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CRITERIA FOR AN EXPONENTIAL DICHOTOMY OF DIFFERENCE EQUATIONS

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In this paper we give sufficient and necessary conditions for exponential dichotomy of a linear difference equation having the form

$$x(n+1) = A(n)\,x(n)\,, \tag{1}$$

where $A(n)$ is a $k \times k$ invertible matrix for $n \in N$ such that

$$|A(n)| \leqq M \quad \text{for} \quad n = 1, 2, 3, \ldots, \quad M \geqq 1 \tag{2}$$

with elements $a_{ij}(n)$ real functions on $N = \{0, 1, \ldots\}$. In what follows we denote by $|\cdot|$ any convenient norm either of a vector or of a matrix.

The difference equation (1) is said to *possess an exponential dichotomy on the set* N if there exist a projection P, that is a matrix such that $P^2 = P$ and constants $K > 0$, $0 < p < 1$ such that

$$\begin{aligned} &|X(n)\,P| \leqq Kp^{n-m}|X(m)\,P|\,, && n \geqq m \geqq 0\,, \\ &|X(n)\,(I-P)| \leqq Kp^{m-n}|X(m)\,(I-P)|\,, && m \geqq n \geqq 0\,, \end{aligned} \tag{3}$$

where $X(n)$ is the matrix solution $X(n) = A(n-1)\ldots A(o)$, $X(o) = I$. Since $A(n)$ is a $k \times k$ invertible and bounded matrix, it can be easily proved that this definition is equivalent to Henry's definition [3, p. 229].

First we prove a lemma which we use in the following.

Lemma 1. *Suppose that* (1) *has exponential dichotomy for* $n \geqq T$, $T \in N$. *Then* (1) *has exponential dichotomy for* $n \geqq 0$.

Proof. From (2) we have

$$|X(n)\,(I-P)| \leqq M^n|(I-P)|\,. \tag{4}$$

Since (1) has exponential dichotomy for $m \geqq T$ we have

$$|X(T)\,(I-P)| \leqq Kp^{m-T}|X(m)\,(I-P)|\,. \tag{5}$$

Let $0 \leqq n \leqq T \leqq m$. Then, by (4), (5), we have

$$|X(n)(I-P)| \leqq M^n|(I-P)| \leqq \frac{M^T|(I-P)|}{|X(T)(I-P)|}|X(T)(I-P)| \leqq$$

$$\leqq \frac{K(Mp^{-1})^T|(I-P)|}{|X(T)(I-P)|}p^{m-n}|X(m)(I-P)|.$$

Let $0 \leqq n \leqq m \leqq T$ and $K_1 = \min\{|X(n)(I-P)|: 0 \leqq n \leqq T\}$. We have $|X(n)(I-P)| \neq 0$ for every $n \in N$. So $K_1 \neq 0$. Therefore

$$|X(n)(I-P)| \leqq M^T|(I-P)| \leqq K_1^{-1}M^T|(I-P)|\,|X(m)(I-P)|.$$

Since $0 \leqq n \leqq m \leqq T$ we have $1 \leqq p^{-T+m-n}$ and therefore

$$|X(n)(I-P)| \leqq K_1^{-1}(Mp^{-1})^T|(I-P)|\,p^{m-n}|X(m)(I-P)|.$$

In the same manner one can prove the first inequality of the exponential dichotomy.

We prove now our main results. The following two propositions are the discrete analogue of those which have been proved by Coppel [2, p. 14] in the continuous case. The adaptation from the continuous to the discrete case is not direct but requires some special devises.

Proposition 1. *Suppose that* (1) *has exponential dichotomy. Then there exist constants* $0 < \theta < 1$, $T > 0$, $T \in N$ *such that*

$$|x(n)| \leqq \theta \sup\{|x(u)| : |u-n| \leqq T,\quad u, n \in N,\quad n \geqq T\}.$$

Proof. We set

$$x_1(n) = X(n)\,p\xi,\quad x_2(n) = X(n)(I-P)\,\xi.$$

Then

$$x(n) = x_1(n) + x_2(n).$$

First consider the case $|x_2(m)| \geqq |x_1(m)|$, for some $m \in N$. From (3), for $n \geqq m \geqq 0$, we have

$$|x_2(m)| \leqq Kp^{n-m}|x_2(n)| \quad \text{or} \quad |x_2(n)| \geqq K^{-1}p^{-(n-m)}|x_2(m)|$$

$$|x_1(n)| \leqq Kp^{n-m}|x_1(m)| \quad \text{or} \quad -|x_1(n)| \geqq -Kp^{n-m}|x_1(m)|.$$

