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ARCHIMEDEAN CLASSES IN AN ORDERED SEMIGROUP IV

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In our previous papers [2], [3] and [4], we studied archimedean classes in an ordered semigroup. The difficulty occurs because of the fact that the set product of two archimedean classes is not necessarily contained in a single archimedean class.

In this paper, we propose to set up the notion of modified archimedean classes of two archimedean classes. Fortunately, for each pair of modified archimedean classes, their set product is contained in some modified archimedean class. In § 1, we prove this fact. Using the results in § 1, in § 2 we study the behavior of the set product of a finite number of archimedean classes.

The terminologies and notations of our previous papers [2], [3] and [4] are used throughout. In particular, we denote by S an ordered semigroup and by $\mathscr{C}$ the set of all archimedean classes of S.

§ 1

Let $C \in \mathscr{C}$. First we suppose that C is a torsion free archimedean class of S. Then by [2] Theorem 3.5, the δ-class $C\delta$ contains at most two elements of $\mathscr{C}$. We define $m_{\pm}(C)$ when and only when $C\delta$ has two elements of $\mathscr{C}$. Thus, in this case, $C\delta$ consists of two torsion free archimedean classes A and B such that $A < B$. Now we define $m_{\pm}(C)$ as the set of all elements x of S such that the archimedean class X of S which contains x lies between A and B. It can be easily seen that $m_{\pm}(C)$ is the set of all x such that $a \leqq x \leqq b$ for some $a \in A$ and $b \in B$. Also we have either $C = A$ or $C = B$ and $m_{\pm}(C) = m_{\pm}(A) = m_{\pm}(B)$. Now $m(C) \in \{C, m_{\pm}(C)\}$ is called a *modified archimedean class* of C.

Next we suppose that C is a periodic archimedean class of S. Then C contains the unique idempotent, say e. We denote $m_0(C) = \{e\}$. Also we denote by C_+ and C_- the set of all nonnegative elements and the set of all nonpositive elements of C, respectively. We define $m_+(C)$ when and only when there exists an idempotent f of S such that $e < f$, $e\, \mathscr{D}_E\, f$ and e and f are consecutive in $e\mathscr{D}_E$. In this case we define $m_+(C) = [e, f]$ as the set of all elements x of S which lie between e and f. Similarly we define $m_-(C)$ when and only when there exists an idempotent g of S such that $g < e$, $g\, \mathscr{D}_E\, e$ and g and e are consecutive in $e\mathscr{D}_E$. In this case we define $m_-(C) =$

$= [g, e]$ as the set of all elements x of S which lie between g and e. Now $m(C) \in \in \{m_0(C), C_+, C_-, C, m_+(C), m_-(C)\}$ is called a *modified archimedean class* of C.

Lemma 1.1. *Let $C \in \mathscr{C}$. Then a modified archimedean class $m(C)$ of C is a convex subsemigroup of S.*

Proof. If $m(C) = C$, then the assertion follows from [2] Lemma 1.3. Suppose C is a torsion free archimedean class and $m(C) = m_\pm(C)$. Let $x, y \in m_\pm(C)$. Then $a_1 \leqq x \leqq b_1$ and $a_2 \leqq y \leqq b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Here $a_1a_2 \in A$, $b_1b_2 \in B$ and $a_1a_2 \leqq xy \leqq b_1b_2$. Hence $xy \in m_\pm(C)$ and so $m_\pm(C)$ is a subsemigroup of S. If, in addition, $z \in S$ and $x \leqq z \leqq y$, then $a_1 \leqq x \leqq z \leqq y \leqq b_2$ and so $z \in \in M_\pm(C)$. Hence $m_\pm(C)$ is convex.

Suppose C is a periodic archimedean class with idempotent e. $m_0(C)$ is clearly a convex subsemigroup of S. Suppose $m(C) = C_+$. Since $e \in C_+$, it follows from [1] Lemma 1 that C_+ is a subsemigroup of S. If $x, y \in C_+$, $z \in S$ and $x \leqq z \leqq y$, then $x^n = e$ for some natural number n and $z \leqq y \leqq e$ and so $e = x^n \leqq z^n \leqq e^n = e$. Hence $z \leqq e$ and $z^n = e$ and so $z \in C_+$. Hence C_+ is convex. Similarly C_- is a convex subsemigroup of S. Let $m(C) = m_+(C)$. Then there exists an idempotent f of S such that $e < f$, $e \mathscr{D}_E f$, e and f are consecutive in $e\mathscr{D}_E$ and $m_+(C) = [e, f]$. Let $x, y \in \in m_+(C)$. Then $e \leqq x \leqq f$ and $e \leqq y \leqq f$ and so $e = e^2 \leqq xy \leqq f^2 = f$. Also $e \in m_+(C)$ and so $m_+(C)$ is a subsemigroup of S. If, in addition, $z \in S$ and $x \leqq z \leqq y$, then $e \leqq x \leqq z \leqq y \leqq f$ and so $z \in m_+(C)$. Hence $m_+(C)$ is convex. Similarly $m_-(C)$ is a convex subsemigroup of S.

Lemma 1.2. *Let $C \in \mathscr{C}$, let $m(C)$ be a modified archimedean class of C, let $x \in m(C)$ and let X be the archimedean class containing x. Then $C\delta \preccurlyeq X\delta$.*

Proof. If $m(C) = C$, then we have $X = C$ and so $C\delta = X\delta$. Suppose C is a torsion free archimedean class and $m(C) = m_\pm(C)$. Thus there exist torsion free archimedean classes A and B in $C\delta$ such that $A < B$. Since $x \in m_\pm(C)$, we have $A \leqq X \leqq B$. Hence by [2] Lemma 5.6, we have $C\delta = C\delta \wedge C\delta = A\delta \wedge B\delta \preccurlyeq X\delta$.

Suppose C is a periodic archimedean class with idempotent e. If $m(C) \in \{m_0(C), C_+, C_-\}$, then $m(C) \subseteq C$ and so $X = C$. Hence $C\delta = X\delta$. Suppose $m(C) = m_+(C)$. Then there exists an idempotent f such that $e < f$, $e \mathscr{D}_E f$, e and f are consecutive in $e\mathscr{D}_E$ and $m_+(C) = [e, f]$. We denote by F the archimedean class containing f. Since $x \in [e, f]$, we have $e \leqq x \leqq f$ and so $C \leqq X \leqq F$. Also by [2] Theorem 3.3, we have $C \, \delta \, F$ and so by [2] Lemma 5.6, we have $C\delta = C\delta \wedge F\delta \preccurlyeq X\delta$. In the case when $m(C) = m_-(C)$, we can prove $C\delta \preccurlyeq X\delta$ in a similar way.

Lemma 1.3. *Let $C, D \in \mathscr{C}$ such that $C\delta \wedge D\delta$ is torsion free and let $m(C)$ and $m(D)$ be modified archimedean classes of C and D, respectively.*

(1) *If $C \, \delta \, D$, $m(C) = C$, $m(D) = D$ and $C = D$, then $m(C)m(D) \subseteq C = = C * D$.*

(2) *If $C \, \delta \, D$, $m(C) = C$, $m(D) = D$ and $C \neq D$, then $m(C)\, m(D) \subseteq m_\pm(C) = = m_\pm(D) = m_\pm(C * D)$.*

(3) *If* $C \delta D$ *and either* $m(C) = m_{\pm}(C)$ *or* $m(D) = m_{\pm}(D)$, *then* $m(C)\, m(D) \subseteq$ $\subseteq m_{\pm}(C) = m_{\pm}(D) = m_{\pm}(C * D)$.

(4) *If* $C \operatorname{non}\delta D$ *and* $C \gamma D$, *then* $m(C)\, m(D) \subseteq m(C) = m(C * D)$.

(5) *If* $C \operatorname{non}\delta D$ *and* $D \gamma C$, *then* $m(C)\, m(D) \subseteq m(D) = m(C * D)$.

Proof. (1) By Lemma 1.1, C is a subsemigroup of S. Hence by [2] Lemma 5.8, $m(C)\, m(D) = CD = C^2 \subseteq C = C * C = C * D$.

(2) Since $C \delta D$, we have $C\delta = D\delta = C\delta \wedge D\delta$ and so C and D are torsion free archimedean classes and $m_{\pm}(C) = m_{\pm}(D)$. Also we have $m(C) = C \subseteq m_{\pm}(C)$ and $m(D) = D \subseteq m_{\pm}(D)$. Hence by Lemma 1.1, we have $m(C)\, m(D) \subseteq m_{\pm}(C)\, m_{\pm}(D) =$ $= (m_{\pm}(C))^2 \subseteq m_{\pm}(C) = m_{\pm}(D)$. Further if $C\delta = D\delta$ is of L-type, we have $C * D =$ $= C$ and if it is of R-type, we have $C * D = D$. Hence always we have $m_{\pm}(C) =$ $= m_{\pm}(D) = m_{\pm}(C * D)$.

Similarly we can prove (3).

(4) Since $C \gamma D$, it follows from [2] Lemma 4.7 that $C\delta \preccurlyeq D\delta$ and so $C\delta =$ $= C\delta \wedge D\delta$. Hence C is a torsion free archimedean class and so $m(C) \in \{C, m_{\pm}(C)\}$. Also by [2] Theorem 6.1, we have $C = C * D$.

1° Case: $m(C) = C$ and $m(D) = D$. Then by [2] Theorem 6.1, we have $m(C)\, m(D) = CD \subseteq C = m(C) = m(C * D)$.

2° Case: $m(C) = m_{\pm}(C)$ and $m(D) = D$. Thus there exist two torsion free archimedean classes A and B in $C\delta$ such that $A < B$ and so $A\delta = B\delta = C\delta \neq D\delta$. Also since $C \gamma D$, we have $A \gamma D$ and $B \gamma D$. Let $x \in m_{\pm}(C)$ and let $y \in D$. Then $a \leqq x \leqq b$ for some $a \in A$ and $b \in B$ and so $ay \leqq xy \leqq by$. But by [2] Theorem 6.1, we have $ay \in AD \subseteq A$ and by $\in BD \subseteq B$ and so $xy \in m_{\pm}(C)$. Hence $m(C)\, m(D) =$ $= m_{\pm}(C)\, D \subseteq m_{\pm}(C) = m_{\pm}(C * D)$.

3° Case: $m(C) = C$ and $m(D)$ arbitrary. Let $x \in C$ and $y \in m(D)$ and let Y be the archimedean class containing y. Then by Lemma 1.2, we have $C\delta \prec D\delta \preccurlyeq Y\delta$ and so $C \operatorname{non}\delta Y$ and $C \gamma Y$. Hence by 1°, we have $xy \in CY \subseteq C$ and so $m(C)\, m(D) =$ $= Cm(D) \subseteq C = m(C) = m(C * D)$.

4° Case: $m(C) = m_{\pm}(C)$ and $m(D)$ arbitrary. In a similar way we can prove $m(C)\, m(D) = m_{\pm}(C)\, m(D) \subseteq m_{\pm}(C) = m(C) = m(C * D)$.

We can prove (5) similarly.

Lemma 1.4. *Let* $C_1, \ldots, C_n \in \mathscr{C}$. *Then*

$$(C_1 * \ldots * C_n)\, \delta = C_1\delta \wedge \ldots \wedge C_n\delta\,.$$

Proof. If $n = 1$, then the assertion is evident. If $n = 2$, then by the definition of the operation $*$, we have $(C_1 * C_2)\, \delta = C_1\delta \wedge C_2\delta$. Hence by induction, the assertion holds for every natural number n.

Lemma 1.5. *Let* $C, D \in \mathscr{C}$ *such that* $C\delta \wedge D\delta$ *is a periodic* δ*-class of L-type,* $C\delta \wedge D\delta \prec C\delta$ *and* $C * D \neq D$. *Let* $m(C)$ *and* $m(D)$ *be modified archimedean classes of* C *and* D, *respectively. Then* $m(C)\, m(D) = m_0(C * D)$.

Proof. First suppose $C \leqq D$. Then $C \leqq C * D \leqq D$. But by Lemma 1.4, we have $(C * D)\,\delta = C\delta \wedge D\delta \prec C\delta$ and by assumption, $C * D \neq D$. Hence $C < C * D < D$. Since $(C * D)\,\delta = C\delta \wedge D\delta$ is a periodic δ-class of L-type, $C * D$ is a periodic archimedean class with idempotent, say h and the $\mathscr{D}_E$-class $h\mathscr{D}_E$ is of L-type.

1° Case: $m(C) = C$ and $m(D) = D$. Let $x \in C$ and $y \in D$. Since $C < C * D < D$, we have $x < h < y$. Hence by [2] Lemma 6.3, we have $h = xh \leqq xy$. On the other hand, $(C * D)\,\delta = C\delta \wedge D\delta \preccurlyeq D\delta$ and so by [2] Lemma 4.7, we have $C * D\,\gamma\, D$. Hence by [2] Theorem 2.7, we have $xy \leqq hy = h$. Hence $xy = h$ and so $m(C)\,m(D) = CD = \{h\} = m_0(C * D)$.

2° Case: $m(C) = C$ and D is torsion free. Let $x \in C$ and $y \in m(D)$ and let Y be the archimedean class containing y. Since $D\delta$ is a torsion free δ-class, we have $C\delta \wedge D\delta \neq D\delta$ and by Lemma 1.2, $D\delta \preccurlyeq Y\delta$. Hence $(C * D)\,\delta = C\delta \wedge D\delta \prec D\delta = D\delta \wedge Y\delta$ and so by [2] Lemma 5.6, there is no $Z \in \mathscr{C}$ such that $Z \in C\delta \wedge D\delta$ and Z lies between D and Y. In particular $C * D$ does not lie between D and Y and, since $C * D < D$, we have $C * D < Y$. Hence $C < C * D < Y$ and by [2] Lemma 5.6, we have $C\delta \wedge Y\delta \preccurlyeq (C * D)\,\delta = C\delta \wedge D\delta \preccurlyeq C\delta \wedge Y\delta$. Hence $C\delta \wedge D\delta = C\delta \wedge Y\delta$ and so

$$C * D = \min\{Z \in \mathscr{C};\ C \leqq Z \leqq D \text{ and } Z \in C\delta \wedge D\delta\}$$

$$= \min\{Z \in \mathscr{C};\ C \leqq Z \leqq Y \text{ and } Z \in C\delta \wedge Y\delta\} = C * Y\,.$$

Hence $C * Y = C * D < Y$ and also $C\delta \wedge Y\delta = C\delta \wedge D\delta \prec C\delta$. Hence by 1°, $xy \in CY = m_0(C * Y) = m_0(C * D)$. Hence $m(C)\,m(D) = C\,m(D) = m_0(C * D)$.

3° Case: $m(C) = C$, D is periodic and $m(D) \in \{m_0(D), D_+, D_-\}$. We have $m(D) \subseteq D$ and by 1°, we have $m(C)\,m(D) = C\,m(D) \subseteq CD = m_0(C * D)$ and so $m(C)\,m(D) = m_0(C * D)$.

4° Case: $m(C) = C$, D is periodic with idempotent e' and $m(D) = m_+(D)$. Thus there exists an idempotent f' of S such that $e' < f'$, $e'\,\mathscr{D}_E\,f'$, e' and f' are consecutive in $e'\mathscr{D}_E$ and $m_+(D) = [e', f']$. Let $x \in C$ and $y \in m_+(D)$ and let Y be the archimedean class containing y. Then $e' \leqq y$ and so $C < C * D < D \leqq Y$. Hence in a similar way to 2°, we can show that $C\delta \wedge Y\delta = C\delta \wedge D\delta$ and so $C\delta \wedge Y\delta \prec C\delta$. Also

$$C * Y = \min\{Z \in \mathscr{C};\ C \leqq Z \leqq Y \text{ and } Z \in C\delta \wedge Y\delta\}$$

$$= \min\{Z \in \mathscr{C};\ C \leqq Z \leqq Y \text{ and } Z \in C\delta \wedge D\delta\} \leqq C * D \leqq C * Y$$

by [2] Theorem 5.11. Hence $C * Y = C * D$ and so by 1°, we have $xy \in CY = m_0(C * Y) = m_0(C * D)$. Hence $m(C)\,m(D) = C\,m_+(D) = m_0(C * D)$.

5° Case: $m(C) = C$, D is periodic with idempotent e' and $m(D) = m_-(D)$. Thus there exists an idempotent g' of S such that $g' < e'$, $g'\,\mathscr{D}_E\,e'$, g' and e' are consecutive in $e'\mathscr{D}_E$ and $m_-(D) = [g', e']$. We denote by G the archimedean class containing g'. Let $x \in C$ and $y \in m_-(D)$. Then $g' \leqq y \leqq e'$. Since $g'\,\mathscr{D}_E\,e'$, it follows from [2] Theorem 3.3 that $G\,\delta\,D$. By way of contradiction we assume $G < C * D$. Then, since $G < C * D < D$, it follows from [2] Lemma 5.6 that $D\delta = D\delta \wedge G\delta \preccurlyeq$

$\preccurlyeq (C * D)\,\delta = C\delta \wedge D\delta \preccurlyeq D\delta$ and so $(C * D)\,\delta = D\delta = G\delta$. Hence by [2] Theorem 3.3, we have $h\,\mathscr{D}_E\,g'\,\mathscr{D}_E\,e'$. But, since $G < C * D < D$, we have $g' < h < e'$, which contradicts that g' and e' are consecutive in $e'\mathscr{D}_E$. Hence we have $C < C * D \leqq \leqq G$ and so $x < h \leqq g' \leqq y \leqq e'$. By 1° we have $xe' \in CD = m_0(C * D) = \{h\}$ and by [2] Lemma 6.3, we have $xh = h$. Hence $h = xh \leqq xy \leqq xe' = h$ and so $xy = h$. Hence $m(C)\,m(D) = Cm_-(D) = \{h\} = m_0(C * D)$.

6° Case: $m(C)$ and $m(D)$ are arbitrary. Let $x \in m(C)$ and $y \in m(D)$ and let X be the archimedean class containing x. Then by Lemma 1.2, we have $C\delta \preccurlyeq X\delta$ and so $(C * D)\,\delta = C\delta \wedge D\delta \prec C\delta = C\delta \wedge X\delta$. Hence by [2] Lemma 5.6, there is no $Z \in C\delta \wedge D\delta$ which lies between C and X. In particular, $C * D$ does not lie between C and X. Since $C < C * D$, we have $X < C * D < D$. Hence by [2] Lemma 5.6, $X\delta \wedge D\delta \preccurlyeq (C * D)\,\delta = C\delta \wedge D\delta \preccurlyeq X\delta \wedge D\delta$ and so $X\delta \wedge D\delta = C\delta \wedge D\delta$. Hence

$$\begin{aligned} C * D &= \min\{Z \in \mathscr{C};\ C \leqq Z \leqq D \text{ and } Z \in C\delta \wedge D\delta\} \\ &= \min\{Z \in \mathscr{C};\ X \leqq Z \leqq D \text{ and } Z \in X\delta \wedge D\delta\} = X * D \end{aligned}$$

and so $X * D = C * D < D$. Also we have $X\delta \wedge D\delta = C\delta \wedge D\delta \prec C\delta \preccurlyeq X\delta$. Hence by $1° \sim 5°$, we have $xy \in X\,m(D) = m_0(X * D) = m_0(C * D) = \{h\}$ and so $xy = h$. Hence $m(C)\,m(D) = \{h\} = m_0(C * D)$.

The case when $D \leqq C$ can be treated similarly.

Lemma 1.6. *Let $C, D \in \mathscr{C}$ such that $C\delta \wedge D\delta$ is a periodic δ-class of L-type, $C\delta \wedge D\delta \prec C\delta$ and $C * D = D$. Let $m(C)$ be a modified archimedean class of C. Then D is a periodic archimedean class. Also if $C \leqq D$, then*

(1) $m(C)\,m_0(D) = m(C)\,D_- = m(C)\,m_+(D) = m_0(D) = m_0(C * D)$;
(2) $m(C)\,D_+ \subseteq D_+ = (C * D)_+,\ m(C)\,D \subseteq D_+ = (C * D)_+$;
(3) $m(C)\,m_-(D) \subseteq m_-(D) = m_-(C * D)$,

and if $D \leqq C$, then

(4) $m(C)\,m_0(D) = m(C)\,D_+ = m(C)\,m_-(D) = m_0(D) = m_0(C * D)$;
(5) $m(C)\,D_- \subseteq D_- = (C * D)_-,\ m(C)\,D \subseteq D_- = (C * D)_-$;
(6) $m(C)\,m_+(D) \subseteq m_+(D) = m_+(C * D)$.

