathscr{L}_1$ be a lattice and let h be a graph isomorphism of $\mathscr{L}^0$ onto $\mathscr{L}_1$. The lattice $\mathscr{L}^0$ belongs to C_1, hence h is either an isomorphism or a dual isomorphism of $\mathscr{L}^0$ onto $\mathscr{L}_1$. The lattice $\mathscr{L} \times \mathscr{L}^\sim$ is clearly isomorphic with its dual, hence $\mathscr{L}^0$ is isomorphic with its dual as well. Therefore $\mathscr{L}^0$ is isomorphic to $\mathscr{L}_1$ and thus $\mathscr{L}^0 \in C_0$. Hence we have proved the following generalization of (B) (cf. § 1):

3.2. Proposition. *Each lattice can be embedded as a sublattice into a lattice belonging to $C_0 \cap C_1$.*

Prime intervals $[u_1, v_1]$ and $[u_2, v_2]$ of a lattice $\mathscr{L}$ are said to be *equivalent* if, whenever h is a graph isomorphism of $\mathscr{L}$ onto a lattice $\mathscr{L}_1$, then we have either (i) $h(u_1) \prec h(v_1)$ and $h(u_2) \prec h(v_2)$, or (ii) $h(u_1) \succ h(v_1)$ and $h(u_2) \succ h(v_2)$. The lattice $\mathscr{L}$ belongs to C_1 if and only if any two prime intervals of $\mathscr{L}$ are equivalent.

Proposition 3.2 can be sharpened by applying the following construction. Let $\mathscr{L} = (L; \leqq)$ be a lattice. Let $\mathscr{A} = (A; \leqq)$ be a lattice isomorphic to $\mathscr{L}'$; we suppose that $L \cap A = \emptyset$. Since $\mathscr{L}$ is embedded into $\mathscr{L}'$, there exists an isomorphism φ of $\mathscr{L}$ into $\mathscr{A}$ such that $\varphi(L)$ is a sublattice of $\mathscr{A}$. For each $x \in L$ we denote $\varphi(x) = x'$.

Put $A_1 = L \cup A$. We define a partial order $\leqq_1$ on A_1 as follows. Let $p, q \in A_1$. We put $p \leqq_1 q$ if one of the following conditions is valid:

(i) $p, q \in L$ or $p, q \in A$ and $p \leqq q$ holds in $\mathscr{L}$ or in $\mathscr{A}$, respectively.

(ii) $p \in L$, $q \in A$ and $p' \leqq q$.

Then

a) $\mathscr{A}_1 = (A_1; \leqq_1)$ is a lattice, and

b) both $\mathscr{L}$ and $\mathscr{A}$ are convex sublattices of $\mathscr{A}_1$. We shall verify, for instance, that for any $p, q \in A_1$ the meet $a \wedge b$ in $\mathscr{A}_1$ exists. The proof concerning the existence of $p \vee q$ can be performed analogously; the assertion b) easily follows from a).

Let $p, q \in A_1$. If both p and q belong to L, then let z be the greatest lower bound of the set $\{p, q\}$ in $\mathscr{L}$; clearly z is also the greatest lower bound of the set $\{p, q\}$ in $\mathscr{A}_1$. The case when both p and q belong to $\mathscr{A}$ is similar. Hence it suffices to consider the case when $p \in L$ and $q \in A$.

Let M be the set of all lower bounds of the set $\{p, q\}$ in $\mathscr{A}_1$. In view of (i) and (ii) we have $M \subseteq L$. As we have already remarked above, for each $a \in L' \setminus L$ there are elements $u, v \in L$ such that $u \prec a \prec v$; hence for each $y \in L'$ there is $u \in L$ with $u \leqq y$. Thus for each $y_1 \in A$ there is $u \in \varphi(L)$ such that $u \leqq y_1$. Put $y_1 = q$, $u_1 = \varphi^{-1}(u)$ and $z = u_1 \wedge p$. Then $z' \leqq u_1' = u \leqq q$, hence $z \leqq_1 q$ and therefore $z \in M$. Thus $M \neq \emptyset$.