Therefore

$$|x(n)| = |x_1(n) + x_2(n)| \geqq |x_2(n)| - |x_1(n)| \geqq K^{-1}p^{-(n-m)}|x_2(m)| - Kp^{n-m}|x_1(m)|,$$

$$|x(n)| \geqq (K^{-1}p^{-(n-m)} - Kp^{n-m})\,|x_2(m)|,\quad n \geqq m \geqq 0$$

or

$$|x(n)| \geqq \tfrac{1}{2}(K^{-1}p^{-(n-m)} - Kp^{n-m})\,|x(m)|,\quad n \geqq m \geqq 0 \tag{6}$$

Now consider the case $|x_2(m)| \leqq |x_1(m)|$, for some $m \in N$. Similarly, we get

$$|x(n)| \geqq \tfrac{1}{2}(K^{-1}(p^{-1})^{m-n} - Kp^{m-n})\,|x(m)|,\quad m \geqq n \geqq 0. \tag{7}$$

We choose $T \in N$ sufficiently large and $0 < \theta < 1$ so that

$$\tfrac{1}{2}(K^{-1}(p^{-1})^T - Kp^T) \geqq \theta^{-1} .$$

From (6) and (7) we obtain that

$$|x(m)| \leqq \theta \sup \{|x(n)| : |m - n| \leqq T, \quad n, m \in N , \quad m \geqq T\} .$$

Next we prove a proposition, which is actually the converse of Proposition 1.

Proposition 2. *Suppose that there exist constants $T \geqq 1$, $T \in N$, and $0 < \theta < 1$ such that*

$$(8) \qquad |x(n)| \leqq \theta \sup \{|x(u)|: |u - n| \leqq T, \ u, n \in N, \ n \geqq T\} .$$

Then (1) *has an exponential dichotomy.*

Proof. Let U be the set of such $u \in R^k$ that the solution x of (1) fulfilling $x(0) = u$ is bounded. Obviously, U is a linear space.

Let x be a solution of (1) with $x(0) \in U$. Since a contradiction with (8) results from $\limsup_{n\to\infty} |x(n)| > 0$, we have

$$\lim_{n\to\infty} |x(n)| = 0 .$$

For any $m \in N$ we conclude again by (8) that $\max \{|x(m)|, |x(m + 1)|, \ldots \ldots, |x(m + T - 1)|\} = \max \{|x(n)|: n = m, m + 1, \ldots\}$,

$$|x(n)| \leqq \theta \max \{|x(m)|, |x(m + 1)|, \ldots, |x(m + T - 1)|\}$$

for $n = m + T, m + T + 1, m + T + 2, \ldots$, and by induction

$$(9) \qquad |x(n)| \leqq \theta^k \max \{|x(m)|, |x(m + 1)|, \ldots, |x(m + T - 1)|\}$$

for $n = m + kT, m + kT + 1, m + kT + 2, \ldots, k = 1, 2, 3, \ldots$. By (2) and (9) we have

$$(10) \qquad |x(n)| \leqq Kp^{n-m}|x(m)| \quad \text{for} \quad n \geqq m \geqq 0 \quad \text{with} \quad p = \theta^{1/T} ,$$
$$K = M^{T-1}\theta^{-1} .$$

Let x be a solution of (1) with $x(0) \in R^k \setminus U$. Since x is unbounded, there exists such an $s(x(0)) \in N$ (we shall write s instead of $s(x(0))$) that $|x(s)| \geqq M^T|x(0)|$, $|x(n)| < M^T|x(0)|$ for $n = 0, 1, \ldots, s - 1$. By (2) we have $s \geqq T$ and (8) implies that a sequence of integers $t_1, t_2, t_3, \ldots$ exists such that $t_1 = s$, $t_i < t_{i+1} \leqq t_i + T$, $|x(t_{i+1})| \geqq \theta^{-1}|x(t_i)|$, $|x(n)| < \theta^{-1}|x(t_i)|$ for $t_i \leqq n < t_{i+1}$, $i = 1, 2, 3, \ldots$.