Proof. Since $D\delta = (C * D)\,\delta = C\delta \wedge D\delta$ is a periodic δ-class of L-type, D is a periodic archimedean class with idempotent, say e', and also the D_E-class $e'\mathscr{D}_E$ is of L-type. First suppose $C \leqq D$. Then since $D\delta = C\delta \wedge D\delta \prec C\delta$, we have $C < D$.

(1) 1° Case: $m(C) = C$. Let $x \in C$ and $y \in D_-$. Then $x < e' \leqq y$ and by [2] Lemma 1.4, we have $xy \leqq e'y = e'$. On the other hand, it follows from [2] Lemma 6.3 that $e' = xe' \leqq xy$. Hence $xy = e'$ and so $m(C)\,D_- = CD_- = \{e'\} = m_0(D) = m_0(C * D)$. Since $m_0(D) \subseteq D_-$, we have $m(C)\,m_0(D) = C\,m_0(D) = m_0(D) = m_0(C * D)$. Finally let $x \in C$ and $y \in m_+(D)$. Thus there exists an idempotent f' of S such that $e' < f'$, $e'\,\mathscr{D}_E f'$, e' and f' are consecutive in $e'\mathscr{D}_E$ and $m_+(D) = [e', f']$. Hence $e' \leqq y \leqq f'$ and $xe' \in C\,m_0(D) = m_0(D) = \{e'\}$. We denote by F the archimedean class containing f'. Then by [2] Theorem 3.3, we have $D\,\delta\,F$ and

so $C\delta \wedge F\delta = C\delta \wedge D\delta \prec C\delta$. Also $C < D < F$ and by [2] Theorem 5.11,

$$C * F = \min\{Z \in \mathscr{C};\ C \leqq Z \leqq F \text{ and } Z \in C\delta \wedge F\delta\}$$
$$= \min\{Z \in \mathscr{C};\ C \leqq Z \leqq F \text{ and } Z \in C\delta \wedge D\delta\} \leqq C * D \leqq C * F\,.$$

Hence $C * F = C * D = D < F$. Hence by Lemma 1.5, we have $xf' \in CF = m_0(C * F) = m_0(D) = \{e'\}$ and so $xf' = e'$. Hence $e' = xe' \leqq xy \leqq xf' = e'$ and so $xy = e'$. Hence $m(C)\, m_+(D) = C\, m_+(D) = \{e'\} = m_0(D) = m_0(C * D)$.

2° Case: $m(C)$ is arbitrary and $m(D) \in \{m_0(D), D_-, m_+(D)\}$. Let $x \in m(C)$ and $y \in m(D)$ and let X be the archimedean class containing x. Then by Lemma 1.2, we have $D\delta = (C * D)\,\delta = C\delta \wedge D\delta \prec C\delta = C\delta \wedge X\delta$ and by [2] Lemma 5.6, there is no $Z \in \mathscr{C}$ such that $Z \in D\delta$ and Z lies between C and X. In particular, D does not lie between C and X and, since $C < D$, we have $X < D$. Also $X\delta \wedge D\delta = D\delta = C\delta \wedge D\delta \prec C\delta \preccurlyeq X\delta$ and

$$D = C * D = \min\{Z \in \mathscr{C};\ C \leqq Z \leqq D \text{ and } Z \in C\delta \wedge D\delta\}$$
$$= \min\{Z \in \mathscr{C};\ C \leqq Z \leqq D \text{ and } Z \in D\delta\}$$
$$= \min\{Z \in \mathscr{C};\ X \leqq Z \leqq D \text{ and } Z \in D\delta\}$$
$$= \min\{Z \in \mathscr{C};\ X \leqq Z \leqq D \text{ and } Z \in X\delta \wedge D\delta\} = X * D\,.$$

Hence by 1°, $xy \in X\, m(D) = m_0(D)$ and so $m(C)\, m(D) = m_0(D) = m_0(C * D)$.

(2) 1° Case: $m(C) = C$. Let $x \in C$ and $y \in D_+$. Then $x < y \leqq e'$ and by (1) $xe' \in C\, m_0(D) = m_0(D) = \{e'\}$. Hence $xy \leqq xe' = e'$. Since D is a periodic archimedean class with idempotent e', we have $y^n = e'$ for some natural number n. First suppose $xy \leqq yx$. Then $x^n e' = x^n y^n \leqq (xy)^n \leqq e'^n = e'$ and by (1) $x^n e' \in C\, m_0(D) = m_0(D) = \{e'\}$. Hence $(xy)^n = e'$. Next suppose $yx \leqq xy$. Then $e'x^n = y^n x^n \leqq (xy)^n \leqq e'$. But $D\delta = (C * D)\,\delta = C\delta \wedge D\delta \prec C\delta$ and by [2] Lemma 4.7, we have $D\, \gamma\, C$. Also since $D\delta = C\delta \wedge D\delta$ is of L-type, the $\mathscr{D}_E$-class $e'\mathscr{D}_E$ is of L-type. Hence by [2] Theorem 2.7, we have $e'x^n = e'$ and so $(xy)^n = e'$. Hence we always have $(xy)^n = e'$ and so $xy \in D$. But since $xy \leqq e'$, we have $xy \in D_+$. Hence $m(C)\, D_+ = CD_+ \subseteq D_+ = (C * D)_+$. Also by (1)

$$m(C)\, D = CD = C(D_+ \cup D_-) =$$
$$= CD_+ \cup CD_- \subseteq D_+ \cup m_0(D) = D_+ = (C * D)_+\,.$$

2° Case: $m(C)$ is arbitrary and $m(D) \in \{D_+, D\}$. In a similar way to (1) 2°, we can prove that $m(C)\, m(D) \subseteq D_+ = (C * D)_+$.

(3) 1° Case: $m(C) = C$. Let $x \in C$ and $y \in m_-(D)$. Thus there exists an idempotent g' of S such that $g' < e'$, $g'\, \mathscr{D}_E\, e'$, g' and e' are consecutive in $e'\mathscr{D}_E$ and $m_-(D) = [g', e']$. We denote by G the archimedean class containing g'. Then $G < D$ and by [2] Theorem 3.3, we have $G\, \delta\, D$ and so $C\delta \wedge D\delta = (C * D)\,\delta = D\delta = G\delta$. But

$$D = C * D = \min\{Z \in \mathscr{C};\ C \leqq Z \leqq D \text{ and } Z \in C\delta \wedge D\delta\}$$
$$= \min\{Z \in \mathscr{C};\ C \leqq Z \leqq D \text{ and } Z \in D\delta\}$$

and so there exists no $Z \in \mathscr{C}$ such that $C \leqq Z < D$ nad $Z \in D\delta$. Hence we have $G < C < C * D = D$ and so $x \in [g', e'] = m_-(D)$. Hence by Lemma 1.1, $xy \in \in m_-(D)\, m_-(D) \subseteq m_-(D)$ and so $m(C)\, m_-(D) = C\, m_-(D) \subseteq m_-(D) = m_-(C * D)$.

2° Case: $m(C)$ is arbitrary. In a similar way to (1) 2°, we can prove $m(C)\, m_-(D) \subseteq \subseteq m_-(D) = m_-(C * D)$.

In the case when $D \leqq C$, we can prove (4), (5) and (6) similarly.

Lemma 1.7. *Let $C, D \in \mathscr{C}$ such that $C\delta \wedge D\delta$ is a periodic δ-class of L-type and $C\delta = C\delta \wedge D\delta$. Let $m(D)$ be a modified archimedean class of D. Then $C = C * D$, which is a periodic archimedean class. Also*

(1) $m_0(C)\, m(D) = m_0(C) = m_0(C * D)$;

(2) $m_+(C)\, m(D) \subseteq m_+(C) = m_+(C * D)$;

(3) $m_-(C)\, m(D) \subseteq m_-(C) = m_-(C * D)$.

Proof. Since $C\delta = C\delta \wedge D\delta$ is a periodic δ-class of L-type, we have $C * D = C$. Also $C = C * D$ is a periodic archimedean class with idempotent, say e, and the $\mathscr{D}_E$-class $e\mathscr{D}_E$ is of L-type.

(1) We have $m_0(C) = \{e\}$. Let $y \in m(D)$ and let Y be the archimedean class containing y. Then by Lemma 1.2, we have $C\delta = C\delta \wedge D\delta \preccurlyeq D\delta \preccurlyeq Y\delta$ and so by [2] Lemma 4.7, we have $C\, \gamma\, Y$. Hence by [2] Theorem 2.7, we have $ey = e$ and so $m_0(C)\, m(D) = \{e\} = m_0(C) = m_0(C * D)$.

(2) There exists an idempotent f of S such that $e < f$, $e\, \mathscr{D}_E f$, e and f are consecutive in $e\mathscr{D}_E$ and $m_+(C) = [e, f]$. We denote by F the archimedean class containing f. By [2] Theorem 3.3, we have $C\, \delta\, F$ and so $F\delta = C\delta = C\delta \wedge D\delta = = F\delta \wedge D\delta$ which is a peripdic δ-class of L-type. Hence by (1) we have $m_0(C)\, m(D) = = m_0(C) = \{e\}$ and $m_0(F)\, m(D) = m_0(F) = \{f\}$. Now let $x \in m_+(C)$ and $y \in m(D)$. Then $e \leqq x \leqq f$ and so $ey \leqq xy \leqq fy$. Since $ey \in m_0(C)\, m(D) = \{e\}$ and $fy \in \in m_0(F)\, m(D) = \{f\}$, we have $e \leqq xy \leqq f$ and so $xy \in [e, f] = m_+(C)$. Hence $m_+(C)\, m(D) \subseteq m_+(C) = m_+(C * D)$.

(3) can be proved in a similar way.

Lemma 1.8. *Let $C \in \mathscr{C}$ such that $C\delta$ is a periodic δ-class of L-type. Then*

(1) *if* $m(C) \in \{m_0(C), m_+(C), m_-(C)\}$, *then* $C\, m(C) = m_0(C) = m_0(C * C)$;

(2) $CC \subseteq C = C * C$, $CC_+ \subseteq C_+ = (C * C)_+$, $CC_- \subseteq C_- = (C * C)_-$;

(3) *if* $m(C) \in \{m_0(C), m_+(C), m_-(C), C_-\}$, *then* $C_+\, m(C) = m_0(C) = m_0(C * C)$;

(4) *if* $m(C) \in \{C, C_+\}$, *then* $C_+\, m(C) \subseteq C_+ = (C * C)_+$;

(5) *if* $m(C) \in \{m_0(C), m_+(C), m_-(C), C_+\}$, *then* $C_-\, m(C) = m_0(C) = m_0(C * C)$;

(6) *if* $m(C) \in \{C, C_-\}$, *then* $C_-\, m(C) \subseteq C_- = (C * C)_-$.

Proof. Since $C\delta$ is a periodic δ-class of L-type, C is a periodic archimedean class with idempotent, say e, and the $\mathscr{D}_E$-class $e\mathscr{D}_E$ is of L-type. By [2] Lemma 5.8, we have $C = C * C$.

(1) First suppose $m(C) = m_0(C)$. By [2] Lemma 1.4, e is the zero element of C and so $C\, m_0(C) = C\{e\} = \{e\} = m_0(C) = m_0(C * C)$. Next suppose $m(C) = m_+(C)$.

Thus there exists an idempotent f of S such that $e < f$, $e \mathscr{D}_E f$, e and f are consecutive in $e\mathscr{D}_E$ and $m_+(C) = [e, f]$. We denote by F the archimedean class containing f. Then by [2] Theorem 3.3, we have $C \,\delta\, F$ and so $F * C = F$. Since e and f are consecutive in $e\mathscr{D}_E$, there is no idempotent h of S such that $e < h < f$ and $e \mathscr{D}_E h$. Hence by [2] Lemma 6.7, CF is contained in a single archimedean class and so by [4] Theorem 2, $CF \subseteq C_- \subseteq C$. Now let $x \in C$ and $y \in m_+(C) = [e, f]$. Then $e \leqq y \leqq f$ and $xe \in C\, m_0(C) = m_0(C) = \{e\}$. Also by [2] Theorem 2.7, we have $fx = f$ and so $(xf)^2 = xfxf = xf$. Moreover, since $xf \in CF \subseteq C$, xf is an idempotent in C and so $xf = e$. Hence $e = xe \leqq xy \leqq xf = e$ and so $xy = e$. Hence $C\, m_+(C) = \{e\} = m_0(C) = m_0(C * C)$. Similarly we can prove that $C\, m_-(C) = m_0(C) = m_0(C * C)$.

(2) Since C is a subsemigroup of S, we have $CC \subseteq C = C * C$. Let $x \in C$ and $y \in C_+$. Then $y \leqq e$ and so $xy \leqq xe = e$. Also $xy \in CC \subseteq C$ and so $xy \in C_+$. Hence $CC_+ \subseteq C_+ = (C * C)_+$. Similarly we can prove that $CC_- \subseteq C_- = (C * C)_-$.

(3) By (1), we have $C_+\, m_0(C) \subseteq C\, m_0(C) = m_0(C)$, $C_+\, m_+(C) \subseteq C\, m_+(C) = m_0(C)$ and $C_+\, m_-(C) \subseteq C\, m_-(C) = m_0(C)$ and so $C_+\, m_0(C) = C_+\, m_+(C) = C_+\, m_-(C) = m_0(C) = m_0(C * C)$.

Let $x \in C_+$ and $y \in C_-$. Then $x, y \in C$, $x \leqq e$ and $e \leqq y$. Hence $e = xe \leqq xy \leqq ey = e$ and so $xy = e$. Hence $C_+ C_- = \{e\} = m_0(C) = m_0(C * C)$.

(4) By Lemma 1.1, C_+ is a subsemigroup of S and so $C_+ C_+ \subseteq C_+ = (C * C)_+$. Also by (3), $C_+ C = C_+(C_+ \cup C_-) = C_+ C_+ \cup C_+ C_- \subseteq C_+ \cup m_0(C) = C_+$ and so $C_+ C \subseteq C_+ = (C * C)_+$.

(5) and (6) can be proved similarly.

Lemma 1.9. *Let $C, D \in \mathscr{C}$ such that $C\delta \wedge D\delta$ is a periodic δ-class of L-type and $C\delta = C\delta \wedge D\delta$. Let $m(D)$ be a modified archimedean class of D.*

(1) *If $C < D$, then $C_+\, m(D) = m_0(C) = m_0(C * D)$.*

(2) *If $D < C$, then $C_-\, m(D) = m_0(C) = m_0(C * D)$.*

Proof. Since $C\delta = C\delta \wedge D\delta$ is a periodic δ-class of L-type, C is a periodic archimedean class with idempotent, say e, and the $\mathscr{D}_E$-class $e\mathscr{D}_E$ is of L-type. Also by Lemma 1.7, we have $C = C * D$.

(1) Suppose $C < D$. Let $x \in C_+$ and $y \in m(D)$ and let Y be the archimedean class containing y. Then by Lemma 1.2, we have $C\delta = C\delta \wedge D\delta \preccurlyeq D\delta \preccurlyeq Y\delta$. By way of contradiction we assume $y < e$. Then we have $Y \leqq C < D$ and by [2] Lemma 5.6, we have $D\delta = D\delta \wedge Y\delta \preccurlyeq C\delta \preccurlyeq D\delta$. Hence $D\delta = C\delta$ and so D is a periodic archimedean class with idempotent, say e'. Since $m(D)$ contains an element $y \in Y$ such that $Y < D$, we must have $m(D) = m_-(D)$. Hence there exists an idempotent g' of S such that $g' < e'$, $g' \mathscr{D}_E e'$, g' and e' are consecutive in $e'\mathscr{D}_E$ and $m_-(D) = [g', e']$. Since $y \in m(D) = m_-(D)$, we have $g' \leqq y < e < e'$. Further since $D \,\delta\, C$, it follows from Theorem 3.3 that $e \mathscr{D}_E e'$, which contradicts that g' and e' are consecutive in $e'\mathscr{D}_E$. Hence we have $e \leqq y$. Also since $x \in C_+$, we have $x \leqq e$. Hence $e = xe \leqq xy \leqq ey$. But since $C\delta \preccurlyeq D\delta$, it follows from [2] Lemma 4.7 that

$C \gamma D$ and so by [2] Theorem 2.7, we have $ey = e$. Hence $xy = e$ and so $C_+ \, m(D) = \{e\} = m_0(C) = m_0(C * D)$.

(2) can be proved similarly.

Lemma 1.10. *Let $C, D \in \mathscr{C}$ such that $C\delta \wedge D\delta$ is a periodic δ-class of L-type and $C\delta = C\delta \wedge D\delta$. Let e and h be the idempotents of C and $D * C$, respectively.*

(1) *If $C < D$, then the following conditions are equivalent:*
  (i) *CD is contained in a single archimedean class;*
  (ii) *$CD \subseteq C_-$;*
  (iii) *$ch = e$ for every $c \in C_-$.*

(2) *If $D < C$, then the following conditions are equivalent:*
  (iv) *CD is contained in a single archimedean class;*
  (v) *$CD \subseteq C_+$;*
  (vi) *$ch = e$ for every $c \in C_+$.*

Proof. Since $(D * C)\,\delta = D\delta \wedge C\delta = C\delta$, C and $D * C$ are really periodic archimedean classes. Also by [2] Theorem 3.3, $e \, \mathscr{D}_E \, h$ and, since $C\delta = C\delta \wedge D\delta$ is of L-type, the $\mathscr{D}_E$-class $e\mathscr{D}_E$ is of L-type.

(1) Suppose $C < D$. First suppose CD is contained in a single archimedean class. Then by [2] Theorem 6.6 and [4] Theorem 2, we have $CD \subseteq C_-$. Next suppose $CD \subseteq C_-$. Let $c \in C_-$. Since $C < D$, we have $C \leqq D * C \leqq D$ and so there exists $d \in D$ such that $h \leqq d$. Also since $c \in C_-$, we have $e \leqq c$. Hence $e = eh \leqq ch \leqq cd$. Since $cd \in CD \subseteq C_- \subseteq C$ and C is convex, we have $ch \in C$. But by [2] Theorem 2.7, $hc = h$ and so $(ch)^2 = chch = ch$. Hence ch is an idempotent of C and so $ch = e$. Finally suppose that $ch = e$ for every $c \in C_-$. Then by [2] Lemma 6.7, CD is contained in a single archimedean class.

(2) can be proved similarly.

Lemma 1.11. *Let $C, D \in \mathscr{C}$ such that $C\delta \wedge D\delta$ is a periodic δ-class of L-type, $C\delta = C\delta \wedge D\delta$ and CD is contained in a single archimedean class and let $m(D)$ be a modified archimedean class of D.*

(1) *Suppose that $C < D$ and in the case when D is a periodic archimedean class and $m(D) = m_+(D)$ we have $C\delta \neq D\delta$. Then $C_- \, m(D) \subseteq C_- = (C * D)_-$ and $C \, m(D) \subseteq C_- = (C * D)_-$.*

(2) *Suppose that $D < C$ and in the case when D is a periodic archimedean class and $m(D) = m_-(D)$ we have $C\delta \neq D\delta$. Then $C_+ \, m(D) \subseteq C_+ = (C * D)_+$ and $C \, m(D) \subseteq C_+ = (C * D)_+$.*

Proof. By Lemma 1.7, we have $C = C * D$. Also C is a periodic archimedean class with idempotent, say e. Since $C\delta = C\delta \wedge D\delta$ is of L-type, the $\mathscr{D}_E$-class $e\mathscr{D}_E$ is of L-type.

(1) 1° Case: $m(D) = D$. Then by Lemma 1.10, we have $C \, m(D) = CD \subseteq C_- = (C * D)_-$ and $C_- \, m(D) \subseteq C \, m(D) \subseteq C_- = (C * D)_-$.

2° Case: D is a periodic archimedean class and $m(D) \in \{D_+, D_-, m_0(D)\}$. Since

$m(D) \subseteq D$, we have by 1° $C\,m(D) \subseteq CD \subseteq C_- = (C * D)_-$ and $C_-\,m(D) \subseteq$ $\subseteq C_-D \subseteq C_- = (C * D)_-$.