Let $z_1, z_2 \in M$. Let z_3 be the join in $\mathscr{L}$ of the pair (z_1, z_2). Then we have $z_3 \leqq_1 p$. Moreover, $z_3' = z_1' \vee z_2'$ holds in the lattice $\mathscr{A}$ (because $\varphi(L)$ is a sublattice of $\mathscr{A}$).

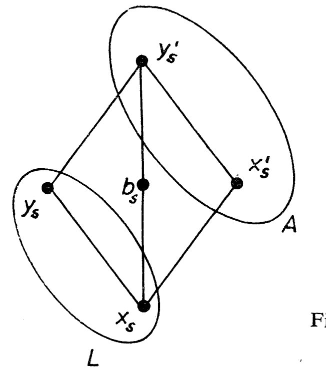

Fig. 3.2.

In view of $z_1' \leqq q$ and $z_2' \leqq q$ we obtain $z_3' = z_1' \vee z_2' \leqq q$, whence $z_3' \leqq q$. Therefore $z_3 \in M$. Since $M \subseteq L$ and $\mathscr{L}$ is of locally finite length, we infer that M possesses a greatest element z_0. Thus $z_0 = p \wedge q$ in $\mathscr{A}_1$.

Let $[x_s, y_s]_{s \in S}$ be the set of all prime intervals of $\mathscr{L}$. Assume that b_s $(s \in S)$ are distinct elements which do not belong to A_1. Put $B = \{b_s\}_{s\in S}$, $A_2 = A_1 \cup B$. Let $u, v \in A_2$. We put $u \leqq_2 v$ if some of the following conditions holds:

(i_1) $u, v \in A_1$ and $u \leqq_1 v$;

(ii_1) $u = b_s \in B$, $v \in A$ and $y'_s \leqq v$.

(iii_1) $v = b_s \in B$, $u \in L$ and $u \leqq x_s$.

(Cf. Fig. 3.2.) Then $\mathscr{A}_2 = (A_2; \leqq_2)$ is a lattice and $\mathscr{L}$ is a convex sublattice of $\mathscr{A}_2$.

If we consider the set $M = \{x_s, y_s, x'_s, y'_s, b_s\}$ (where $[x_s, y_s]$ is a prime interval of $\mathscr{L}$), then $(M; \leqq_2)$ is a convex sublattice of $\mathscr{A}_2$ and in view of Lemma 1.3, [7] all prime intervals of $(M; \leqq_2)$ are mutually equivalent. In particular, each prime interval of $\mathscr{L}$ is equivalent to some prime interval of the lattice $\mathscr{A}$.

Since $\mathscr{A}$ is isomorphic to $\mathscr{L}'$ and $\mathscr{L}' \in C_1$, any two prime intervals of $\mathscr{A}$ are equivalent. Thus any two prime intervals of the lattice $\mathscr{A}_2$ are equivalent. This yields that $\mathscr{A}_2$ belongs to the class C_1. Hence we have

3.3. Proposition. *Each lattice can be embedded as a convex sublattice into a lattice belonging to C_1.*

This sharpens Theorem (A) (cf. § 1).

Let $\mathscr{L}$ and $\mathscr{A}_2$ be as above. We denote $\mathscr{A}_2 = a(\mathscr{L})$. If $\mathscr{L}$ is self-dual, then $a(\mathscr{L})$ is self-dual as well and, in view of 3.3, in this case the lattice $a(\mathscr{L})$ belongs to C_0.

Let us consider the lattice $a(\mathscr{L} \times \mathscr{L}^\sim)$. The lattice $\mathscr{L}$ is embedded as a convex sublattice into $a(\mathscr{L} \times \mathscr{L}^\sim)$. Hence $\mathscr{L} \times \mathscr{L}^\sim$ is self-dual and in view of 3.3, $a(\mathscr{L} \times \mathscr{L}^\sim)$ belongs to $C_1 \cap C_0$. Therefore we have

3.4. Theorem. *Each lattice can be embedded as a convex sublattice into a lattice belonging to $C_0 \cap C_1$.*

This sharpens Theorem (B) and generalizes Theorem (C) above (cf. § 1).

4. WEAK DIRECT PRODUCT DECOMPOSITIONS

In this section we shall apply Theorem 2.3 for investigating weak direct product decompositions of partially ordered sets and graphs.

We begin by recalling some definitions (cf. also Grätzer [4]).