Let $s \leqq n < m$. Find $i, j \in N$ such that $t_i \leqq n < t_{i+1}$, $t_{j-1} < m \leqq t_j$. Then we have

$$|x(m)| \geqq M^{-T+1}|x(t_j)| \geqq M^{-T+1}\theta^{-j+i}|x(t_i)| \geqq M^{-2(T-1)}\theta^{-j+i}|x(n)| .$$

Since $(j - i)\, T \geqq m - n$, we have

$$(11) \qquad |x(n)| \leqq Kp^{m-n}|x(m)| \quad \text{with} \quad p = \theta^{1/T} , \quad K = M^{2(T-1)} .$$

Let V be a complementary space to U in R^k (i.e. $R^k = U + V$). Put $S = \sup\{s(v): v \in V \setminus \{0\}\}$. Since $S = \sup\{s(v): v \in V, |v| = 1\}$, we obtain by a compactness argument that $S < \infty$.

Let P be the projection on U along V. Then from (10), (11) the dichotomy (3) holds for $n, m \geqq S$ with $p = \theta^{1/T}$, $K = M^{2(T-1)}\theta^{-1}$, and by Lemma 1 equation 1 has an exponential dichotomy for $n, m \geqq 0$ (provided that $|H| = \sup\{|Hy|: y \in R^k, |y| \leqq 1\}$ holds for $k \times k$ matrices H). The proof is completed.

We can apply the above propositions to prove the following proposition (cf. Palmer [4, p. 187] for the continuous case).

Proposition 3. *Suppose that $A(n)$ is a $k \times k$ bounded upper triangular and invertible matrix for all $n \in N$. Then (1) has exponential dichotomy if and only if the corresponding diagonal system*

$$x(n+1) = \operatorname{diag}(\alpha_{11}(n), \ldots \alpha_{kk}(n))\, x(n) \tag{12}$$

has an exponential dichotomy.

Proof. Suppose that (12) has an exponential dichotomy and let $x(n) = \operatorname{diag}(1, \beta, \beta^2, \ldots, \beta^{k-1})\, y(n)$ be a β transformation, according to Bylov [1, p. 605]. Let

$$y(n+1) = \begin{pmatrix} \alpha_{11}(n) & \beta\alpha_{12}(n) & \ldots & \beta^{k-1}\alpha_{1k}(n) \\ 0 & \alpha_{22}(n) & \ldots & \beta^{k-2}\alpha_{2k}(n) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \alpha_{kk}(n) \end{pmatrix} y(n)\,. \tag{13}$$

From the fact that (12) has an exponential dichotomy and the roughness of the exponential dichotomy [3, p. 232] (13) has an exponential dichotomy provided β is taken sufficiently small. Since (1) is kinematically similar to (13), also (1) has an exponential dichotomy.

Conversely, suppose that (1) has exponential dichotomy. We show, by induction, that (12) has an exponential dichotomy. It is obvious for $k = 1$. Considering it is true for $k - 1$ we show that it is true for all $k \in N$. Since (1) has an exponential dichotomy according to Proposition 1 there exist $T \geqq 1$, $0 \leqq \theta < 1$ such that for any solution of (1):

$$|x(n)| \leqq \theta \sup\{|x(s)|: |s - n| \leqq T\}\,.$$

This is true for all solutions of (1) and, therefore, also for those solutions of (1), which have the last coordinate equal to zero. Hence, by Proposition 2, the equation

$$x(n+1) = \begin{pmatrix} \alpha_{11}(n) & \alpha_{12}(n) & \ldots & \alpha_{1,k-1}(n) \\ 0 & \alpha_{22}(n) & \ldots & \alpha_{2,k-1}(n) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \alpha_{k-1,k-1}(n) \end{pmatrix} x(n)$$

has an exponential dichotomy. Therefore the diagonal system $x(n + 1) =$

$= \operatorname{diag}(\alpha_{11}(n), \ldots, \alpha_{k-1,k-1}(n))\, x(n)$ has an exponential dichotomy. According to [3, p. 230] the equation

$$x(n+1) = A(n)\, x(n) + g(n),$$

where $g^*(n) = [0, 0, \ldots, f(n)]$ has a bounded solution for every bounded function $f(n)$. So the equation

$$x(n+1) = \alpha_{kk}(n)\, x(n) + f(n) \tag{14}$$

has a bounded solution for every bounded function $f(n)$. Therefore by [3, p. 230] the homogeneous equation of (14) has an exponential dichotomy and the proof is completed.

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