3° Case: D is a periodic archimedean class with idempotent e' and $m(D) = m_-(D)$. Thus there exists an idempotent g' of S such that $g' < e'$, $g'\,\mathscr{D}_E\,e'$, g' and e' are consecutive in $e'\mathscr{D}_E$ and $m_-(D) = [g', e']$. We denote by G the archimedean class containing g'. Then by [2] Theorem 3.3, we have $G\;\delta\;D$. By way of contradiction we assume $G < C$. Then since $G < C < D$, it follows from [2] Lemma 5.6 that we have $D\delta = G\delta \wedge D\delta \preccurlyeq C\delta = C\delta \wedge D\delta \preccurlyeq D\delta$ and so $C\delta = D\delta = G\delta$. Hence by [2] Theorem 3.3, $e, g', e' \in e'\mathscr{D}_E$ and since $G < C < D$, we have $g' < e < e'$, which contradicts that g' and e' are consecutive in $e'\mathscr{D}_E$. Hence we have $C \leqq G < D$ and so $e \leqq g' < e'$. Now let $x \in C$ and $y \in m_-(D)$. Then $e \leqq g' \leqq y \leqq e'$ and so $e = xe \leqq xy \leqq xe'$. Here $e \in C_-$ and by 1°, $xe' \in CD \subseteq C_-$. But by Lemma 1.1, C_- is convex and so $xy \in C_-$. Hence $C\,m(D) = C\,m_-(D) \subseteq C_- = (C * D)_-$ and also $C_-\,m(D) = C_-\,m_-(D) \subseteq C\,m_-(D) \subseteq C_- = (C * D)_-$.

4° Case: D is a periodic archimedean class and $m(D) = m_+(D)$. By assumption, we have $C\delta \neq D\delta$ and, since $C\delta = C\delta \wedge D\delta \preccurlyeq D\delta$, we have $C\delta \prec D\delta$. Now let $x \in C$ and $y \in m_+(D)$. We denote by Y the archimedean class containing y. Then by Lemma 1.2, we have $D\delta \preccurlyeq Y\delta$ and so $(C * D)\,\delta = C\delta \wedge D\delta = C\delta \prec D\delta =$ $= D\delta \wedge Y\delta$. Hence by [2] Lemma 5.6, there exists no $Z \in \mathscr{C}$ such that $Z \in C\delta$ and Z lies between D and Y. In particular, C does not lie between D and Y and, since $C < D$, we have $C < Y$. Also

$$\begin{aligned} Y * C &= \max\{Z \in \mathscr{C};\; C \leqq Z \leqq Y \text{ and } Z \in C\delta \wedge Y\delta\} \\ &= \max\{Z \in \mathscr{C};\; C \leqq Z \leqq Y \text{ and } Z \in C\delta\} \\ &= \max\{Z \in \mathscr{C};\; C \leqq Z \leqq D \text{ and } Z \in C\delta\} \\ &= \max\{Z \in \mathscr{C};\; C \leqq Z \leqq D \text{ and } Z \in C\delta \wedge D\delta\} = D * C\,. \end{aligned}$$

Since $(D * C)\,\delta = C\delta \wedge D\delta$, $D * C = Y * C$ is a periodic archimedean class with idempotent, say h. Since CD is contained in a single archimedean class, it follows from Lemma 1.10 that $ch = e$ for every $c \in C_-$ and, applying Lemma 1.10 again, we have $xy \in CY \subseteq C_-$. Hence $C\,m(D) = C\,m_+(D) \subseteq C_- = (C * D)_-$ and also $C_-\,m(D) = C_-\,m_+(D) \subseteq C\,m_+(D) \subseteq C_- = (C * D)_-$.

5° Case: D is a torsion free archimedean class and $m(D) = m_\pm(D)$. Since $C\delta =$ $= C\delta \wedge D\delta$ is a periodic δ-class and $D\delta$ is a torsion free δ-class, we have $C\delta \neq D\delta$. In a similar way to 4°, we can prove that $C\,m(D) = C\,m_\pm(D) \subseteq C_- = (C * D)_-$ and $C_-\,m(D) = C_-\,m_\pm(D) \subseteq C_- = (C * D)_-$.

(2) can be proved similarly.

Theorem 1.12. *Let $C, D \in \mathscr{C}$ such that $C\delta = D\delta$ is a periodic δ-class of L-type.*

(1) *Suppose $C < D$. We denote $m_+(D) = [e', f']$. Thus e' is the idempotent of D and f' is the idempotent of S such that $e' < f'$, $e'\,\mathscr{D}_E\,f'$ and e' and f' are consecutive in $e'\mathscr{D}_E$. We denote by F the archimedean class containing f'.*

(i) *If CF is contained in a single archimedean class, then* $C\,m_+(D) = m_0(C) = m_0(C * D)$ *and* $C_-\,m_+(D) = m_0(C) = m_0(C * D)$.

(ii) *If CF is not contained in a single archimedean class, then* $C\,m_+(D) \subseteq m_+(C) = m_+(C * D)$ *and* $C_-\,m_+(D) \subseteq m_+(C) = m_+(C * D)$.

(2) *Suppose $D < C$. We denote $m_-(D) = [g', e']$. Thus e' is the idempotent of D and g' is the idempotent of S such that $g' < e'$, $g'\,\mathscr{D}_E\,e'$ and g' and e' are consecutive in $e'\mathscr{D}_E$. We denote by G the archimedean class containing g'.*

(i) *If CG is contained in a single archimedean class, then* $C\,m_-(D) = m_0(C) = m_0(C * D)$ *and* $C_+\,m_-(D) = m_0(C) = m_0(C * D)$.

(ii) *If CG is not contained in a single archimedean class, then* $C\,m_-(D) \subseteq m_-(C) = m_-(C * D)$ *and* $C_+\,m_-(D) \subseteq m_-(C) = m_-(C * D)$.

Proof. We denote by e the idempotent of the periodic archimedean class C.

(1) Since $e'\,\mathscr{D}_E\,f'$, it follows from [2] Theorem 3.3 that $C\;\delta\;D\;\delta\;F$ and $C = C * D$. Now let $x \in C$ and $y \in m_+(D)$. Then since $C < D < F$, we have $e < e' \leqq y \leqq f'$ and so $e = xe \leqq xy \leqq xf'$.

(i) Suppose CF is contained in a single archimedean class. Then by Lemma 1.10, $xf' \in CF \subseteq C_- \subseteq C$. But by [2] Theorem 2.7, we have $f'x = f'$ and so $(xf')^2 = xf'xf' = xf'$. Hence xf' is an idempotent of C and so $xf' = e$. Hence $xy = e$. Hence $C\,m_+(D) = \{e\} = m_0(C) = m_0(C * D)$ and also, since $C_-\,m_+(D) \subseteq C\,m_+(D)$, we have $C_-\,m_+(D) = m_0(C) = m_0(C * D)$.

(ii) Next suppose that CF is not contained in a single archimedean class. Then by [4] Theorems 3 and 4, there exists an idempotent f of S such that $e < f$, $e\,\mathscr{D}_E\,f$, e and f are consecutive in $e\mathscr{D}_E$ and $CF \subseteq [e,f]$. Hence $xf' \in CF \subseteq [e,f]$ and so $e \leqq xy \leqq xf' \leqq f$. Hence $xy \in [e,f] = m_+(C)$. Hence $C\,m_+(D) \subseteq m_+(C) = m_+(C * D)$ and also $C_-\,m_+(D) \subseteq C\,m_+(D) \subseteq m_+(C) = m_+(C * D)$.

(2) can be proved similarly.

Lemma 1.13. *Let $C, D \in \mathscr{C}$ such that $C\delta \wedge D\delta$ is a periodic δ-class of L-type, $C\delta = C\delta \wedge D\delta$ and CD is not contained in a single archimedean class and let $m(D)$ be a modified archimedean class of D.*

(1) *If $C < D$, then* $C_-\,m(D) \subseteq m_+(C) = m_+(C * D)$ *and* $C\,m(D) \subseteq m_+(C) = m_+(C * D)$.

(2) *If $D < C$, then* $C_+\,m(D) \subseteq m_-(C) = m_-(C * D)$ *and* $C\,m(D) \subseteq m_-(C) = m_-(C * D)$.

Proof. By Lemma 1.7, we have $C = C * D$. We denote by e the idempotent of the periodic archimedean class C.

(1) 1° Case: $m(D) = D$. By [2] Theorem 6.8 and [4] Theorems 3 and 4, there exists an idempotent f of S such that $e < f$, $e\,\mathscr{D}_E\,f$, e and f are consecutive in $e\mathscr{D}_E$ and $CD \subseteq [e,f]$. Hence $C\,m(D) = CD \subseteq [e,f] = m_+(C) = m_+(C * D)$ and also $C_-\,m(D) = C_-D \subseteq CD \subseteq m_+(C) = m_+(C * D)$.

2° Case: D is a periodic archimedean class and $m(D) \in \{D_+, D_-, m_0(D)\}$. We have $m(D) \subseteq D$ and by 1°, $C_- m(D) \subseteq C_- D \subseteq m_+(C) = m_+(C * D)$ and $C\, m(D) \subseteq CD \subseteq m_+(C) = m_+(C * D)$.

3° Case: D is a periodic archimedean class with idempotent e' and $m(D) = m_+(D)$. Thus there exists an idempotent f' of S such that $e' < f'$, $e' \mathscr{D}_E f'$, e' and f' are consecutive in $e'\mathscr{D}_E$ and $m_+(D) = [e', f']$. We denote by F the archimedean class containing f'. Since $e' \mathscr{D}_E f'$, it follows from [2] Theorem 3.3 that $D\, \delta\, F$ and so $C\delta = C\delta \wedge D\delta = C\delta \wedge F\delta$ is of L-type. Also we have $C < D < F$. Since $(D * C)\, \delta = D\delta \wedge C\delta$ and $(F * C)\, \delta = F\delta \wedge C\delta = D\delta \wedge C\delta$, $D * C$ and $F * C$ are periodic archimedean classes. We denote by h and k the idempotents of $D * C$ and $F * C$, respectively. Since CD is not contained in a single archimedean class, it follows from [2] Lemma 6.7 that there exists an idempotent f of S such that $e < f$, $e \mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$ and also there exists $c \in C_- \setminus \{e\}$ such that $ch = f$. Since $D < F$, it follows from [2] Lemma 5.7 that $D * C \leqq F * C$. Hence $h \leqq k$ and so $e < ch \leqq ck$. Hence by Lemma 1.10, CF is not contained in a single archimedean class and so by [2] Theorem 6.8 and [4] Theorems 3 and 4, $CF \subseteq [e, f]$. Now let $x \in C$ and $y \in m_+(C) = [e', f']$. Then $e < e' \leqq y \leqq f'$ and so $e = xe \leqq xy \leqq xf'$ with $xf' \in CF \subseteq [e, f]$. Hence $xy \in [e, f] = m_+(C)$ and so $C\, m(D) = C\, m_+(D) \subseteq m_+(C) = m_+(C * D)$ and also $C_-\, m(D) = C_-\, m_+(D) \subseteq C\, m_+(D) \subseteq m_+(C) = m_+(C * D)$.

4° Case: D is a periodic archimedean class with idempotent e' $m(D) = m_-(D)$. Thus there exists an idempotent g' of S such that $g' < e'$, $g' \mathscr{D}_E e'$, g' and e' are consecutive in $e'\mathscr{D}_E$ and $m_-(D) = [g', e']$. We denote by G the archimedean class containing g'. Since $g' \mathscr{D}_E e'$, it follows from [2] Theorem 3.3 that $G\, \delta\, D$. By way of contradiction we assume $G < C$. Then since $G < C < D$, it follows from [2] Lemma 5.6 that $D\delta = D\delta \wedge G\delta \preccurlyeq C\delta = C\delta \wedge D\delta \preccurlyeq D\delta$ and so $C\delta = D\delta$. Hence by [2] Theorem 3.3, we have $g', e, e' \in e'\mathscr{D}_E$. But since $G < C < D$, we have $g' < e < e'$, which contradicts that g' and e' are consecutive in $e'\mathscr{D}_E$. Hence we have $C \leqq G < D$. Since CD is not contained in a single archimedean class, it follows from [2] Theorem 6.8 and [4] Theorems 3 and 4 that there exists an idempotent f of S such that $e < f$, $e \mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$ and $CD \subseteq [e, f]$. Now let $x \in C$ and $y \in m_-(D) = [g', e']$. Then we have $e \leqq g' \leqq y \leqq e'$ and so $e = xe \leqq xy \leqq xe'$ with $xe' \in CD \subseteq [e, f]$. Hence we have $xy \in [e, f] = m_+(C)$. Hence $C\, m(D) = C\, m_-(D) \subseteq m_+(C) = m_+(C * D)$ and also $C_-\, m(D) = C_-\, m_-(D) \subseteq C\, m_-(D) \subseteq m_+(C) = m_+(C * D)$.

5° Case: D is a torsion free archimedean class and $m(D) = m_\pm(D)$. Since $D\delta$ is a torsion free δ-class, $C\delta = C\delta \wedge D\delta$ is a periodic δ-class and $C\delta = C\delta \wedge D\delta \preccurlyeq D\delta$, we have $C\delta \prec D\delta$. Now let $x \in C$ and $y \in m_\pm(D)$. We denote by Y the archimedean class containing y. Then by Lemma 1.2, we have $D\delta \preccurlyeq Y\delta$ and so $(D * C)\, \delta = D\delta \wedge C\delta = C\delta \prec D\delta = D\delta \wedge Y\delta$. Hence by [2] Lemma 5.6, there is no $Z \in \mathscr{C}$ such that $Z \in C\delta$ and Z lies between D and Y. In particular, C does not lie between D and Y and, since $C < D$, we have $C < Y$. Also

$$\begin{aligned} Y * C &= \max\{Z \in \mathscr{C};\ C \leqq Z \leqq Y \text{ and } Z \in Y\delta \wedge C\delta\} \\ &= \max\{Z \in \mathscr{C};\ C \leqq Z \leqq Y \text{ and } Z \in C\delta\} \\ &= \max\{Z \in \mathscr{C};\ C \leqq Z \leqq D \text{ and } Z \in C\delta\} \\ &= \max\{Z \in \mathscr{C};\ C \leqq Z \leqq D \text{ and } Z \in D\delta \wedge C\delta\} = D * C\,. \end{aligned}$$

We denote by h the idempotent of the periodic archimedean class $D * C = Y * C$. Since CD is not contained in a single archimedean class, it follows from Lemma 1.10 that there exists $c \in C_-$ such that $ch \neq e$ and so by Lemma 1.10 again, CY is not contained in a single archimedean class. Hence by [2] Theorem 6.8 and [4] Theorems 3 and 4, there exists an idempotent f of S such that $e < f$, $e\,\mathscr{D}_E\,f$ and e and f are consecutive in $e\mathscr{D}_E$ and $CY \subseteq [e, f]$. Hence $xy \in CY \subseteq [e, f] = m_+(C)$ and so $C\,m(D) = C\,m_\pm(D) \subseteq m_+(C) = m_+(C * D)$ and also $C_-\,m(D) = C_-\,m_\pm(D) \subseteq \subseteq C\,m_\pm(D) \subseteq m_+(C) = m_+(C * D)$.

(2) can be proved similarly.

Theorem 1.14. *Let $C, D \in \mathscr{C}$ and let $m(C)$ and $m(D)$ be modified archimedean classes of C and D, respectively. Then there exists a modified archimedean class $m(C * D)$ of $C * D$ such that $m(C)\,m(D) \subseteq m(C * D)$.*

Proof. Case: $C\delta \wedge D\delta$ is torsion free. Then by [2] Theorem 6.1, if $C \operatorname{non}\delta\, D$, then we have either $C\,\gamma\,D$ or $D\,\gamma\,C$. Hence by Lemma 1.3, we have the assertion.

Case: $C\delta \wedge D\delta$ is periodic of L-type and $C\delta \neq C\delta \wedge D\delta$. Then we have $C\delta \wedge \wedge D\delta \prec C\delta$. If $C * D \neq D$, then the assertion holds by Lemma 1.5. If $C * D = D$, then the assertion holds by Lemma 1.6.

Case: $C\delta \wedge D\delta$ is periodic of L-type and $C\delta = C\delta \wedge D\delta$. Then C is a periodic archimedean class and so $m(C) \in \{C, C_+, C_-, m_0(C), m_+(C), m_-(C)\}$. If $m(C) \in \in \{m_0(C), m_+(C), m_-(C)\}$, then the assertion holds by Lemma 1.7. If $m(C) \in \in \{C, C_+, C_-\}$ and $D = C$, then the assertion holds by Lemma 1.8. If either $m(C) = = C_+$ and $C < D$ or $m(C) = C_-$ and $D < C$, then the assertion holds by Lemma 1.9. If either $m(C) \in \{C, C_-\}$ and $C < D$ or $m(C) \in \{C, C_+\}$ and $D < C$, then the assertion holds by Lemmas 1.11, 1.12 and 1.13.

Case: $C\delta \wedge D\delta$ is periodic of R-type. We have the assertion dually.

Corollary 1.15. *Let $C_1, \ldots, C_n \in \mathscr{C}$ and, for each natural number $1 \leqq i \leqq n$, let $m(C_i)$ be a modified archimedean class of C_i. Then there exists a modified archimedean class $m(C_1 * \ldots * C_n)$ of $C_1 * \ldots * C_n$ such that $m(C_1) \ldots m(C_n) \subseteq \subseteq m(C_1 * \ldots * C_n)$.*

In particular, for $C_1, \ldots, C_n \in \mathscr{C}$, there exists a modified archimedean class $m(C_1 * \ldots * C_n)$ of $C_1 * \ldots * C_n$ such that $C_1 \ldots C_n \subseteq m(C_1 * \ldots * C_n)$.

§ 2

Lemma 2.1. *Let $C_1, \ldots, C_n \in \mathscr{C}$ such that $C_1\delta \wedge \ldots \wedge C_n\delta$ is a torsion free δ-class. Then there exists a natural number $1 \leqq i \leqq n$ such that $C_i\delta = C_1\delta \wedge \ldots \wedge C_n\delta$.*

Proof. We put $C = C_1 * \ldots * C_n$, $D_1 = \max\{C_1, \ldots, C_n\}$ and $D_2 = \min\{C_1, \ldots, C_n\}$. Then $D_1 = C_j$ and $D_2 = C_k$ for some natural numbers $1 \leqq j \leqq n$ and $1 \leqq k \leqq n$. By Lemma 1.4, $C\delta = (C_1 * \ldots * C_n)\,\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ and so C is a torsion free archimedean class. Hence C is either positive torsion free or negative torsion free. First suppose C is positive torsion free. By [2] Theorem 5.11, we have $C = C_1 * \ldots * C_n \leqq D_1 * \ldots * D_1 = D_1 = C_j$. But $C\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_j\delta$ and so by [2] Lemma 4.7, we have $C\,\gamma\,C_j$. Hence by [2] Lemma 5.1, we have $C = C_j$ and so $C_j\delta = C\delta = C_1\delta \wedge \ldots \wedge C_n\delta$. Next suppose C is negative torsion free. Then we obtain in a similar wat that $C = C_k$ and $C_k\delta = C_1\delta \wedge \ldots \wedge C_n\delta$.