Let us consider a direct product $\prod_{i\in I} \mathscr{L}_i$ of partially ordered sets $\mathscr{L}_i = (L_i; \leqq)$. Let $L_w \neq \emptyset$ be a subset of the cartesian product $\prod_{i\in I} L_i$ such that

(i) if $x, y \in L_w$, then the set $\{i \in I: x(i) \neq y(i)\}$ is finite;

(ii) if $x \in L_w$, $y \in \prod_{i\in I} L_i$ and if the set $\{i \in I: x(i) \neq y(i)\}$ is finite, then $y \in L_w$.

The partially ordered set $\mathscr{L}_w = (L_w; \leqq)$ is said to be a *weak direct product of the system* $\{\mathscr{L}_i\}_{i\in I}$. We denote this fact by writing $\mathscr{L}_w = (w) \prod_{i\in I} \mathscr{L}_i$. (Cf. also [5], [6].)

Similarly, consider a direct product $\prod_{i\in I} \mathscr{G}_i$ of graphs $\mathscr{G}_i = (G_i; E_i)$ and let G_w be a subset of the cartesian product $\prod_{i\in I} G_i$ fulfilling the conditions (i) and (ii) (with L_w and L_i replaced by G_w and G_i). Let $\mathscr{G} = (G_w, E_w)$ be the graph whose set of vertices is G_w and whose set of edges E_w consists of those pairs (x, y) of elements of G_w which have the following property: there exists $j \in I$ such that $(x(j), y(j)) \in E_j$, and $x(i) = y(i)$ for each $i \in I$ with $i \neq j$. Then $\mathscr{G}$ is said to be the weak direct product of the system $\{\mathscr{G}_i\}_{i \in I}$; we write $\mathscr{G} = (w) \prod_{i \in I} \mathscr{G}_i$. Weak direct products of graphs were investigated by Miller [10].

The following lemma generalizes Corollary 2.2.

4.1. Lemma. *Let $\varphi\colon \mathscr{L} \to (w) \prod_{i \in I} \mathscr{L}_i$ be a weak direct product representation of a partially ordered set $\mathscr{L}$. Then $\varphi\colon \mathscr{G}(\mathscr{L}) \to (w) \prod_{i \in I} \mathscr{G}(\mathscr{L}_i)$ is a weak direct product representation of the graph $\mathscr{G}(L)$.*

This is an immediate consequence of the above definitions.

Let $\mathscr{L} = (L; \leqq)$ be a partially ordered set and suppose that $\mathscr{L}$ has a weak direct representation

$$\varphi\colon \mathscr{L} \to (w) \prod_{i \in I} \mathscr{L}_i\,, \tag{4.1}$$

where $\mathscr{L}_i = (L_i; \leqq_i)$. Let j be a fixed element of the set I. Put $I(j) = I \setminus \{j\}$. Let L_j^* be the set of all elements y of the cartesian product $\prod_{i \in I(j)} L_i$ which have the following property: there exists $x \in L$ such that $\varphi(x)(i) = y(i)$ for each $i \in I(j)$. Put $\mathscr{L}_j^* = (L_j^*, \leqq)$. Then $\mathscr{L}_j^* = (w) \prod_{i \in I(j)} \mathscr{L}_i$ and we have a direct product representation

$$\varphi_j\colon \mathscr{L} \to \mathscr{L}_j \times \mathscr{L}_j^*\,. \tag{4.2}$$

If $x \in L$, then the component of $\varphi(x)$ in $\mathscr{L}_j$ with respect to the weak direct product decomposition (4.1) is the same as the component of $\varphi_j(x)$ in L_j with respect to the direct product decomposition (4.2).

An analogous assertion is valid for weak direct product decompositions of graphs.

The following theorem generalizes Corollary 2.3.