Theorem 2.2. *Suppose that $C_1, \ldots, C_n \in \mathscr{C}$ such that $C_1\delta \wedge \ldots \wedge C_n\delta$ is a torsion free δ-class and there exists an archimedean class A such that for every natural number $1 \leqq i \leqq n$ such that $C_i\delta = C_1\delta \wedge \ldots \wedge C_n\delta$, we have $C_i = A$. Then*

$$C_1 \ldots C_n \subseteq A = C_1 * \ldots * C_n \,.$$

Proof. By Lemma 2.1, there really exists a natural number $1 \leqq i \leqq n$ such that $C_i\delta = C_1\delta \wedge \ldots \wedge C_n\delta$. Hence by assumption $A\delta = C_i\delta = C_1\delta \wedge \ldots \wedge C_n\delta$. In particular A is a torsion free archimedean class. Now we denote by j the least natural number $1 \leqq j \leqq n$ such that $C_1\delta \wedge \ldots \wedge C_j\delta = C_1\delta \wedge \ldots \wedge C_n\delta$. We show that

$$(*) \qquad C_1 \ldots C_j \subseteq A = C_1 * \ldots * C_j \,.$$

If $j = 1$, then $C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ and so, by assumption, we have $C_1 = A$. Hence $(*)$ holds. Next suppose $j > 1$. Then for every natural number p such that $1 \leqq p \leqq j - 1$, we have $A\delta = C_1\delta \wedge \ldots \wedge C_n\delta \prec C_1\delta \wedge \ldots \wedge C_{j-1}\delta \preccurlyeq C_p\delta$ by the minimality of j and so $A \neq C_p$. On the other hand $C_1\delta \wedge \ldots \wedge C_j\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ is torsion free δ-class and so by Lemma 2.1, we have $C_i\delta = C_1\delta \wedge \ldots \wedge C_j\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ for some $1 \leqq i \leqq j$ and then by assumption, $C_i = A$. Hence. we must have $i = j$ and so we have $C_j = A$. Also since $C_j\delta = A\delta = C_1\delta \wedge \ldots \wedge C_n\delta \prec C_1\delta \wedge \ldots \wedge C_{j-1}\delta = (C_1 * \ldots * C_{j-1})\,\delta$, we have C_j nonδ $C_1 * \ldots * C_{j-1}$ and also by [2] Lemma 4.7, $C_j\,\gamma\,C_1 * \ldots * C_{j-1}$. By Corollary 1.15, there exists a modified archimedean class $m(C_1 * \ldots * C_{j-1})$ of $C_1 * \ldots * C_{j-1}$ such that $C_1 \ldots C_{j-1} \subseteq m(C_1 * \ldots * C_{j-1})$. Hence by Lemma 1.3, we have $C_1 \ldots C_j \subseteq m(C_1 * \ldots * C_{j-1})\,C_j \subseteq C_j = A$. Further by [2] Theorem 6.1, we have $A = C_j = (C_1 * \ldots * C_{j-1}) * C_j$. Thus we obtain $(*)$.

Now let k be a natural number such that $j \leqq k < n$ and $C_1 \ldots C_k \subseteq A = C_1 * \ldots * C_k$. Then $A\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_{k+1}\delta$. First suppose $A\delta \prec C_{k+1}\delta$. Then A nonδ C_{k+1} and $A\,\gamma\,C_{k+1}$. Hence by [2] Theorem 6.1,

$$C_1 \ldots C_k C_{k+1} \subseteq A C_{k+1} \subseteq A = A * C_{k+1} = C_1 * \ldots * C_k * C_{k+1} \,.$$

Next suppose $A\delta = C_{k+1}\delta$. Then $C_{k+1}\delta = A\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ and so by assumption, $C_{k+1} = A$. Hence

$$C_1 \ldots C_k C_{k+1} \subseteq A C_{k+1} = AA \subseteq A = A * A = C_1 * \ldots * C_k * C_{k+1} \,.$$

By induction, we obtain $C_1 \ldots C_n \subseteq A = C_1 * \ldots * C_n$.

Theorem 2.3. *Suppose that $C_1, \ldots, C_n \in \mathscr{C}$ such that $C_1\delta \wedge \ldots \wedge C_n\delta$ is a torsion free δ-class and there exist two distinct archimedean classes A and B such that there exist natural numbers $1 \leqq i \leqq n$ and $1 \leqq j \leqq n$ such that $C_i\delta = C_j\delta = C_1\delta \wedge \ldots \wedge C_n\delta$, $C_i = A$ and $C_j = B$. Then $C_1 \ldots C_n$ is not contained in a single archimedean class. Also in this case, we have*

$$C_1 \ldots C_n \subseteq m_{\pm}(A) = m_{\pm}(B) = m_{\pm}(C_1 * \ldots * C_n)\,.$$

Proof. Since $A\delta = C_i\delta = C_1\delta \wedge \ldots \wedge C_n\delta = C_j\delta = B\delta$, A and B are torsion free archimedean classes which lie in the same δ-class. Hence by [2] Theorem 3.5, the δ-class $A\delta = B\delta$ consists of exactly two elements A and B. Without loss of generality, we can assume $A < B$. Then A is negative torsion free and B is positive torsion free. We put $D_1 = \min\{C_1, \ldots, C_n\}$ and $D_2 = \max\{C_1, \ldots, C_n\}$. Then $D_1 = C_p$ and $D_2 = C_q$ for some natural numbers $1 \leqq p \leqq n$ and $1 \leqq q \leqq n$. Hence $A\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_p\delta = D_1\delta$ and $B\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_q\delta = D_2\delta$ and so by [2] Lemma 4.7, we have $A\,\gamma\,D_1$ and $B\,\gamma\,D_2$. Also we have $D_1 \leqq C_i = A < B = C_j \leqq D_2$ and so by [2] Lemma 5.1, we have $A = D_1$ and $B = D_2$. Hence for every natural number $1 \leqq r \leqq n$, we have $A = D_1 \leqq C_r \leqq D_2 = B$. Now $m_{\pm}(A) = m_{\pm}(B)$ and this is the set of elements x of S such that the archimedean class containing the element x lies between A and B. Thus $C_r \subseteq m_{\pm}(A)$ for every natural number $1 \leqq r \leqq n$. Also by [3] Theorem 9, there exists an o-homomorphism v of $m_{\pm}(A)$ into the additive ordered group of real numbers such that

if $x \in A$, then $v(x) < 0$;
if $x \in m_{\pm}(A) \setminus (A \cup B)$, then $v(x) = 0$;
if $x \in B$, then $v(x) > 0$.

Now we take arbitrarily $c_r \in C_r$ for each natural number $1 \leqq r \leqq n$. Since $C_i = A$ and $C_j = B$, we have $v(c_i) < 0$ and $v(c_j) > 0$. Now let $v(c_1) + \ldots + s\,v(c_i) + \ldots + v(c_n)$ and $v(c_1) + \ldots + t\,v(c_j) + \ldots + v(c_n)$ be sums arising from $v(c_1) + \ldots + v(c_n)$ by replacing $v(c_i)$ by $s\,v(c_i)$ and $v(c_j)$ by $t\,v(c_j)$, respectively, leaving other terms unchanged. Then since $v(c_i) < 0$ and $v(c_j) > 0$, we can consider, by taking s and t sufficiently large,

$$v(c_1 \ldots c_i^s \ldots c_n) = v(c_1) + \ldots + s\,v(c_i) + \ldots + v(c_n) < 0\,,$$
$$v(c_1 \ldots c_j^t \ldots c_n) = v(c_1) + \ldots + t\,v(c_j) + \ldots + v(c_n) > 0\,.$$

Hence $c_1 \ldots c_i^s \ldots c_n \in A$ and $c_1 \ldots c_j^t \ldots c_n \in B$. But since both $c_1 \ldots c_i^s \ldots c_n$ and and $c_1 \ldots c_j^t \ldots c_n$ are elements of $C_1 \ldots C_n$, $C_1 \ldots C_n$ is not contained in a single archimedean class.

Since $C_r \subseteq m_{\pm}(A)$ for every natural number $1 \leqq r \leqq n$, it follows from Lemma 1.1 that $C_1 \ldots C_n \subseteq m_{\pm}(A) = m_{\pm}(B)$. Also by Lemma 1.4, we have $(C_1 * \ldots * C_n)\,\delta = C_1\delta \wedge \ldots \wedge C_n\delta = A\delta$ and so $C_1 * \ldots * C_n \in A\delta$. But since $A\delta$ consists of exactly two elements A and B, we have $C_1 * \ldots * C_n = A$ or $C_1 * \ldots * C_n = B$ and in both cases, we have $m_{\pm}(C_1 * \ldots * C_n) = m_{\pm}(A) = m_{\pm}(B)$.

Corollary 2.4. *Let $C_1, \ldots, C_n \in \mathscr{C}$ such that $C_1\delta \wedge \ldots \wedge C_n\delta$ is a torsion free*

δ-class. Then $C_1 \dots C_n$ is contained in a single archimedean class if and only if the set

$$\{C_i;\ i \text{ is a natural number},\ 1 \leqq i \leqq n \text{ and } C_i\delta = C_1\delta \wedge \dots \wedge C_n\delta\}$$

is a one-element subset of $\mathscr{C}$.

Lemma 2.5. *Let $C_1, \dots, C_n \in \mathscr{C}$ such that $C_1\delta \wedge \dots \wedge C_n\delta$ is a periodic δ-class. Then $m_0(C_1 * \dots * C_n) \subseteq C_1 \dots C_n$.*

Proof. For each natural number $1 \leqq i \leqq n$, we put

$m(C_i) = m_0(C_i)$ if C_i is a periodic archimedean class;

$m(C_i) = C_i$ if C_i is a torsion free archimedean class.

Then always we have $m(C_i) \subseteq C_i$. Here we only consider the case when the periodic δ-class $C_1\delta \wedge \dots \wedge C_n\delta$ is of L-type. Let k be the least natural number $1 \leqq k \leqq n$ such that $C_1\delta \wedge \dots \wedge C_k\delta = C_1\delta \wedge \dots \wedge C_n\delta$.

Case: $k > 1$. By the minimality of k, we have $(C_1 * \dots * C_{k-1})\,\delta * C_k\delta = C_1\delta \wedge \dots \wedge C_k\delta = C_1\delta \wedge \dots \wedge C_n\delta \prec C_1\delta \wedge \dots \wedge C_{k-1}\delta = (C_1 * \dots * C_{k-1})\,\delta$. Also by Corollary 1.15, there exists a modified archimedean class $m(C_1 * \dots * C_{k-1})$ of $C_1 * \dots * C_{k-1}$ such that $m(C_1) \dots m(C_{k-1}) \subseteq m(C_1 * \dots * C_{k-1})$. First suppose that $(C_1 * \dots * C_{k-1}) * C_k \neq C_k$. Then by Lemma 1.5, we have $m(C_1) \dots m(C_{k-1})\,.\,m(C_k) \subseteq m(C_1 * \dots * C_{k-1})\, m(C_k) = m_0(C_1 * \dots * C_{k-1} * C_k)$ and so $m(C_1) \dots m(C_k) = m_0(C_1 * \dots * C_k)$. Next suppose that $(C_1 * \dots * C_{k-1}) * C_k = C_k$. Then $C_k\delta = (C_1 * \dots * C_k)\,\delta = C_1\delta \wedge \dots \wedge C_k\delta = C_1\delta \wedge \dots \wedge C_n\delta$ and so C_k is a periodic archimedean class. Hence by definition, we have $m(C_k) = m_0(C_k)$ and by Lemma 1.6, we have $m(C_1) \dots m(C_{k-1})\, m(C_k) \subseteq m(C_1 * \dots * C_{k-1})\, m_0(C_k) = m_0(C_1 * \dots * C_{k-1} * C_k)$ and so $m(C_1) \dots m(C_k) = m_0(C_1 * \dots * C_k)$.

Case: $k = 1$. Then $C_1\delta = C_1\delta \wedge \dots \wedge C_n\delta$ and so C_1 is a periodic archimedean class and so $m(C_1) = m_0(C_1)$.

Thus always we have $m(C_1) \dots m(C_k) = m_0(C_1 * \dots * C_k)$. Now let p be a natural number such that $k \leqq p < n$ and $m(C_1) \dots m(C_p) = m_0(C_1 * \dots * C_p)$. Then by Lemma 1.7, we have $m(C_1) \dots m(C_p)\, m(C_{p+1}) = m_0(C_1 * \dots * C_p)\, m(C_{p+1}) = m_0(C_1 * \dots * C_p * C_{p+1})$. Hence by induction, we obtain $m(C_1) \dots m(C_n) = m_0(C_1 * \dots * C_n)$. Since $m(C_i) \subseteq C_i$ for every natural number $1 \leqq i \leqq n$, we have $m_0(C_1 * \dots * C_n) = m(C_1) \dots m(C_n) \subseteq C_1 \dots C_n$.

Theorem 2.6. *Let $C_1, \dots, C_n \in \mathscr{C}$ such that $C_1 \dots C_n$ is contained in a single archimedean class. Then $C_1 \dots C_n \subseteq C_1 * \dots * C_n$.*

Proof. First suppose that $C_1\delta \wedge \dots \wedge C_n\delta$ is torsion free. Then by Corollary 2.4, there exists an archimedean class A such that for every natural number $1 \leqq i \leqq n$ such that $C_i\delta = C_1\delta \wedge \dots \wedge C_n\delta$ we have $C_i = A$. Hence by Theorem 2.2, we have $C_1 \dots C_n \subseteq C_1 * \dots * C_n$.

Next suppose that $C_1\delta \wedge \dots \wedge C_n\delta$ is periodic. Then by Lemma 2.5, $C_1 \dots C_n$ contains the idempotent of the archimedean class $C_1 * \dots * C_n$ and since $C_1 \dots C_n$ is contained in a single archimedean class, we have $C_1 \dots C_n \subseteq C_1 * \dots * C_n$.

Lemma 2.7. *Let a be an element of finite order n of S. If there exists an idempotent g of S such that $a^n \mathscr{D}_E g$ and a lies between a^n and g, then $n \leqq 2$.*

Proof. Suppose $1 < n$. We only consider the case when a is positive, that is, $a < a^2$. Then we have $g < a < a^2 \leqq a^n$. By [2] Lemmas 1.6 and 1.7, we have $a^n \mathscr{L} g$ or $a^n \mathscr{R} g$. For the sake of definiteness, we assume $a^n \mathscr{R} g$. Then $a^n g = g$ and $ga^n = a^n$. Then $g = g^2 \leqq ag \leqq a^n g = g$ and so $g = ag$. Hence $gaga = ga$, $ga^2ga^2 = ga^2$ and so ga and ga^2 are idempotents of S. We have $a < ga^2$, since $ga^2 \leqq a$ would imply $a^n = ga^n \leqq \ldots \leqq ga^2 \leqq a$, which is a contradiction. If $a \leqq ga$, then $a^3 \leqq (ga)^3 = ga \leqq a^2 \leqq a^3$ and if $ga \leqq a$, then $a^3 \leqq (ga^2)^3 = ga^2 = (ga)\, a \leqq \leqq a^2 \leqq a^3$. Hence always we have $a^2 = a^3$.

Lemma 2.8. *Let C be a periodic archimedean class in S.*

(1) *If there exists $D \in C$ such that $C < D$ and $C \,\delta\, D$, then $C^2 \subseteq C_+$.*

(2) *If there exists $D \in C$ such that $D < C$ and $C \,\delta\, D$, then $C^2 \subseteq C_-$.*

Proof. (1) Since C is periodic and $C \,\delta\, D$, it follows from [2] Theorem 3.2 that D is also a periodic archimedean class. We denote by e and f the idempotents in C and D, respectively. Then by [2] Theorem 3.3, we have $e \mathscr{D}_E f$. Also since $C < D$, we have $e < f$. Now let $x, y \in C_-$. Then we have $e \leqq x^2 \leqq x < f$ and $e \leqq y^2 \leqq \leqq y < f$ and so by Lemma 2.7, we have $x^2 = y^2 = e$. But we have either $x \leqq y$ or $y \leqq x$. If $x \leqq y$, then $e = x^2 \leqq xy \leqq y^2 = e$ and if $y \leqq x$, then $e = y^2 \leqq xy \leqq \leqq x^2 = e$. Hence always we have $xy = e$ and so $C_-^2 = m_0(C)$. Hence by Lemma 1.8, we have

$$C^2 = (C_+ \cup C_-)^2 = C_+^2 \cup C_+C_- \cup C_-C_+ \cup C_-^2 \subseteq$$

$$\subseteq C_+ \cup m_0(C) \cup m_0(C) \cup m_0(C) = C_+ .$$

(2) can be proved similarly.

Lemma 2.9. *Let $C_1, \ldots, C_n \in \mathscr{C}$ such that $C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ is a δ-class of L-type.*

(1) *Suppose that $C_1 \ldots C_n$ is not contained in a single archimedean class and $C_1 \ldots C_n \subseteq m_+(C_1)$. Thus, denoting by e the idempotent of the periodic archimedean class C_1, there exists an idempotent f of S such that $e < f$, $e \mathscr{D}_E f$, e and f are consecutive in $e\mathscr{D}_E$ and $m_+(C_1) = [e, f]$. Let k be the least natural number such that $C_1 \ldots C_k$ is not contained in a single archimedean class. Then*

(i) $k > 1$.

(ii) *For every natural number $1 \leqq i \leqq k - 1$, we have $C_1 \ldots C_i \subseteq C_1$.*

(iii) *For every natural number $1 \leqq i \leqq k - 1$, we have $C_1 < C_{i+1}$. In particular $C_1 < C_k$.*

(iv) *We denote by h the idempotent of the periodic archimedean class $C_k * C_1$.*

Then $e < f < h$ *and* $h \mathscr{D}_E e$.

(v) $C_1 \dots C_k \subseteq m_+(C_1)$.

(vi) *If* $x_1 \in C_1, \dots, x_{k-1} \in C_{k-1}$, $y_k \in C_k$ *such that* $x_1 \dots x_{k-1} y_k \notin C_1$, *then* $x_1 \dots \dots x_{k-1} h = f$. *Also there exists* $x_k \in C_k$ *such that* $x_1 \dots x_{k-1} x_k = f$.

(vii) *There exist* $x_1 \in C_1, \dots, x_n \in C_n$ *such that* $x_1 \dots x_n = f$.

(2) *Suppose that* $C_1 \dots C_n$ *is not contained in a single archimedean class and* $C_1 \dots C_n \subseteq m_-(C_1)$. *Thus, denoting by* e *the idempotent of the periodic archimedean class* C_1, *there exists an idempotent* g *of* S *such that* $g < e$, $g \mathscr{D}_E e$, g *and* e *are consecutive in* $e\mathscr{D}_E$ *and* $m_-(C_1) = [g, e]$. *Let* k *be the least natural number such that* $C_1 \dots C_k$ *is not contained in a single archimedean class. Then*

(i) $k > 1$.

(ii) *For every natural number* $1 \leqq i \leqq k - 1$, *we have* $C_1 \dots C_i \subseteq C_1$.

(iii) *For every natural number* $1 \leqq i \leqq k - 1$, *we have* $C_{i+1} < C_1$. *In particular* $C_k < C_1$.

(iv) *We denote by* h *the idempotent of the periodic archimedean class* $C_k * C_1$. *Then* $h < g < e$ *and* $h \mathscr{D}_E e$.

(v) $C_1 \dots C_k \subseteq m_-(C_1)$.

(vi) *If* $x_1 \in C_1, \dots, x_{k-1} \in C_{k-1}$, $y_k \in C_k$ *such that* $x_1 \dots x_{k-1} y_k \notin C_1$, *then* $x_1 \dots x_{k-1} h = g$. *Also there exists* $x_k \in C_k$ *such that* $x_1 \dots x_{k-1} x_k = g$.

(vii) *There exist* $x_1 \in C_1, \dots, x_n \in C_n$ *such that* $x_1 \dots x_n = g$.

Proof. (1) (i). Since $C_1 \dots C_k$ is not contained in a single archimedean class, it is clear that $k > 1$.

(ii) Let $1 \leqq i \leqq k - 1$. Then by the minimality of k, $C_1 \dots C_i$ is contained in a single archimedean class. Hence by Theorem 2.6, we have $C_1 \dots C_i \subseteq C_1 * \dots * C_i$. If $i = 1$, then trivially we have $C_1 \dots C_i \subseteq C_1$. Suppose $i \geqq 2$. Then we have $C_1\delta = C_1\delta \wedge \dots \wedge C_n\delta \preccurlyeq C_2\delta \wedge \dots \wedge C_i\delta = (C_2 * \dots * C_i)\,\delta$ and $C_1\delta \wedge (C_2 * \dots * C_i)\,\delta = C_1\delta$ is of L-type. Hence $C_1 \dots C_i \subseteq C_1 * \dots * C_i = C_1 * (C_2 * \dots * C_i) = C_1$.