4.2. Theorem. *Let $\mathscr{L} = (L; \leqq)$ be a lattice. Let $\mathscr{G}_i = (G_i, E_i)$ $(i \in I)$ be graphs and let $\varphi\colon \mathscr{G}(\mathscr{L}) \to (w) \prod_{i \in I} \mathscr{G}_i$ be a weak direct product representation of the graph $\mathscr{G}(\mathscr{L})$. Then there exist partial orders $\leqq_i$ on $G_i (i \in I)$ such that* (i) *$\varphi\colon \mathscr{L} \to (w) \prod_{i \in I} \mathscr{L}_i$ is a weak direct product representation of $\mathscr{L}$, where $\mathscr{L}_i = (G_i; \leqq_i)$, and* (ii) *$\mathscr{G}(\mathscr{L}_i) = \mathscr{G}_i$ for each $i \in I$.*

Proof. Under analogous notations as above, for each $j \in I$ we have the direct representation

$$\varphi_j\colon \mathscr{G}(\mathscr{L}) \to \mathscr{G}_j \times \mathscr{G}_j^* \tag{4.3}$$

such that for each $x \in L$ the relation $\varphi(x)(j) = (\varphi_j(x))_j$ is valid, the symbol on the right hand side denoting the component of $\varphi_j(x)$ in the direct factor G_j. Now from (4.3) and (D) we infer that φ_j turns out to be also a direct product representation for the partially ordered set $\mathscr{L}$:

$$\varphi_j\colon \mathscr{L} \to \mathscr{L}_j \times \mathscr{L}_j^*\,, \tag{4.4}$$

where $\mathscr{L}_j = (G_j; \leqq_j)$ is a partially ordered set with $\mathscr{G}(\mathscr{L}_j) = \mathscr{G}_j$. Let $x, y \in L$.

From (4.4) we obtain that

$$x \leqq y \Rightarrow \varphi(x)(j) \leqq_j \varphi(y)(j) \quad \text{for each} \quad j \in I\,.$$

Conversely, assume that $\varphi(x)(j) \leqq_j \varphi(y)(j)$ is valid for each $j \in I$. There exists a finite set $J = \{j(1), j(2), \ldots, j(n)\} \subseteq I$ such that $\varphi(x)(i) = \varphi(y)(i)$ for each $i \in I \setminus J$. The case $J = \emptyset$ is trivial; suppose that $J \neq \emptyset$. From (4.4) we obtain by induction that there is a direct product representation

$$\varphi_J\colon \mathscr{L} \to \mathscr{L}_{j(1)} \times \mathscr{L}_{j(2)} \times \ldots \times \mathscr{L}_{j(n)} \times \mathscr{L}_J^* \tag{4.5}$$

such that for each $t \in L$, the component of $\varphi(t)$ in $\mathscr{L}_{j(k)}$ with respect to (4.4) is the same as the component of $\varphi(t)$ in $\mathscr{L}_{j(k)}$ with respect to (4.5); moreover, the components of $\varphi(x)$ and $\varphi(y)$ in $\mathscr{L}_J^*$ coincide. Hence from (4.5) we infer that $x \leq y$. Thus φ is a weak direct product representation of $\mathscr{L}$.

4.3. Theorem. *The class C_0 is closed with respect to weak direct products.*

The proof is analogous to that of 3.1 (with the distinction that instead of 2.3 we apply Theorem 4.2).

5. DIRECTLY INDECOMPOSABLE FACTORS

A partially ordered set $\mathscr{L}$ is said to be *directly indecomposable* if, whenever $\mathscr{L}$ is isomorphic to a direct product $\mathscr{L}_1 \times \mathscr{L}_2$ of partially ordered sets $\mathscr{L}_i = (L_i; \leqq)$ $(i = 1, 2)$, then either card $L_1 = 1$ or card $L_2 = 1$.

5.1. Proposition. (Cf. [5], [6].) *Let $\mathscr{L}$ be a lattice. Then there exists a weak direct representation $\varphi\colon \mathscr{L} \to (w)\prod_{i\in I} \mathscr{L}_i$ such that all $\mathscr{L}_i$ are directly indecomposable.*

Also, the weak direct representation with directly indecomposable factors is uniquely determined up to isomorphism (cf. [5], p. 410).