(iii) By way of contradiction we assume $C_{i+1} \leqq C_1$ for some natural number $1 \leqq i \leqq k - 1$. Let $x_1 \in C_1, \dots, x_n \in C_n$. Then by (ii), we have $x_1 \dots x_i \in C_1 \dots \dots C_i \subseteq C_1$. First suppose $C_{i+1} < C_1$. Then we have $x_{i+1} < e$ and so $x_1 \dots \dots x_i x_{i+1} \leqq x_1 \dots x_i e = e$. Next suppose $C_{i+1} = C_1$. We denote by F the archimedean class containing the element f. Then $C_1 < F$. Also since $e \mathscr{D}_E f$, it follows from [2] Theorem 3.3 that $C_1 \,\delta\, F$. Hence by Lemma 2.8, we have $x_1 \dots x_i x_{i+1} \in C_1^2 \subseteq (C_1)_+$ and so $x_1 \dots x_i x_{i+1} \leqq e$. Thus always we have $x_1 \dots x_{i+1} \leqq e$. If $i + 1 = n$, then we have $x_1 \dots x_n \leqq e$. Suppose $i + 1 < n$. Then for every natural number $1 \leqq j \leqq n$, we have $C_1\delta = C_1\delta \wedge \dots \wedge C_n\delta \preccurlyeq C_j\delta$ and by [2] Lemma 4.7, we have $C_1 \gamma C_j$. Also since $C_1\delta$ is a δ-class of L-type, the D_E-class $e\mathscr{D}_E$ is of L-type. Hence by [2] Lemma 2.7, we have $ex_j = e$ and so $x_1 \dots x_n \leqq ex_{i+2} \dots \dots x_n = e$. Thus always we have $x_1 \dots x_n \leqq e$. On the other hand $x_1 \dots x_n \in C_1 \dots \dots C_n \subseteq m_+(C_1) = [e, f]$, we have $e \leqq x_1 \dots x_n$. Hence $x_1 \dots x_n = e$ and so

$C_1 \ldots C_n = \{e\} = m_0(C_1) \subseteq C_1$, which contradicts that $C_1 \ldots C_n$ is not contained in a single archimedean class.

(iv) Since $C_1\delta \preccurlyeq C_j\delta$ for every $1 \leqq j \leqq n$, it follows from [2] Theorem 2.7 that $ey_2 \ldots y_k = e$ for every $y_2 \in C_2, \ldots, y_k \in C_k$ and so $e \in C_1 \ldots C_k$. Since $C_1 \ldots C_k$ is not contained in a single archimedean class, there exist $x_1 \in C_1, \ldots, x_k \in C_k$ such that $x_1 \ldots x_k \notin C_1$. But by (ii) $x_1 \ldots x_k = (x_1 \ldots x_{k-1})\, x_k \in C_1C_k$ and so C_1C_k contains an element which does not belong to C_1. On the other hand, since $C_1\delta \preccurlyeq C_k\delta$, we have $e = ey_k \in C_1 \cap C_1C_k$. Hence C_1C_k is not contained in a single archimedean class. Also by (iii) we have $C_1 < C_k$ and, since $C_1\delta \preccurlyeq C_k\delta$, we have $C_1\delta = C_1\delta \wedge C_k\delta$. Hence by [2] Lemma 6.7, we have $e < f < h$. Moreover since $C_1\delta = C_1\delta \wedge C_k\delta = (C_k * C_1)\,\delta$, it follows from [2] Theorem 3.3 that $h \mathscr{D}_E e$.

(v) For every natural number $1 \leqq j \leqq n$, we have $C_1\delta \preccurlyeq C_j\delta$ and, since $C_1\delta$ is of L-type, we have $C_1 * C_j = C_1$. Hence $C_1 * \ldots * C_k = C_1$. Now by Corollary 1.15, there exists a modified archimedean class $m(C_1 * \ldots * C_k) = m(C_1)$ such that $C_1 \ldots C_k \subseteq m(C_1)$. But since $C_1 \ldots C_k$ is not contained in a single archimedean class, we have either $m(C_1) = m_+(C_1)$ or $m(C_1) = m_-(C_1)$. By way of contradiction, we assume $m(C_1) = m_-(C_1)$. Let $z_1 \in C_1, \ldots, z_k \in C_k$. Then $z_1 \ldots z_k \in C_1 \ldots C_k \subseteq m(C_1) = m_-(C_1)$ and so $z_1 \ldots z_k \leqq e$. On the other hand, by (iii) we have $C_1 < C_2$ and so $e < z_2$. Also by [2] Theorem 2.7 we have $ez_3 = \ldots = ez_k = e$ and so $z_1z_2 \ldots \ldots z_k \geqq z_1e \ldots z_k = e \ldots z_k = e$. Hence $z_1 \ldots z_k = e$ and so $C_1 \ldots C_k = \{e\} \subseteq C_1$, which is a contradiction. Hence $C_1 \ldots C_k \subseteq m(C_1) = m_+(C_1)$.

(vi) Suppose that $x_1 \in C_1, \ldots, x_{k-1} \in C_{k-1}$, $y_k \in C_k$ such that $x_1 \ldots x_{k-1}y_k \notin C_1$. Let F and X be archimedean classes which contain f and $x_1 \ldots x_{k-1}y_k$, respectively. Then since $x_1 \ldots x_{k-1}y_k \in C_1 \ldots C_k \subseteq [e, f]$ and $x_1 \ldots x_{k-1}y_k \notin C_1$, we have $C_1 < X \leqq F$. We have $C_1\delta \preccurlyeq C_k\delta$. First suppose $C_1\delta = C_k\delta$. Then $C_k * C_1 = C_k$ and so C_k is a periodic archimedean class with idempotent h and in particular $y_kh = h$. Next suppose $C_1\delta \prec C_k\delta$. Then $C_k\delta \wedge (C_k * C_1)\,\delta = C_k\delta \wedge C_k\delta \wedge C_1\delta = C_1\delta \prec C_k\delta$ and $C_k * (C_k * C_1) = C_k * C_1$ and so by Lemma 1.6, $y_kh \in C_km_0(C_k * C_1) = m_0(C_k * C_1) = \{h\}$. Hence always we have $y_kh = h$. We have $e \mathscr{D}_E f$ and by (iv), $e \mathscr{D}_E h$ and since $C_1\delta$ is of L-type, we have $e \mathscr{L} f \mathscr{L} h$. Since $x_1 \ldots x_{k-1}y_k \leqq f < h$, we have $(x_1 \ldots x_{k-1}y_k)^2 \leqq (x_1 \ldots x_{k-1}y_k)\, h = x_1 \ldots x_{k-1}h \leqq fh = f$. Also since $(x_1 \ldots x_{k-1}y_k)^2 \in X$, we have $e < (x_1 \ldots x_{k-1}y_k)^2 \leqq x_1 \ldots x_{k-1}h$. Now $(C_k * C_1)\,\delta = C_1\delta$ and by [2] Theorem 2.7, we have $(x_1 \ldots x_{k-1}h)^2 = x_1 \ldots x_{k-1}(hx_1 \ldots \ldots x_{k-1})\, h = x_1 \ldots x_{k-1}h$ and so $x_1 \ldots x_{k-1}h$ is an idempotent of S. Further we have $(x_1 \ldots x_{k-1}h)\, e = (x_1 \ldots x_{k-1})\,(he) = x_1 \ldots x_{k-1}h$ and $e(x_1 \ldots x_{k-1}h) = (ex_1 \ldots x_{k-1})\, h = eh = e$ and so $x_1 \ldots x_{k-1}h \mathscr{D}_E e$. Since e and f are consecutive in $e\mathscr{D}_E$, we have $x_1 \ldots x_{k-1}h = f$.

If $C_1\delta = C_k\delta$, then since $C_k * C_1 = C_k$, we can put $x_k = h$ and then $x_1 \ldots x_{k-1}x_k = x_1 \ldots x_{k-1}h = f$. Next suppose $C_1\delta \prec C_k\delta$. By (iii), we have $C_1 < C_k$ and so $C_1 \leqq C_k * C_1 \leqq C_k$. But since $(C_k * C_1)\,\delta = C_k\delta \wedge C_1\delta = C_1\delta \prec C_k\delta$, we have $C_k * C_1 \neq C_k$ and so $C_k * C_1 < C_k$. We take $x_k \in C_k$ arbitrarily. Then $h < x_k$ and so $f = x_1 \ldots x_{k-1}h \leqq x_1 \ldots x_{k-1}x_k$. On the other hand, by (v) we have $x_1 \ldots$

$\ldots x_{k-1}x_k \in C_1 \ldots C_k \subseteq m_+(C_1) = [e, f]$ and so $x_1 \ldots x_{k-1}x_k \leqq f$. Hence $x_1 \ldots \ldots x_{k-1}x_k = f$.

(vii) We have $e \in C_1$ and for every natural number $1 \leqq j \leqq n$, we have $C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta$. Hence for every $z_2 \in C_2, \ldots, z_k \in C_k$ we have $e = ez_2 \ldots z_k \in C_1 \ldots C_k$. But since $C_1 \ldots C_k$ is not contained in a single archimedean class, there exist $x_1 \in C_1, \ldots, x_{k-1} \in C_{k-1}$, $y_k \in C_k$ such that $x_1 \ldots x_{k-1}y_k \notin C_1$. Hence by (vi) there exists $x_k \in C_k$ such that $x_1 \ldots x_{k-1}x_k = f$. Since $e \mathscr{D}_E f$, it follows from [2] Theorem 3.3 that $F\delta = C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_j\delta$ for every natural number $1 \leqq j \leqq n$. We take $x_{k+1} \in C_{k+1}, \ldots, x_n \in C_n$ arbitrarily. Then by [2] Theorem 2.7, we have $x_1 \ldots x_n = (x_1 \ldots x_k)\, x_{k+1} \ldots x_n = fx_{k+1} \ldots x_n = f$.

(2) can be proved similarly.

Lemma 2.10. *Let $C_1, \ldots, C_n \in \mathscr{C}$ such that $C_1\delta \wedge \ldots \wedge C_n\delta$ is a periodic δ-class of L-type. We denote by e the idempotent of the periodic archimedean class $C_1 * \ldots * C_n$. Also let h be the least natural number such that $C_1\delta \wedge \ldots \wedge C_h\delta = C_1\delta \wedge \ldots \wedge C_n\delta$.*

(1) *For the following three conditions* (i), (ii) *and* (iii), (i) *implies* (ii) *and* (ii) *implies* (iii).

(i) *$C_1 \ldots C_n$ is not contained in a single archimedean class and $C_1 \ldots C_n \subseteq m_+(C_1 * \ldots * C_n)$.*

(ii) *$C_1 * \ldots * C_h = C_h = C_h * \ldots * C_n$, $C_h \ldots C_n$ is not contained in a single archimedean class, $C_h \ldots C_h \subseteq m_+(C_h) = m_+(C_h * \ldots * C_n)$ and if $h > 1$, then $C_h < C_1 * \ldots * C_{h-1}$.*

(iii) *There exists an idempotent f of S such that $e < f$, $e \mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$ and also there exist $x_1 \in C_1, \ldots, x_n \in C_n$ such that $x_1 \ldots x_n = f$.*

(2) *For the following three conditions* (iv), (v) *and* (vi), (iv) *implies* (v) *and* (v) *implies* (vi).

(iv) *$C_1 \ldots C_n$ is not contained in a single archimedean class and $C_1 \ldots C_n \subseteq m_-(C_1 * \ldots * C_n)$.*

(v) *$C_1 * \ldots * C_h = C_h = C_h * \ldots * C_n$, $C_h \ldots C_n$ is not contained in a single archimedean class, $C_h \ldots C_n \subseteq m_-(C_h) = m_-(C_h * \ldots * C_n)$ and if $h > 1$, then $C_1 * \ldots * C_{h-1} < C_1$.*

(vi) *There exists an idempotent g of S such that $g < e$, $g \mathscr{D}_E e$ and g and e are consecutive in $e\mathscr{D}_E$ and also there exist $x_1 \in C_1, \ldots, x_n \in C_n$ such that $x_1 \ldots x_n = g$.*

Proof. By Lemma 1.4, $(C_1 * \ldots * C_n)\, \delta = C_1\delta \wedge \ldots \wedge C_n\delta$ and so $C_1 * \ldots * C_n$ is really a periodic archimedean class.

(1) First suppose (i) holds. If $h = 1$, then for every natural number $1 \leqq j \leqq n$, we have $C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_j\delta$ and so $C_1 * C_j = C_1$. Hence $C_1 = C_1 * \ldots * C_n$. The remaining conditions of (ii) are evident. Suppose $h > 1$. Then by the definition of h, we have $(C_1 * \ldots * C_{h-1})\, \delta \wedge C_h\delta = C_1\delta \wedge \ldots \wedge C_{h-1}\delta \wedge C_h\delta = C_1\delta \wedge \ldots \wedge C_n\delta \prec C_1\delta \wedge \ldots \wedge C_{h-1}\delta = (C_1 * \ldots * C_{h-1})\, \delta$. Also by

Corollary 1.15, there exists a modified archimedean class $m(C_1 * \dots * C_{h-1})$ of $C_1 * \dots * C_{h-1}$ such that $C_1 \dots C_{h-1} \subseteq m(C_1 * \dots * C_{h-1})$. Now by way of contradiction, we assume $C_1 * \dots * C_{h-1} * C_h \neq C_h$. Then by Lemma 1.5, we have $C_1 \dots \dots C_{h-1}C_h \subseteq m(C_1 * \dots * C_{h-1})\, C_h = m_0(C_1 * \dots * C_{h-1} * C_h)$. Also for every natural number $1 \leqq j \leqq n$, we have $(C_1 * \dots * C_h)\,\delta = C_1\delta \wedge \dots \wedge C_h\delta = C_1\delta \wedge \dots \wedge C_n\delta \preccurlyeq C_j\delta$ and so $(C_1 * \dots * C_h)\,\delta \wedge C_j\delta = (C_1 * \dots * C_h)\,\delta$. Hence by Lemma 1.7, we have $m_0(C_1 * \dots * C_h)\, C_j = m_0(C_1 * \dots * C_h)$ and so $C_1 \dots \dots C_h C_{h+1} \dots C_n \subseteq m_0(C_1 * \dots * C_h)\, C_{h+1} \dots C_n = m_0(C_1 * \dots * C_h) \subseteq C_1 * \dots \dots * C_h$, which contradicts that $C_1 \dots C_n$ is not contained in a single archimedean class. Hence we have $C_1 * \dots * C_h = C_h$. Also for every natural number $1 \leqq j \leqq n$, we have $C_h\delta = (C_1 * \dots * C_h)\,\delta = C_1\delta \wedge \dots \wedge C_h\delta = C_1\delta \wedge \dots \wedge C_n\delta \preccurlyeq C_j\delta$ and so $C_h * C_j = C_h$. Hence $C_h * \dots * C_n = C_h$. We have $C_h = C_1 * \dots * C_h = (C_1 * \dots \dots * C_{h-1}) * (C_h * \dots * C_n) = C_1 * \dots * C_n$ and so e is the idempotent of C_h. By way of contradiction, we assume $C_1 * \dots * C_{h-1} \leqq C_h$. Let $x_1 \in C_1, \dots, x_n \in C_n$. Then by Lemma 1.6, we have $x_1 \dots x_{h-1}x_h \in m(C_1 * \dots * C_{h-1})\, C_h \subseteq (C_h)_+$ and so $x_1 \dots x_{h-1}x_h \leqq e$. Hence by [2] Theorem 2.7, we have $x_1 \dots x_n = x_1 \dots x_h \dots x_n \leqq \leqq e \dots x_n = e$. On the other hand, since $C_1 \dots C_n \subseteq m_+(C_1 * \dots * C_n)$, there exists an idempotent f of S such that $e < f$, $e\,\mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$ and also $C_1 \dots C_n \subseteq [e, f]$. Hence $x_1 \dots x_n \in C_1 \dots C_n \subseteq [e, f]$ and so $e \leqq x_1 \dots x_n$. Hence $x_1 \dots x_n = e$ and so $C_1 \dots C_n = \{e\} \subseteq C_h$. which contradicts that $C_1 \dots C_n$ is not contained in a single archimedean class. Hence we have $C_h < C_1 * \dots * C_{h-1}$. By way of contradiction we assume $C_h \dots C_n$ is contained in a single archimedean class. Then by Lemma 2.6, we have $C_h \dots C_n \subseteq C_h * \dots * C_n = C_h$. Hence by Lemma 1.6, $C_1 \dots C_n = (C_1 \dots C_{h-1})\,(C_h \dots C_n) \subseteq m(C_1 * \dots * C_{h-1})\, C_h \subseteq (C_h)_- \subseteq C_h$, which contradicts that $C_1 \dots C_n$ is not contained in a single archimedean class. Hence $C_h \dots C_n$ is not contained in a single archimedean class. Finally, by way of contradiction, we assume that $C_h \dots C_n$ is not contained in $m_+(C_h * \dots * C_n)$. Then, since $C_h \dots C_n$ is not contained in a single archimedean class, we must have $C_h \dots C_n \subseteq \subseteq m_-(C_h * \dots * C_n) = m_-(C_h)$. Hence by Lemma 1.6 again, we have $C_1 \dots C_n \subseteq \subseteq m(C_1 * \dots * C_{h-1})\, m_-(C_h) = m_0(C_h) \subseteq C_h$, which contradicts that $C_1 \dots C_n$ is not contained in a single archimedean class. Hence $C_h \dots C_n \subseteq m_+(C_h * \dots * C_n) = = m_+(C_h)$.

Next suppose (ii) holds. Since $C_h * \dots * C_n = C_h = C_1 * \dots * C_h = C_1 * \dots \dots * C_h * \dots * C_n$, e is the idempotent of C_h. Since $C_h \dots C_n \subseteq m_+(C_h) = = m_+(C_h * \dots * C_n)$, there exists an idempotent f of S such that $e < f$, $e\,\mathscr{D}_E f$, e and f are consecutive in $e\mathscr{D}_E$ and $C_h \dots C_n \subseteq [e, f]$. If $h = 1$, then $C_1\delta = C_1\delta \wedge \dots \dots \wedge C_n\delta$, $C_1 \dots C_n$ is not contained in a single archimedean class and $C_1 \dots C_n \subseteq \subseteq m_+(C_1)$. Hence by Lemma 2.9, there exist $y_1 \in C_1, \dots, y_n \in C_n$ such that $y_1 \dots y_n = = f$. Next suppose $h > 1$. Then $C_h\delta = (C_h * \dots * C_n)\,\delta = C_h\delta \wedge \dots \wedge C_n\delta$ and so by Lemma 2.9, there exist $y_h \in C_h, \dots, y_n \in C_n$ such that $y_h \dots y_n = f$. Now by the definition of h, we have $C_h\delta = C_1\delta \wedge \dots \wedge C_h\delta = C_1\delta \wedge \dots \wedge C_n\delta \prec C_1\delta \wedge \dots \dots \wedge C_{h-1}\delta$. Since $C_h < C_1 * \dots * C_{h-1}$, we have

$$C_h = (C_1 * \ldots * C_{h-1}) * C_h =$$
$$= \max\{Z \in \mathscr{C};\ C_h \leqq Z \leqq C_1 * \ldots * C_{h-1} \text{ and } Z \in (C_1 * \ldots * C_{h-1})\,\delta \wedge C_h\delta\}$$
$$= \max\{Z \in \mathscr{C};\ C_h \leqq Z \leqq C_1 * \ldots * C_{h-1} \text{ and } Z \in C_h\delta\}$$

and so there exists no $Z \in \mathscr{C}$ such that $C_h < Z \leqq C_1 * \ldots * C_{h-1}$ and $Z \in C_h\delta$. We denote by F the archimedean class containing f. Then since $e < f$, we have $C_h < F$ and, since $e\,\mathscr{D}_E f$, it follows from [2] Theorem 3.3 that $C_h\,\delta\,F$. Hence we have $C_1 * \ldots * C_{h-1} < F$. Further since e and f are consecutive in $e\mathscr{D}_E$, there exists no $Z \in \mathscr{C}$ such that $C_h < Z < F$ and $Z \in C_h\delta$. Also we have $(C_1 * \ldots * C_{h-1})\,\delta \wedge \wedge\, F\delta = (C_1 * \ldots * C_{h-1})\,\delta \wedge C_h\delta = C_h\delta$. Hence

$$(C_1 * \ldots * C_{h-1}) * F$$
$$= \min\{Z \in \mathscr{C};\ C_1 * \ldots * C_{h-1} \leqq Z \leqq F \text{ and } Z \in (C_1 * \ldots * C_{h-1})\,\delta \wedge F\delta\}$$
$$= \min\{Z \in \mathscr{C};\ C_1 * \ldots * C_{h-1} \leqq Z \leqq F \text{ and } Z \in C_h\delta\} = F\,.$$

We take $y_1 \in C_1, \ldots, y_{h-1} \in C_{h-1}$ arbitrarily. Then, by Corollary 1.15, there exists a modified archimedean class $m(C_1 * \ldots * C_{h-1})$ of $C_1 * \ldots * C_{h-1}$ such that $C_1 \ldots C_{h-1} \subseteq m(C_1 * \ldots * C_{h-1})$. Hence by Lemma 1.6, $y_1 \ldots y_n = y_1 \ldots y_{h-1} f \in \in C_1 \ldots C_{h-1}\, m_0(F) \subseteq m(C_1 * \ldots * C_{h-1})\, m_0(F) = m_0(F) = \{f\}$ and so $y_1 \ldots y_n = = f$.

(2) can be proved similarly.

Theorem 2.11. *Let $C_1, \ldots, C_n \in \mathscr{C}$ such that $C_1\delta \wedge \ldots \wedge C_n\delta$ is a periodic δ-class of L-type. We denote by e_1 the idempotent of the periodic archimedean class $C_1 * \ldots * C_n$. Also let h be the least natural number such that $C_1\delta \wedge \ldots \wedge C_h\delta = = C_1\delta \wedge \ldots \wedge C_n\delta$.*

(1) *$C_1 \ldots C_n$ is not contained in a single archimedean class and $C_1 \ldots C_n \subseteq \subseteq m_+(C_1 * \ldots * C_n)$ if and only if there exists an idempotent f_1 of S such that $e_1 < f_1$, $e_1\,\mathscr{D}_E f_1$ and e_1 and f_1 are consecutive in $e_1\mathscr{D}_E$ and satisfies either*

(i) *$h > 1$, $C_1 * \ldots * C_h = C_h = C_h * \ldots * C_n$, $C_h < C_1 * \ldots * C_{h-1}$, $C_h \ldots C_n$ is not contained in a single archimedean class and $C_h \ldots C_n \subseteq m_+(C_h) = m_+(C_h * \ldots \ldots * C_n)$, or*

(ii) *$h = 1$, $C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta$, $C_1 < C_2$ and satisfies either*

(a) *C_1C_2 is not contained in a single archimedean class and $C_1C_2 \subseteq \subseteq m_+(C_1 * C_2) = m_+(C_1) = m_+(C_1 * \ldots * C_n)$, or*

(b) *C_1C_2 is contained in a single archimedean class and $C_2\delta \wedge \ldots \wedge C_n\delta = = C_1\delta \wedge \ldots \wedge C_n\delta$. Also, denoting by e_2 the idempotent of the periodic archimedean class $C_2 * C_1$, there exists an idempotent f_2 of S such that $e_2 < f_2$, $e_2\,\mathscr{D}_E f_2$, e_2 and f_2 are consecutive in $e_2\mathscr{D}_E$ and $xf_2 = f_1$ for some $x \in (C_1)_-$ and satisfies either*

(b_1) *$C_2 * \ldots * C_n \neq C_2 * C_1$ and f_2 is the idempotent of the periodic archimedean class $C_2 * \ldots * C_n$, or*

(b_2) *$C_2 * \ldots * C_n = C_2 * C_1$, $C_2 \ldots C_n$ is not contained in a single archimedean. class and $C_2 \ldots C_n \subseteq m_+(C_2 * \ldots * C_n) = m_+(C_2 * C_1)$.*

(2) *$C_1 \ldots C_n$ is not contained in a single archimedean class and $C_1 \ldots C_n \subseteq \subseteq m_-(C_1 * \ldots * C_n)$ if and only if there exists an idempotent g_1 of S such that $g_1 < e_1$, $g_1 \mathscr{D}_E e_1$ and g_1 and e_1 are consecutive in $e_1\mathscr{D}_E$ and satisfies either*

(i) *$h > 1$, $C_1 * \ldots * C_h = C_h = C_h * \ldots * C_n$, $C_1 * \ldots * C_{h-1} < C_h$, $C_h \ldots C_n$ is not contained in a single archimedean class and $C_h \ldots C_n \subseteq m_-(C_h) = m_-(C_h * \ldots \ldots * C_n)$, or*

(ii) *$h = 1$, $C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta$, $C_2 < C_1$ and satisfies either*

(a) *C_1C_2 is not contained in a single archimedean class and $C_1C_2 \subseteq \subseteq m_-(C_1 * C_2) = m_-(C_1) = m_-(C_h * \ldots * C_n)$, or*

(b) *C_1C_2 is contained in a single archimedean class and $C_2\delta \wedge \ldots \wedge C_n\delta = = C_1\delta \wedge \ldots \wedge C_n\delta$. Also, denoting by e_2 the idempotent of the periodic archimedean class $C_2 * C_1$, there exists an idempotent g_2 of S such that $g_2 < e_2$, $g_2 \mathscr{D}_E e_2$, g_2 and e_2 are consecutive in $e_2\mathscr{D}_E$ and $xg_2 = g_1$ for some $x \in (C_1)_+$ and satisfies either*

(b_1) *$C_2 * \ldots * C_n \neq C_2 * C_1$ and g_2 is the idempotent of the periodic archimedean class $C_2 * \ldots * C_n$, or*

(b_2) *$C_2 * \ldots * C_n = C_2 * C_1$, $C_2 \ldots C_n$ is not contained in a single archimedean class and $C_2 \ldots C_n \subseteq m_-(C_2 * \ldots * C_n) = m_-(C_2 * C_1)$.*

Proof. By Lemma 1.4, $(C_1 * \ldots * C_n)\,\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ and so $C_1 * \ldots * C_n$ is really a periodic archimedean class.

(1) First suppose that $C_1 \ldots C_n$ is not contained in a single archimedean class and $C_1 \ldots C_n \subseteq m_+(C_1 * \ldots * C_n)$. Then there exists an idempotent f_1 of S such that $e_1 < f_1$, $e_1 \mathscr{D}_E f_1$, e_1 and f_1 are consecutive in $e_1\mathscr{D}_E$ and $m_+(C_1 * \ldots * C_n) = = [e_1, f_1]$.

(i) Suppose $h > 1$. Then by Lemma 2.10, $C_1 * \ldots * C_h = C_h = C_h * \ldots * C_n$, $C_h < C_1 * \ldots * C_{h-1}$, $C_h \ldots C_n$ is not contained in a single archimedean class and $C_h \ldots C_n \subseteq m_+(C_h) = m_+(C_h * \ldots * C_n)$.

(ii) Suppose $h = 1$. Then we have $C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta$. Also for every natural number $1 \leqq j \leqq n$, we have $C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_j\delta$ and so $C_1 * C_j = C_1$. Hence $C_1 * \ldots * C_n = C_1$ and e_1 is the idempotent of C_1. Hence by Lemma 2.9, we have $C_1 < C_2$.

(a) Suppose C_1C_2 is not contained in a single archimedean class. Then denoting by k the least natural number such that $C_1 \ldots C_k$ is not contained in a single archimedean class, we have $k = 2$. Hence by Lemma 2.9, we have $C_1C_2 \subseteq m_+(C_1) = = m_+(C_1 * C_2) = m_+(C_1 * \ldots * C_n)$.

(b) Suppose C_1C_2 is contained in a single archimedean class. Since $(C_2 * C_1)\,\delta = = C_2\delta \wedge C_1\delta = C_1\delta$, $C_2 * C_1$ is really a periodic archimedean class. Now by way of contradiction, we assume $C_1\delta \wedge \ldots \wedge C_n\delta \neq C_2\delta \wedge \ldots \wedge C_n\delta$. Then $C_1\delta = = C_1\delta \wedge \ldots \wedge C_n\delta \prec C_2\delta \wedge \ldots \wedge C_n\delta$. Let $x_1 \in C_1$, $x_2 \in C_2, \ldots, x_n \in C_n$. We denote by X the archimedean class which contains $x_2 \ldots x_n$. By Corollary 1.15, there exists a modified archimedean class $m(C_2 * \ldots * C_n)$ of $C_2 * \ldots * C_n$ such that

$C_2 \ldots C_n \subseteq m(C_2 * \ldots * C_n)$ and so $x_2 \ldots x_n \in m(C_2 * \ldots * C_n)$. Hence by Lemma 1.2, $C_2\delta \wedge \ldots \wedge C_n\delta = (C_2 * \ldots * C_n)\,\delta \preccurlyeq X\delta$ and so $C_1\delta \prec C_2\delta \wedge \ldots \wedge C_n\delta = C_2\delta \wedge (C_2\delta \wedge \ldots \wedge C_n\delta) \preccurlyeq C_2\delta \wedge X\delta$. Hence by [2] Lemma 5.6, there exists no $Z \in \mathscr{C}$ such that $Z \in C_1\delta$ and Z lies between C_2 and X. In particular, C_1 does not lie between C_2 and X and, since $C_1 < C_2$, we have $C_1 < X$. Also since $C_1\delta \prec C_2\delta$ and $C_1\delta \prec X\delta$, we have

$$\begin{aligned} X * C_1 &= \max\{Z \in \mathscr{C};\ C_1 \leqq Z \leqq X \ \text{ and } \ Z \in C_1\delta \wedge X\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_1 \leqq Z \leqq X \ \text{ and } \ Z \in C_1\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_1 \leqq Z \leqq C_2 \ \text{ and } \ Z \in C_1\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_1 \leqq Z \leqq C_2 \ \text{ and } \ Z \in C_1\delta \wedge C_2\delta\} = C_2 * C_1\,. \end{aligned}$$

Since e_2 is the idempotent of $C_2 * C_1$, e_2 is the idempotent of $X * C_1$. Since C_1C_2 is contained in a single archimedean class, it follows from Lemma 1.10 that $xe_2 = e_1$ for every $x \in (C_1)_-$ and so by Lemma 1.10 again, $C_1X \subseteq (C_1)_- \subseteq C_1$. Hence $x_1x_2 \ldots x_n \in C_1X \subseteq C_1$ and so $C_1C_2 \ldots C_n \subseteq C_1$, which contradicts that $C_1 \ldots C_n$ is not contained in a single archimedean class. Hence we have $C_2\delta \wedge \ldots \wedge C_n\delta = C_1\delta \wedge \ldots \wedge C_n\delta = C_1\delta$.

(b_1) Suppose $C_2 * \ldots * C_n \neq C_2 * C_1$. By way of contradiction, we assume $C_1\delta = C_2\delta$. Then for every $1 \leqq j \leqq n$, we have $C_2\delta = C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_j\delta$ and so $C_2 * C_1 = C_2 = C_2 * \ldots * C_n$ which is a contradiction. Hence $C_1\delta \prec C_2\delta$. Since $C_1 < C_2$, we have $C_1 \leqq C_2 * C_1 \leqq C_2$ but, since $(C_2 * C_1)\,\delta = C_2\delta \wedge C_1\delta = C_1\delta \prec C_2\delta$, we have $C_2 * C_1 < C_2$. Also

$$\begin{aligned} C_2 * C_1 &= \max\{Z \in \mathscr{C};\ C_1 \leqq Z \leqq C_2 \ \text{and} \ Z \in C_2\delta \wedge C_1\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_1 \leqq Z \leqq C_2 \ \text{and} \ Z \in C_1\delta\} \end{aligned}$$

and so there exists no $Z \in \mathscr{C}$ such that $C_2 * C_1 < Z \leqq C_2$ and $Z \in C_1\delta$. By way of contradiction we assume $C_2 * \ldots * C_n \leqq C_2$. Then, since $(C_2 * \ldots * C_n)\,\delta = C_2\delta \wedge \ldots \wedge C_n\delta = C_1\delta \prec C_2\delta$, we have $C_2 * \ldots * C_n < C_2$. Also

$$\begin{aligned} C_2 * \ldots * C_n &= C_2 * (C_2 * \ldots * C_n) \\ &= \max\{Z \in \mathscr{C};\ C_2 * \ldots * C_n \leqq Z \leqq C_2 \ \text{and} \ Z \in (C_2 * \ldots * C_n)\,\delta \wedge C_2\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_2 * \ldots * C_n \leqq Z \leqq C_2 \ \text{and} \ Z \in C_1\delta\} \end{aligned}$$

and so there exists no $Z \in \mathscr{C}$ such that $C_2 * \ldots * C_n < Z \leqq C_2$ and $Z \in C_1\delta$. Hence

$$C_2 * C_1 = \max\{Z \in \mathscr{C};\ Z \leqq C_2 \ \text{and} \ Z \in C_1\delta\} = C_2 * \ldots * C_n\,,$$

which is a contradiction. Hence $C_2 * C_1 < C_2 < C_2 * \ldots * C_n$. Also since $(C_2 * \ldots * C_n)\,\delta = C_1\delta$, $C_2 * \ldots * C_n$ is a periodic archimedean class. We denote by f_2 the idempotent of $C_2 * \ldots * C_n$. Then since $C_2 * C_1 < C_2 * \ldots * C_n$, we have $e_2 < f_2$. Also since $(C_2 * C_1)\,\delta = C_1\delta = (C_2 * \ldots * C_n)\,\delta$, it follows from [2] Theorem 3.3 that $e_2\,\mathscr{D}_E\,f_2$.

$$\begin{aligned} C_2 * \ldots * C_n &= C_2 * (C_2 * \ldots * C_n) \\ &= \min\{Z \in \mathscr{C};\ C_2 \leqq Z \leqq C_2 * \ldots * C_n \ \text{and} \ Z \in C_2\delta \wedge (C_2 * \ldots * C_n)\,\delta\} \\ &= \min\{Z \in \mathscr{C};\ C_2 \leqq Z \leqq C_2 * \ldots * C_n \ \text{and} \ Z \in C_1\delta\} \end{aligned}$$

and so there exists no $Z \in \mathscr{C}$ such that $C_2 \leqq Z < C_2 * \ldots * C_n$ and $Z \in C_1\delta$. Since there exists no $Z \in C$ such that $C_2 * C_1 < Z \leqq C_2$ and $Z \in C_1\delta$, there exists no $Z \in C$ such that $C_2 * C_1 < Z < C_2 * \ldots * C_n$ and $Z \in C_1\delta = (C_2 * C_1)\,\delta$ and so by [2] Theorem 3.3, e_2 and f_2 are consecutive in $e_2\mathscr{D}_E$. Moreover by Lemma 2.9, there exist $x_1 \in C_1$, $x_2 \in C_2, \ldots, x_n \in C_n$ such that $x_1x_2 \ldots x_n = f_1$. But since $C_2 < C_2 * \ldots * C_n$, we have $x_2 < f_2$. Also since $(C_2 * \ldots * C_n)\,\delta = C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_j\delta$ for every $1 \leqq j \leqq n$, it follows from [2] Theorem 2.7 that $f_2x_3 \ldots x_n = f_2$. Hence $f_1 = x_1x_2 \ldots x_n \leqq x_1f_2 \ldots x_n = x_1f_2$. On the other hand by Lemma 2.5, $\{f_2\} = m_0(C_2 * \ldots * C_n) \subseteq C_2 \ldots C_n$ and so there exist $y_2 \in C_2, \ldots, y_n \in C_n$ such that $f_2 = y_2 \ldots y_n$. Hence $x_1f_2 = x_1y_2 \ldots y_n \in C_1C_2 \ldots C_n \subseteq m_+(C_1 * C_2 * \ldots * C_n) = [e_1, f_1]$ and so we have $x_1f_2 \leqq f_1$. Hence we have $x_1f_2 = f_1$. Further by [2] Theorem 2.7, $e_1f_2 = e_1y_2 \ldots y_n = e_1 < f_1 = x_1f_2$ and so we have $e_1 < x_1$. Hence $x_1 \in (C_1)_-$.

(b$_2$) Suppose $C_2 * \ldots * C_n = C_2 * C_1$. Since e_2 is the idempotent of $C_2 * C_1$, $C_1 < C_2$ and C_1C_2 is contained in a single archimedean class, it follows from Lemma 1.10 that $xe_2 = e_1$ for every $x \in (C_1)_-$. Also we have $C_1 \leqq C_2 * C_1 = C_2 * \ldots * C_n$. By way of contradiction, we assume $C_1 = C_2 * \ldots * C_n$. Then by Corollary 1.15, there exists a modified archimedean class $m(C_2 * \ldots * C_n)$ of $C_2 * \ldots * C_n$ such that $C_2 \ldots C_n \subseteq m(C_2 * \ldots * C_n)$. Hence by Lemma 1.8, we see that $C_1C_2 \ldots C_n \subseteq C_1m(C_2 * \ldots * C_n) = C_1m(C_1) \subseteq C_1$, which contradicts that $C_1 \ldots C_n$ is not contained in a single archimedean class. Hence we have $C_1 < C_2 * \ldots * C_n = C_2 * C_1$. We have $(C_2 * \ldots * C_n)\,\delta = C_2\delta \wedge \ldots \wedge C_n\delta = C_1\delta \wedge \ldots \wedge C_n\delta \preccurlyeq C_1\delta$ and so $(C_2 * \ldots * C_n) * C_1 = C_2 * \ldots * C_n$. Since e_2 is also the idempotent of $C_2 * \ldots * C_n$, it follows from Lemma 1.10 that $C_1(C_2 * \ldots * C_n)$ is contained in a single archimedean class. By way of contradiction we assume that $C_2 \ldots C_n$ is contained in a single archimedean class. Then by Theorem 2.6, we have $C_2 \ldots C_n \subseteq C_2 * \ldots * C_n$ and so $C_1C_2 \ldots C_n \subseteq C_1(C_2 * \ldots * C_n)$. But since $C_1(C_2 * \ldots * C_n)$ is contained in a single archimedean class, this is a contradiction. Hence $C_2 \ldots C_n$ is not contained in a single archimedean class. But by Corollary 1.15, there exists a modified archimedean class $m(C_2 * \ldots * C_n)$ of $C_2 * \ldots * C_n$ such that $C_2 \ldots C_n \subseteq m(C_2 * \ldots * C_n)$. Since $C_2 \ldots C_n$ is not contained in a single archimedean class, we must have either $m(C_2 * \ldots * C_n) = m_+(C_2 * \ldots * C_n)$ or $m(C_2 * \ldots * C_n) = m_-(C_2 * \ldots * C_n)$. Now by Lemma 2.9, there exist $x_1 \in C_1$, $x_2 \in C_2, \ldots, x_n \in C_n$ such that $x_1x_2 \ldots x_n = f_1$. But by [2] Theorem 2.7, we have $e_1x_2 \ldots x_n = e_1 < f_1 = x_1x_2 \ldots x_n$ and so $e_1 < x_1$, whence $x_1 \in (C_1)_-$. Hence we have $e_1 = x_1e_2 < f_1 = x_1x_2 \ldots x_n$ and so $e_2 < x_2 \ldots x_n$. Hence $C_2 \ldots C_n$ contains $x_2 \ldots x_n$ such that $e_2 < x_2 \ldots x_n$ and so $m(C_2 * \ldots * C_n) \neq m_-(C_2 * \ldots * C_n)$. Hence we have $C_2 \ldots C_n \subseteq m_+(C_2 * \ldots * C_n) = m_+(C_2 * C_1)$. In particular, there exists an idempotent f_2 of S such that $e_2 < f_2$, $e_2\, \mathscr{D}_E\, f_2$ and e_2 and f_2 are consecutive in $e_2\mathscr{D}_E$ and $m_+(C_2 * \ldots * C_n) = [e_2, f_2]$. Again we consider $x_1 \in C_1$, $x_2 \in C_2, \ldots, x_n \in C_n$ such that $x_1x_2 \ldots x_n = f_1$. We have shown that $x_1 \in (C_1)_-$. We have $x_2 \ldots x_n \in C_2 \ldots C_n \subseteq m_+(C_2 * \ldots * C_n) = [e_2, f_2]$ and so $x_2 \ldots x_n \leqq f_2$. Hence $f_1 = x_1x_2 \ldots x_n \leqq$

$\leqq x_1 f_2$. On the other hand, it follows from Lemma 2.5 that $\{f_2\} = m_0(C_2 * \dots \dots * C_n) \subseteq C_2 \dots C_n$ and so there exist $y_2 \in C_2, \dots, y_n \in C_n$ such that $f_2 = y_2 \dots y_n$. Hence $x_1 f_2 = x_1 y_2 \dots y_n \in C_1 \dots C_n \subseteq m_+(C_1 * \dots * C_n) = [e_1, f_1]$ and so $x_1 f_2 \leqq \leqq f_1$. Hence $x_1 f_2 = f_1$.