5.2. Theorem. *Let $\mathscr{L}$ be a lattice. Then the following conditions are equivalent:* (i) *all directly indecomposable direct factors of $\mathscr{L}$ belong to C_0*; (ii) *$\mathscr{L}$ belongs to C_0.*

Proof. Let (i) be valid. Then in view of 5.1 and 4.3 we have $\mathscr{L} \in C_0$. Conversely, assume that $\mathscr{L} \in C_0$. By way of contradiction, suppose that there exists an indecomposable direct factor $\mathscr{A}$ of $\mathscr{L}$ which does not belong to C_0. Let $\mathscr{L}_i$ $(i \in I)$ be as in 5.1; we may suppose that $\mathscr{A} = \mathscr{L}_i$ for some $i \in I$. Let I_0 be the set of all $j \in I$ such that $\mathscr{L}_j$ is isomorphic to $\mathscr{A}$. Since $\mathscr{A} \notin C_0$, there is a lattice $\mathscr{B}$ such that $\mathscr{G}(\mathscr{A}) \cong \mathscr{G}(\mathscr{B})$ and $\mathscr{B}$ is not isomorphic to $\mathscr{A}$. In view of 4.1 there exists a lattice $\mathscr{L}_1 = (w)\prod_{i\in I} \mathscr{L}_i^1$ such that (i) if $i \in I_0$, then $\mathscr{L}_i^1 \cong \mathscr{B}$; (ii) if $i \in I \setminus I_0$, then $\mathscr{L}_i^1 \cong$ $\cong \mathscr{L}_i$; (iii) $\mathscr{G}(\mathscr{L}) \cong \mathscr{G}(\mathscr{L}_1)$. No directly indecomposable direct factor of $\mathscr{L}_1$ is isomorphic to $\mathscr{A}$, hence $\mathscr{L}_1$ fails to be isomorphic with $\mathscr{L}$, which is a contradiction.

Let us denote by $T(\mathscr{L})$ the set of all nonisomorphic types of lattices whose graphs are isomorphic to the graph of a given lattice $\mathscr{L}$.

5.3. Theorem. *Let $\alpha > 1$ be a cardinal. There exists a lattice $\mathcal{L}$ with $T(\mathcal{L}) \geqq \alpha$.*

Proof. Let I be a set such that card $I = \alpha - 1$ if α is finite and card $I = \alpha$ if α is infinite. For each $i \in I$ let $\mathcal{L}_i$ be the lattice in Fig. 3.1. Let $\mathcal{L}_0 = \prod_{i \in I} \mathcal{L}_i$, $\mathcal{L}_0 = (L_0; \leqq)$. Let $x_0 \in L_0$ with $x(i) = u$ for each $i \in I$. Let $\mathcal{L}$ be the weak product $(w) \prod_{i \in I} \mathcal{L}_i$ such that $\mathcal{L} = (L; \leqq)$, $x_0 \in L$.

If $\mathcal{L}_i^1$ is a lattice such that $\mathcal{G}(\mathcal{L}_i^1) = \mathcal{G}(\mathcal{L}_i)$, then either $\mathcal{L}_i^1 \cong \mathcal{L}_i$ or $\mathcal{L}_i^1 \cong \mathcal{L}_i^{\sim}$. According to 4.1 we have $\mathcal{G}(\mathcal{L}) = (w) \prod_{i \in I} \mathcal{G}(\mathcal{L}_i)$. In view of 4.2 there exists a lattice $\mathcal{L}^2 = (L; \leqq_2)$ such that $\mathcal{L}^2 = (w) \prod_{i \in I} \mathcal{L}_i^2$, where $\mathcal{L}_i^2 \cong \mathcal{L}_i$ or $\mathcal{L}_i^2 \cong \mathcal{L}_i^{\sim}$ for each $i \in I$.

Let us denote by I_1 the set of all $i \in I$ with $\mathcal{L}_i^2 \cong \mathcal{L}_i$. The lattice $\mathcal{L}^2$ is determined up to isomorphism by the power of the set I_1; for distinct cardinalities of I_1 the corresponding lattices are not isomorphic. Hence card $T(\mathcal{L}) \geqq \alpha$.

Let $\{A_i\}_{i \in I}$ be the set of all antichains of a lattice $\mathcal{L}$; we put $b(\mathcal{L}) = \sup \{\text{card } A_i\}_{i \in I}$. The cardinal $b(\mathcal{L})$ is said to be the breadth of $\mathcal{L}$.

We conclude by mentioning without proof that there exists a lattice $\mathcal{L}$ with $b(\mathcal{L}) = 2$ such that (under Continuum Hypothesis) we have card $T(\mathcal{L}) = c$ (the power of the continuum).

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