Conversely suppose that there exists an idempotent f_1 of S such that $e_1 < f_1$, $e_1 \mathscr{D}_E f_1$ and e_1 and f_1 are consecutive in $e_1 \mathscr{D}_E$ and satisfies either the condition (i) or the condition (ii).

Case: the condition (i) is satisfied. By Lemma 2.5, we have $\{e_1\} = m_0(C_1 * \dots \dots * C_n) \subseteq C_1 \dots C_n$ and so $e_1 \in C_1 \dots C_n$. Also by Lemma 2.10, there exist $y_1 \in \in C_1, \dots, y_n \in C_n$ such that $f_1 = y_1 \dots y_n \in C_1 \dots C_n$. Hence $C_1 \dots C_n$ is not contained in a single archimedean class. Also by Corollary 1.15, there exists a modified archimedean class $m(C_1 * \dots * C_n)$ of $C_1 * \dots * C_n$ such that $C_1 \dots C_n \subseteq m(C_1 * \dots * C_n)$. Since $e_1, f_1 \in C_1 \dots C_n \subseteq m(C_1 * \dots * C_n)$, e_1 is the idempotent of $C_1 * \dots * C_n$ and $e_1 < f_1$, we must have $m(C_1 * \dots * C_n) = m_+(C_1 * \dots * C_n)$ and so $C_1 \dots C_n \subseteq \subseteq m_+(C_1 * \dots * C_n)$.

Case: the conditions (ii) and (a) are satisfied. Since $C_1\delta = C_1\delta \wedge \dots \wedge C_n\delta \preccurlyeq \preccurlyeq C_j\delta$ for every natural number $1 \leqq j \leqq n$, we have $C_1 * \dots * C_n = C_1$ and so e_1 is the idempotent of C_1. Also $C_1 C_2$ is not contained in a single archimedean class and $C_1 C_2 \subseteq m_+(C_1) = [e_1, f_1]$. Hence by Lemma 2.9, there exist $y_1 \in C_1$ and $y_2 \in C_2$ such that $y_1 y_2 = f_1$. We denote by F the archimedean class containing f_1. Then by [2] Theorem 3.3, we have $F \,\delta\, C_1$ and so $F\delta = C_1\delta = C_1\delta \wedge \dots \wedge C_n\delta \preccurlyeq \preccurlyeq C_j\delta$ for every natural number $1 \leqq j \leqq n$. We take $x_3 \in C_3, \dots, x_n \in C_n$ arbitrarily. Then by [2] Theorem 2.7, we have $e_1 = e_1 y_2 x_3 \dots x_n \in C_1 \dots C_n$ and $f_1 = f_1 x_3 \dots \dots x_n = y_1 y_2 x_3 \dots x_n \in C_1 \dots C_n$. Hence $C_1 \dots C_n$ is not contained in a single archimedean class and $C_1 \dots C_n \subseteq m_+(C_1) = m_+(C_1 * \dots * C_n)$.

Case: the conditions (ii), (b) and (b_1) are satisfied. We have $C_1 = C_1 * \dots * C_n$ and e_1 is the idempotent of C_1. Also there exists $x_1 \in C_1$ such that $x_1 f_2 = f_1$. Further since f_2 is the idempotent of $C_2 * \dots * C_n$, it follows from Theorem 2.5 that $\{f_2\} = = m_0(C_2 * \dots * C_n) \subseteq C_2 \dots C_n$ and so there exist $x_2 \in C_2, \dots, x_n \in C_n$ such that $f_2 = x_2 \dots x_n$. Hence $f_1 = x_1 f_2 = x_1 x_2 \dots x_n \in C_1 \dots C_n$. On the other hand, since $C_1\delta = C_1\delta \wedge \dots \wedge C_n\delta \preccurlyeq C_j\delta$ for every natural number $1 \leqq j \leqq n$, it follows from [2] Theorem 2.7 that $e_1 = e_1 x_2 \dots x_n \in C_1 \dots C_n$. Hence $C_1 \dots C_n$ is not contained in a single archimedean class and $C_1 \dots C_n \subseteq m_+(C_1) = m_+(C_1 * \dots * C_n)$.

Case: the conditions (ii), (b) and (b_2) are satisfied. We have $C_1 = C_1 * \dots * C_n$ and e_1 is the idempotent of C_1. Since $C_2 * \dots * C_n = C_2 * C_1$, e_2 is the idempotent of $C_2 * \dots * C_n$. Since $C_2 \dots C_n$ is not contained in a single archimedean class and $C_2 \dots C_n \subseteq m_+(C_2 * \dots * C_n) = [e_2, f_2]$, it follows from Lemma 2.10 that there exist $x_2 \in C_2, \dots, x_n \in C_n$ such that $x_2 \dots x_n = f_2$. Also by assumption, there exists $x_1 \in C_1$ such that $x_1 f_2 = f_1$. Hence $f_1 = x_1 f_2 = x_1 x_2 \dots x_n \in C_1 \dots C_n$. Also we have $e_1 = e_1 x_2 \dots x_n \in C_1 \dots C_n$. Hence $C_1 \dots C_n$ is not contained in a single archimedean class and $C_1 \dots C_n \subseteq m_+(C_1) = m_+(C_1 * \dots * C_n)$.

(2) can be proved similarly.

Here we give some examples which show that the cases given in (1) of Theorem 2.11 really exist.

Example 1. Let S_1 be the ordered semigroup with the multiplication given by the following table and with the order $b < x < a < u < c$.

	b	x	a	u	c
b	b	b	b	b	b
x	b	b	b	b	u
a	b	x	a	u	u
u	u	u	u	u	u
c	c	c	c	c	c

In S_1, we put $C_1 = \{a\}$, $C_2 = \{b, x\}$ and $C_3 = \{c\}$. Then we can show that the condition (i) is satisfied.

Example 2. Let S_2 be the ordered semigroup with the multiplication given by the following table and with the order $a < x < u < b$.

	a	x	u	b
a	a	a	a	a
x	a	a	a	u
u	u	u	u	u
b	b	b	b	b

In S_2, we put $C_1 = \{a, x\}$ and $C_2 = \{b\}$. Then we can show that the conditions (ii) and (a) are satisfied.

Example 3. Let S_3 be the ordered semigroup with the multiplication given by the following table and with the order $a < x < u < b < c$.

	a	x	u	b	c
a	a	a	a	a	a
x	a	a	a	x	u
u	u	u	u	u	u
b	u	u	u	b	c
c	c	c	c	c	c

In S_3, we put $C_1 = \{a, x\}$, $C_2 = \{b\}$ and $C_3 = \{c\}$. Then we can show that the conditions (ii), (b) and (b_1) are satisfied.

Example 4. Let S_4 be the ordered semigroup with the multiplication given by the following table and with the order $a < x < y < b < z < u < c$.

	a	x	y	b	z	u	c
a	a	a	a	a	a	a	a
x	a	a	a	a	a	a	b
y	a	a	a	a	x	b	b
b	b	b	b	b	b	b	b
z	b	b	b	b	b	b	u
u	u	u	u	u	u	u	u
c	c	c	c	c	c	c	c

In S_4, we put $C_1 = \{a, x, y\}$, $C_2 = \{b, z\}$ and $C_3 = \{c\}$. Then we can show that the conditions (ii), (b) and (b_2) are satisfied.

Lemma 2.12. (1) *Let $B, C \in \mathscr{C}$ such that $C\delta$ is a periodic δ-class of L-type, $C\delta \prec B\delta$ and $C < B$. Let e be the idempotent of the periodic archimedean class $B * C$ and let f be an idempotent of S such that $e < f$, $e \mathscr{D}_E f$ and e and f are consecutive in $e\mathscr{D}_E$. Then we have $xe = e$ and $xf = f$ for every $x \in B$.*

(2) *Let $B, C \in \mathscr{C}$ such that $C\delta$ is a periodic δ-class of L-type, $C\delta \prec B\delta$ and $B < C$. Let e be the idempotent of the periodic archimedean class $B * C$ and let g be an idempotent of S such that $g < e$, $g \mathscr{D}_E e$ and g and e are consecutive in $e\mathscr{D}_E$ Then we have $xg = g$ and $xe = e$ for every $x \in B$.*

Proof. (1) Since $(B * C)\,\delta = B\delta \wedge C\delta = C\delta$, $B * C$ is really a periodic archimedean class. Since $C < B$, we have $C \leqq B * C \leqq B$, but since $(B * C)\,\delta = C\delta \prec \prec B\delta$, we have $B * C < B$. Now

$$B * C = \max\{Z \in \mathscr{C};\ C \leqq Z \leqq B \text{ and } Z \in B\delta \wedge C\delta\}$$
$$= \max\{Z \in \mathscr{C};\ C \leqq Z \leqq B \text{ and } Z \in C\delta\}$$

and so there exists no $Z \in \mathscr{C}$ such that $B * C < Z \leqq B$ and $Z \in C\delta$. We denote by F the archimedean class containing f. Then, since $e \mathscr{D}_E f$, it follows from [2] Theorem 3.3 that $F\delta = C\delta = (B * C)\,\delta$. Also, since $e < f$, we have $B * C < F$. Hence we have $B < F$. Now let $x \in B$. Then $e < x < f$ and so $e = e^2 \leqq xe \leqq x^2 \leqq xf \leqq f^2 = f$. But since $C\delta$ is of L-type, $e\mathscr{D}_E = f\mathscr{D}_E$ is also of L-type and so by [2] Theorem 2.7, we have $ex = e$ and $fx = f$. Hence $(xe)^2 = xexe = xe$ and $(xf)^2 = xfxf = xf$ and so xe and xf are idempotents of S. Also $(xe)\,e = xe$, $e(xe) = (ex)\,e = e$, $(xf)\,f = xf$ and $f(xf) = (fx)\,f = f$. Hence $xe \mathscr{D}_E e \mathscr{D}_E f \mathscr{D}_E xf$. But, since $e \leqq xe \leqq x^2 \leqq xf \leqq f$, $x^2 \in B$ and e and f are consecutive in $e\mathscr{D}_E$, we have $xe = e$ and $xf = f$.

(2) can be proved similarly.

Theorem 2.13. (1) *Let $C_1, \ldots, C_n \in \mathscr{C}$ such that $C_1\delta \wedge \ldots \wedge C_n\delta$ is a periodic δ-class of L-type. Then $C_1 \ldots C_n$ is not contained in a single archimedean class and $C_1 \ldots C_n \subseteq m_+(C_1 * \ldots * C_n)$ if and only if there exist a natural number*

$m \geqq 2$, $m-1$ *natural numbers* $h_1, \ldots, h_{m-1}$ *such that* $h_1 < \ldots < h_{m-1} < n$ *and* $2m-1$ *elements* $e_1, \ldots, e_m$ *and* $f_1, \ldots, f_{m-1}$ *of S which satisfy*

(I) *for every natural number* $1 \leqq j \leqq m-1$, $C_{h_j}\delta = C_1\delta \wedge \ldots \wedge C_n\delta$;

(II) *for every natural number* $1 \leqq j \leqq m-1$, $C_{h_j} < C_{h_j+1}$;

(III) *if* $h_1 \geqq 2$, *then*

$$C_1\delta \wedge \ldots \wedge C_n\delta \prec C_1\delta \wedge \ldots \wedge C_{h_1-1}\delta;$$

(IV) *for every natural number* $1 \leqq j \leqq m-2$ *such that* $h_j + 1 < h_{j+1}$

$$C_1\delta \wedge \ldots \wedge C_n\delta \prec C_{h_j+1}\delta \wedge \ldots \wedge C_{h_{j+1}-1}\delta\ ;$$

(V) *if* $h_1 \geqq 2$, *then* $C_{h_1} < C_1 * \ldots * C_{h_1-1}$;

(VI) e_1 *is the idempotent of* $C_1 * \ldots * C_n$ *and for every natural number* $1 \leqq j \leqq m-1$, e_{j+1} *is the idempotent of* $C_{h_j+1} * C_{h_j}$;

(VII) *for each natural number* $1 \leqq j \leqq m-1$, f_j *is an idempotent of S such that* $e_j < f_j$, $e_j \mathscr{D}_E f_j$ *and* e_j *and* f_j *are consecutive in* $e_j\mathscr{D}_E$;

(VIII) *for each natural number* $2 \leqq j \leqq m-1$, *there exists* $y_{j-1} \in C_{h_{j-1}}$ *such that* $y_{j-1}f_j = f_{j-1}$;

(IX) $C_1 * \ldots * C_{h_1} = C_{h_1}$;

(X) *either* $f_{m-1} \in C_{h_{m-1}}C_{h_{m-1}+1}$ *or there exist an idempotent* f_m *of S and* $y_{m-1} \in C_{h_{m-1}}$ *such that* $e_m < f_m$, $e_m \mathscr{D}_E f_m$, e_m *and* f_m *are consecutive in* $e_m\mathscr{D}_E$, $y_{m-1}f_m = f_{m-1}$ *and* $f_m \in C_{h_{m-1}+1} * \ldots * C_n$.

(2) *Let* $C_1, \ldots, C_n \in \mathscr{C}$ *such that* $C_1\delta \wedge \ldots \wedge C_n\delta$ *is a periodic* δ*-class of L-type. Then* $C_1 \ldots C_n$ *is not contained in a single archimedean class and* $C_1 \ldots C_n \subseteq m_-(C_1 * \ldots * C_n)$ *if and only if there exist a natural number* $m \geqq 2$, $m-1$ *natural numbers* $h_1, \ldots, h_{m-1}$ *such that* $h_1 < \ldots < h_{m-1} < n$ *and* $2m-1$ *elements* $e_1, \ldots, e_m$ *and* $g_1, \ldots, g_{m-1}$ *of S which satisfy*

(I) *for every natural number* $1 \leqq j \leqq m-1$ $C_{h_j}\delta = C_1\delta \wedge \ldots \wedge C_n\delta$;

(II) *for every natural number* $1 \leqq j \leqq m-1$, $C_{h_j+1} < C_{h_j}$;

(III) *if* $h_1 \geqq 2$, *then*

$$C_1\delta \wedge \ldots \wedge C_n\delta \prec C_1\delta \wedge \ldots \wedge C_{h_1-1}\delta\ ;$$

(IV) *for every natural number* $1 \leqq j \leqq m-2$ *such that* $h_j + 1 < h_{j+1}$

$$C_1\delta \wedge \ldots \wedge C_n\delta \prec C_{h_j+1}\delta \wedge \ldots \wedge C_{h_{j+1}-1}\delta\ ;$$

(V) *if* $h_1 \geqq 2$, *then* $C_1 * \ldots * C_{h_1-1} < C_{h_1}$;

(VI) e_1 *is the idempotent of* $C_1 * \ldots * C_n$ *and for every natural number* $1 \leqq j \leqq m-1$, e_{j+1} *is the idempotent of* $C_{h_j+1} * C_{h_j}$;

(VII) *for each natural number* $1 \leqq j \leqq m-1$, g_j *is an idempotent of S such that* $g_j < e_j$, $g_j \mathscr{D}_E e_j$ *and* g_j *and* e_j *are consecutive in* $e_j\mathscr{D}_E$;

(VIII) *for each natural number* $2 \leqq j \leqq m-1$, *there exists* $y_{j-1} \in C_{h_{j-1}}$ *such that* $y_{j-1}g_j = g_{j-1}$;

(IX) $C_1 * \ldots * C_{h_1} = C_{h_1}$;

(X) *either* $g_{m-1} \in C_{h_{m-1}}C_{h_{m-1}+1}$ *or these exist an idempotent* g_m *of S and* $y_{m-1} \in C_{h_{m-1}}$ *such that* $g_m < e_m$, $g_m \mathscr{D}_E e_m$, g_m *and* e_m *are consecutive in* $e_m\mathscr{D}_E$, $y_{m-1}g_m = g_{m-1}$ *and* $g_m \in C_{h_{m-1}+1} * \ldots * C_n$.

Proof. First we prove the direct part of the theorem by induction on n. If $n = 1$, then the assertion holds tirivially. Now suppose $n \geqq 2$ and suppose that $C_1 \dots C_n$ is not contained in a single archimedean class and $C_1 \dots C_n \subseteq m_+(C_1 * \dots * C_n)$. Then by Theorem 2.11, there exist the idempotent e_1 of S in $C_1 * \dots * C_n$ and an idempotent f_1 of S such that $e_1 < f_1$, $e_1 \mathscr{D}_E f_1$ and e_1 and f_1 are consecutive in $e_1 \mathscr{D}_E$. Also either one of the conditions (i) and (ii) in Theorem 2.11 is satisfied.

Case: the condition (i) is satisfied. We put the least natural number h such that $C_1\delta \wedge \dots \wedge C_h\delta = C_1\delta \wedge \dots \wedge C_n\delta$ by h_1. Then by (i), we have $C_{h_1} * \dots * C_n = (C_1 * \dots * C_{h_1}) * \dots * C_n = C_1 * \dots * C_n$ and so $C_{h_1}\delta \wedge \dots \wedge C_n\delta = C_1\delta \wedge \dots \wedge C_n\delta$ is a periodic δ-class of L-type. Also $C_{h_1} \dots C_n$ is not contained in a single archimedean class and $C_{h_1} \dots C_n \subseteq m_+(C_{h_1} * \dots * C_n)$ and so by induction hypothesis, there exist natural numbers $h_1', h_2, \dots, h_{m-1}$ and elements $e_1, \dots, e_m, f_1, \dots, f_{m-1}$ of S which satisfy the conditions (I)–(X) on $C_{h_1}, \dots, C_n$. By way of contradiction we assume $h_1 \neq h_1'$. Then we have $h_1 < h_1'$ and by (III) on $C_{h_1}, \dots, C_n$ we have $C_{h_1}\delta \wedge \dots \wedge C_n\delta \prec C_{h_1}\delta \wedge \dots \wedge C_{h_1'-1}\delta$. But this is a contradiction, since $C_{h_1}\delta \wedge \dots \wedge C_{h_1'-1}\delta \preccurlyeq C_{h_1}\delta = (C_{h_1} * \dots * C_n)\,\delta = C_{h_1}\delta \wedge \dots \wedge C_n\delta$ by (i). Hence we have $h_1 = h_1'$. By (i), $C_{h_1}\delta = C_{h_1}\delta \wedge \dots \wedge C_n\delta = C_1\delta \wedge \dots \wedge C_n\delta$ and by (I) on $C_{h_1}, \dots, C_n$, for every $2 \leqq j \leqq m-1$, we have $C_{h_j}\delta = C_{h_1}\delta \wedge \dots \wedge C_n\delta = C_1\delta \wedge \dots \wedge C_n\delta$. Hence (I) on $C_1, \dots, C_n$ is satisfied. It is clear that (II) on $C_1, \dots, C_n$ is satisfied. By the definition of h_1 we have (III) on $C_1, \dots, C_n$. Suppose $1 \leqq j \leqq m-2$ and $h_j + 1 < h_{j+1}$. Then $C_1\delta \wedge \dots \wedge C_n\delta = C_{h_1}\delta \wedge \dots \wedge C_n\delta \prec C_{h_j+1}\delta \wedge \dots \wedge C_{h_{j+1}-1}\delta$ and so we have (IV) on $C_1, \dots, C_n$. By (i), we have (V) on $C_1, \dots, C_n$. Since $C_{h_1} * \dots * C_n = C_1 * \dots * C_n$ by (i), the idempotent of $C_{h_1} * \dots * C_n$ is the idempotent e_1 of $C_1 * \dots * C_n$. Thus (VI) on $C_1, \dots, C_n$ follows from (VI) on $C_{h_1}, \dots, C_n$. (VII) and (VIII) on $C_1, \dots, C_n$ are clear. (IX) on $C_1, \dots, C_n$ holds by (i). Finally (X) on $C_1, \dots, C_n$ is clear.

Case: the conditions (ii) and (a) are satisfied. We put $m = 2$ and $h_1 = 1$. Then we obtain (I)–(V) clearly. By assumption e_1 is the idempotent of $C_1 * \dots * C_n$ and f_1 is an idempotent of S such that $e_1 < f_1$, $e_1 \mathscr{D}_E f_1$ and e_1 and f_1 are consecutive in $e_1 \mathscr{D}_E$. Now let e_2 be the idempotent of the periodic archimedean class $C_2 * C_1$. Then we have (VI)–(IX). Let k be the least natural number such that $C_1 \dots C_k$ is not contained in a single archimedean class. By (a), $C_1 C_2$ is not contained in a single archimedean class and so $k = 2$. Hence by Lemma 2.9, there exist $x_1 \in C_1$ and $x_2 \in C_2$ such that $x_1 x_2 = f_1$ and so $f_1 \in C_1 C_2$. Hence we have (X).

Case: the conditions (ii), (b) and (b_1) are satisfied. We put $m = 2$ and $h_1 = 1$. Then we have (I)–(V) clearly. By assumption, e_1 is the idempotent of $C_1 * \dots * C_n$, e_2 is the idempotent of the periodic archimedean class $C_2 * C_1$ and f_1 is an idempotent of S such that $e_1 < f_1$, $e_1 \mathscr{D}_E f_1$ and e_1 and f_1 are consecutive in $e_1 \mathscr{D}_E$. Hence we have (VI)–(IX). Since $C_1 < C_2$, we have $C_1 \leqq C_2 * C_1 \leqq C_2$ but since $C_2 * C_1 \neq C_2 * \dots * C_n$, we have $C_2\delta \neq C_1\delta \wedge \dots \wedge C_n\delta = C_1\delta = C_1\delta \wedge C_2\delta = (C_2 * C_1)\,\delta$ and so $C_2 * C_1 \neq C_2$. Hence $C_2 * C_1 < C_2$. Since $(C_2 * \dots * C_n)\,\delta = C_2\delta \wedge \dots \wedge C_n\delta = C_1\delta \wedge \dots * C_n\delta$, $C_2 * \dots * C_n$ is a periodic archimedean class, whose idempotent we denote by f. We have

$$\begin{aligned} C_2 * C_1 &= \max\{Z \in \mathscr{C};\ C_1 \leqq Z \leqq C_2 \text{ and } Z \in C_2\delta \wedge C_1\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_1 \leqq Z \leqq C_2 \text{ and } Z \in C_1\delta\} \\ &= \max\{Z \in \mathscr{C};\ Z \leqq C_2 \text{ and } Z \in C_1\delta\}. \end{aligned}$$

But if $C_2 * \ldots * C_n \leqq C_2$ were true, then we have

$$\begin{aligned} C_2 * \ldots * C_n &= C_2 * (C_2 * \ldots * C_n) \\ &= \max\{Z \in \mathscr{C};\ C_2 * \ldots * C_n \leqq Z \leqq C_2 \text{ and} \\ &\qquad Z \in (C_2 * \ldots * C_n)\,\delta \wedge C_2\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_2 * \ldots * C_n \leqq Z \leqq C_2 \text{ and } Z \in C_1\delta\} \\ &= \max\{Z \in \mathscr{C};\ Z \leqq C_2 \text{ and } Z \in C_1\delta\} = C_2 * C_1 \end{aligned}$$

which is a contradiction. Hence $C_2 * C_1 < C_2 < C_2 * \ldots * C_n$ and so $e_2 < f$. Also since $(C_2 * C_1)\,\delta = C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta = (C_2 * \ldots * C_n)\,\delta$, it follows from [2] Theorem 3.3 that $e_2\,\mathscr{D}_E f$. Moreover

$$\begin{aligned} C_2 * \ldots * C_n &= C_2 * (C_2 * \ldots * C_n) \\ &= \min\{Z \in \mathscr{C};\ C_2 \leqq Z \leqq C_2 * \ldots * C_n \text{ and} \\ &\qquad Z \in (C_2 * \ldots * C_n)\,\delta \wedge C_2\delta\} \\ &= \min\{Z \in \mathscr{C};\ C_2 \leqq Z \leqq C_2 * \ldots * C_n \text{ and } Z \in C_1\delta\}. \end{aligned}$$

Hence there is no $Z \in \mathscr{C}$ such that $C_2 * C_1 < Z < C_2 * \ldots * C_n$ and $Z \in C_1\delta$. Hence by [2] Theorem 3.3, e_2 and f are consecutive in $e_1\mathscr{D}_E$. Further by (b), we have $y_1 f = f_1$ for some $y_1 \in (C_1)_- \subseteq C_1$. Hence we put $f = f_2$ and we obtain (X).

Case: the conditions (ii), (b) and (b_2) are satisfied. By (b), $C_2\delta \wedge \ldots \wedge C_n\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ is a δ-class of L-type. Also by (b_2), $C_2 \ldots C_n$ is not contained in a single archimedean class and $C_2 \ldots C_n \subseteq m_+(C_2 * \ldots * C_n)$ and so by induction hypothesis, there exist a natural number m' such that $m' \geqq 2$, $m' - 1$ natural numbers $h'_1, \ldots, h'_{m'-1}$ such that $h'_1 < \ldots < h'_{m'-1} < n$ and $2m' - 1$ elements $e'_1, \ldots, e'_{m'}, f'_1, \ldots, f'_{m'-1}$ satisfying the conditions (I)–(X) on $C_2, \ldots, C_n$. We put $m = m' + 1$, $h_1 = 1$, $h_2 = h'_1, \ldots, h_m = h'_{m'}$. By assumption e_1 is the idempotent of $C_1 * \ldots * C_n$ and f_1 is an idempotent of S such that $e_1 < f_1$, $e_1\,\mathscr{D}_E f_1$ and e_1 and f_1 are consecutive in $e_1\mathscr{D}_E$. Also we put $e_2 = e'_1, \ldots, e_m = e'_{m'}$, $f_2 = f'_1, \ldots, f_{m-1} = f'_{m'-1}$. Now we have $C_{h_1}\delta = C_1\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ and for every $2 \leqq j \leqq m - 1$, we have $C_{h_j}\delta = C_{h'_{j-1}}\delta = C_2\delta \wedge \ldots \wedge C_n\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ by (b). Hence we obtain (I) on $C_1, \ldots, C_n$. We have $C_{h_1} = C_1 < C_2 = C_{h_1+1}$ by (ii). Let $2 \leqq j \leqq m - 1$. Then $C_{h_j} = C_{h'_{j-1}} < C_{h'_{j-1}+1} = C_{h_j+1}$ and so we obtain (II) on $C_1, \ldots, C_n$. (III) and (V) on $C_1, \ldots, C_n$ hold trivially. Let $1 \leqq j \leqq m - 2$ such that $h_j + 1 < h_{j+1}$. If $j = 1$, then $2 = h_1 + 1 < h_2 = h'_2$ and by (III) on $C_2, \ldots, C_n$, we have $C_1\delta \wedge \ldots \wedge C_n\delta = C_2\delta \wedge \ldots \wedge C_n\delta \prec C_2\delta \wedge \ldots \wedge C_{h_1'-1}\delta = C_{h_1+1}\delta \wedge \ldots \wedge C_{h_2-1}\delta$. Also if $j \geqq 2$, then by (IV) on $C_2, \ldots, C_n$, we have $C_1\delta \wedge \ldots \wedge C_n\delta = C_2\delta \wedge \ldots \wedge C_n\delta \prec C_{h'_{j-1}+1}\delta \wedge \ldots \wedge C_{h_j'-1}\delta = C_{h_j+1}\delta \wedge \ldots \wedge C_{h_{j+1}-1}\delta$, and so we obtain (IV) on $C_1, \ldots, C_n$. By assumption,

e_1 is the idempotent of $C_1 * \dots * C_n$. By (b_2) we have $C_2 * \dots * C_n = C_2 * C_1$ and so $e_2 = e_1'$ is the idempotent of $C_2 * \dots * C_n = C_2 * C_1$. If $2 \leqq j \leqq m-1$, then $e_{j+1} = e_j'$ is the idempotent of $C_{h'_{j-1}+1} * C_{h'_{j-1}} = C_{h_j+1} * C_{h_j}$. Hence we obtain (VI) on $C_1, \dots, C_n$. By assumption, f_1 is an idempotent of S such that $e_1 < f_1$, $e_1 \mathscr{D}_E f_1$ and e_1 and f_1 are consecutive in $e_1 \mathscr{D}_E$. Let $2 \leqq j \leqq m-1$. Then $f_j = f'_{j-1}$ is an idempotent of S such that $e_j = e'_{j-1} < f'_{j-1} = f_j$, $e_j = e'_{j-1} \mathscr{D}_E f'_{j-1} = f_j$ and $e_j = e'_{j-1}$ and $f_j = f'_{j-1}$ are consecutive in $e_j \mathscr{D}_E = e'_{j-1}\mathscr{D}_E$. Hence (VII) on $C_1, \dots, C_n$ is satisfied. By (b), we have $y_1 f_2 = f_1$ for some $y_1 \in C_1 = C_{h_1}$. Let $3 \leqq j \leqq m-1$. Then $2 \leqq j-1 \leqq m-2 = m'-1$ and there exists $y_{j-1} \in C_{h'_{j-2}} = C_{h_{j-1}}$ such that $y_{j-1} f_j = y_{j-1} f'_{j-1} = f'_{j-2} = f_{j-1}$. Hence we have (VIII) on $C_1, \dots, C_n$. Since $h_1 = 1$, (IX) on $C_1, \dots, C_n$ holds clearly. By (X) on $C_2, \dots, C_n$, either $f_{m-1} = f'_{m'-1} \in C_{h'_{m'-1}} C_{h'_{m'-1}+1} = C_{h_{m-1}} C_{h_{m-1}+1}$ or there exist an idempotent $f_m = f'_{m'}$ of S and $y_{m-1} \in C_{h'_{m'-1}} = C_{h_{m-1}}$ such that $e_m = e'_{m'} < f'_{m'} = f_m$, $e_m = e'_{m'} \mathscr{D}_E f'_{m'} = f_m$, $e_m = e'_{m'}$ and $f_m = f'_{m'}$ are consecutive in $e'_{m'} \mathscr{D}_E = e_m \mathscr{D}_E$, $y_{m-1} f_m = y_{m-1} f'_{m'} = f'_{m'-1} = f_{m-1}$ and $f_m = f'_{m'} \in C_{h'_{m'-1}+1} * \dots * C_n = C_{h_{m-1}+1} * \dots * C_n$. Thus we obtain (X) on $C_1, \dots, C_n$.

Conversely suppose that the conditions (I)–(X) are satisfied. Preliminarily we show that

(*) If $s < h_1$, then $x_s e_1 = e_1$ and $x_s f_1 = f_1$ for every $x_s \in C_s$;

(**) If $2 \leqq j \leqq m-1$ and $h_{j-1} < s < h_j$, then $x_s e_j = e_j$ and $x_s f_j = f_j$ for every $x_s \in C_s$.

Suppose $s < h_1$. Then $2 \leqq h_1$ and by (I) and (III), we have $C_{h_1}\delta = C_1\delta \wedge \dots \wedge C_n\delta \prec C_1\delta \wedge \dots \wedge C_{h_1-1}\delta = C_s\delta \wedge (C_1 * \dots * C_{h_1-1})\delta$, and by [2] Lemma 5.6, there exists no $Z \in \mathscr{C}$ such that Z lies between C_s and $C_1 * \dots * C_{h_1-1}$ and $Z \in C_1\delta \wedge \dots \wedge C_n\delta$. In particular, C_{h_1} does not lie between C_s and $C_1 * \dots * C_{h_1-1}$ and since $C_{h_1} < C_1 * \dots * C_{h_1-1}$ by (V), we have $C_{h_1} < C_s$. Also by (IX),

$$\begin{aligned} C_{h_1} &= (C_1 * \dots * C_{h_1-1}) * C_{h_1} \\ &= \max\{Z \in \mathscr{C};\ C_{h_1} \leqq Z \leqq C_1 * \dots * C_{h_1-1} \text{ and} \\ &\quad Z \in (C_1 * \dots * C_{h_1-1})\delta \wedge C_{h_1}\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_{h_1} \leqq Z \leqq C_1 * \dots * C_{h_1-1} \text{ and} \\ &\quad Z \in C_1\delta \wedge \dots \wedge C_n\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_{h_1} \leqq Z \leqq C_s \text{ and } Z \in C_s\delta \wedge C_{h_1}\delta\} = C_s * C_{h_1}. \end{aligned}$$

Also since $C_{h_1}\delta = C_1\delta \wedge \dots \wedge C_n\delta$, we have $C_1 * \dots * C_n = (C_1 * \dots * C_{h_1}) * \dots * C_n = C_{h_1} * \dots * C_n = C_{h_1}$ and so by (VI) e_1 is the idempotent of $C_{h_1} = C_s * C_{h_1}$. Moreover $C_{h_1}\delta = C_1\delta \wedge \dots \wedge C_n\delta \prec C_1\delta \wedge \dots \wedge C_{h_1-1}\delta \preccurlyeq C_s\delta$. Further by (VII) f_1 is an idempotent of S such that $e_1 < f_1$, $e_1 \mathscr{D}_E f_1$ and e_1 and f_1 are consecutive in $e_1\mathscr{D}_E$. Hence by Lemma 2.12, we have $x_s e_1 = e_1$ and $x_s f_1 = f_1$ for every $x_s \in C_s$. This proves (*).

Now let $2 \leqq j \leqq m-1$ and $h_{j-1} < s < h_j$. Then $1 \leqq j-1 \leqq m-2$ and by (I) and (IV), we have $C_{h_{j-1}}\delta = C_1\delta \wedge \dots \wedge C_n\delta \prec C_{h_{j-1}+1}\delta \wedge \dots \wedge C_{h_j-1}\delta \preccurlyeq$

$\preccurlyeq C_{h_{j-1}+1}\delta \wedge C_s\delta$ and so by [2] Lemma 5.6, there exists no $Z \in C$ such that Z lies between $C_{h_{j-1}+1}$ and C_s and $Z \in C_1\delta \wedge \ldots \wedge C_n\delta$. In particular $C_{h_{j-1}}$ does not lie between $C_{h_{j-1}+1}$ and C_s and since $C_{h_{j-1}} < C_{h_{j-1}+1}$ by (II), we have $C_{h_{j-1}} < C_s$. Also since $C_{h_{j-1}}\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ by (I), we have

$$\begin{aligned} C_s * C_{h_{j-1}} &= \max\{Z \in \mathscr{C};\ C_{h_{j-1}} \leqq Z \leqq C_s \text{ and } Z \in C_{h_{j-1}}\delta \wedge C_s\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_{h_{j-1}} \leqq Z \leqq C_s \text{ and } Z \in C_1\delta \wedge \ldots \wedge C_n\delta\} \\ &= \max\{Z \in \mathscr{C};\ C_{h_{j-1}} \leqq Z \leqq C_{h_{j-1}+1} \text{ and } Z \in C_{h_{j-1}}\delta \wedge C_{h_{j-1}+1}\delta\} \\ &= C_{h_{j-1}+1} * C_{h_{j-1}}. \end{aligned}$$

Hence by (VI) e_j is the idempotent of $C_s * C_{h_{j-1}}$. Moreover $C_{h_{j-1}}\delta = C_1\delta \wedge \ldots \wedge C_n\delta \prec C_{h_{j-1}+1}\delta \wedge \ldots \wedge C_{h_j-1}\delta \preccurlyeq C_s\delta$. Further by (VII), f_j is an idempotent of S such that $e_j < f_j$, $e_j \mathscr{D}_E f_j$ and e_j and f_j are consecutive in $e_j\mathscr{D}_E$. Hence by Lemma 2.12, we have $x_se_j = e_j$ and $x_sf_j = f_j$ for every $x_s \in C_s$. This proves (**).

Now by (VIII) for each natural number $2 \leqq j \leqq m-1$, there exists $y_{j-1} \in C_{h_{j-1}}$ such that $y_{j-1}f_j = f_{j-1}$. For each $1 \leqq i \leqq h_{m-1}$ such that $i \neq h_j$ for all $1 \leqq j \leqq m-1$, we take $x_i \in C_i$ arbitrarily. Then by (*), if $2 \leqq h_1$, then $x_1 \ldots x_{h_1-1}f_1 = f_1$. In the case when $h_1 = 1$, we assume $x_1 \ldots x_{h_1-1}$ is the void symbol and then we have always this relation. Also for each natural number $2 \leqq j \leqq m-1$, it follows from (**) that if $h_{j-1}+1 \leqq h_j - 1$, then $x_{h_{j-1}+1} \ldots x_{h_j-1}f_j = f_j$ and so $y_{j-1}x_{h_{j-1}+1} \ldots x_{h_j-1}f_j = y_{j-1}f_j = f_{j-1}$. In the case when $h_{j-1}+1 = h_j$, we assume $x_{h_{j-1}+1} \ldots x_{h_j-1}$ is the void symbol and then we have always this relation. Thus $f_1 = x_1 \ldots x_{h_1-1} \ldots y_{j-1}x_{h_{j-1}+1} \ldots x_{h_j-1} \ldots y_{m-2}x_{h_{m-2}+1} \ldots x_{h_{m-1}-1}f_{m-1}$. By (X) the following two cases are possible:

Case 1: $f_{m-1} \in C_{h_{m-1}}C_{h_{m-1}+1}$;

Case 2: there exist an idempotent f_m of S and $y_{m-1} \in C_{h_{m-1}}$ such that $e_m < f_m$, $e_m \mathscr{D}_E f_m$, e_m and f_m are consecutive in $e_m\mathscr{D}_E$, $y_{m-1}f_m = f_{m-1}$ and $f_m \in C_{h_{m-1}+1} * \ldots * C_n$.

Case 1: We have $f_{m-1} = z_{h_{m-1}}z_{h_{m-1}+1}$ for some $z_{h_{m-1}} \in C_{h_{m-1}}$ and $z_{h_{m-1}+1} \in C_{h_{m-1}+1}$. We take $x_{h_{m-1}+2} \in C_{h_{m-1}+2}, \ldots, x_n \in C_n$ arbitrarily. We denote by E_{m-1} and F_{m-1} the archimedean classes containing e_{m-1} and f_{m-1}, respectively. Then since $e_{m-1} \mathscr{D}_E f_{m-1}$, it follows from [2] Theorem 3.3 that $E_{m-1}\, \delta\, F_{m-1}$. On the other hand, if $m = 2$, then by (VI), $E_{m-1}\delta = (C_1 * \ldots * C_n)\,\delta = C_1\delta \wedge \ldots \wedge C_n\delta$ and if $m > 2$, then by (I), $E_{m-1}\delta = (C_{h_{j-1}+1} * C_{h_{j-1}})\,\delta = C_{h_{j-1}+1}\delta \wedge C_{h_{j-1}}\delta = C_1\delta \wedge \ldots \wedge C_n\delta$. Thus always we have $F_{m-1}\delta = E_{m-1}\delta = C_1\delta \wedge \ldots \wedge C_n\delta$. Hence by [2] Theorem 2.7, we have $z_{h_{m-1}}z_{h_{m-1}+1}x_{h_{m-1}+2} \ldots x_n = f_{m-1}x_{h_{m-1}+2} \ldots x_n = f_{m-1}$ and so $f_1 = x_1 \ldots x_{h_1-1} \ldots y_{m-2}x_{h_{m-2}+1} \ldots x_{h_{m-1}-1}z_{h_{m-1}}z_{h_{m-1}+1} \ldots x_n \in C_1 \ldots C_n$.

Case 2. By Lemma 2.5, we have $\{f_m\} = m_0(C_{h_{m-1}+1} * \ldots * C_n) \subseteq C_{h_{m-1}+1} \ldots C_n$ and so there exist $z_{h_{m-1}+1} \in C_{h_{m-1}+1}, \ldots, z_n \in C_n$ such that $f_m = z_{h_{m-1}+1} \ldots z_n$. Hence $f_{m-1} = y_{m-1}f_m = y_{m-1}z_{h_{m-1}+1} \ldots z_n$ and so $f_1 = x_1 \ldots x_{h_1-1} \ldots y_{m-2}x_{h_{m-2}+1} \ldots x_{h_{m-1}-1}y_{m-1}z_{h_{m-1}+1} \ldots z_n \in C_1 \ldots C_n$.

Thus in both cases we have $f_1 \in C_1 \ldots C_n$. On the other hand, by Lemma 2.5, we have $\{e_1\} = m_0(C_1 * \ldots * C_n) \subseteq C_1 \ldots C_n$. Hence $e_1, f_1 \in C_1 \ldots C_n$. Hence $C_1 \ldots C_n$ is not contained in a single archimedean class. Also by Corollary 1.16, there exists a modified archimedean class $m(C_1 * \ldots * C_n)$ such that $C_1 \ldots C_n \subseteq$ $\subseteq m(C_1 * \ldots * C_n)$. Since e_1 is the idempotent of $C_1 * \ldots * C_n$ and $e_1 < f_1$, we must have $C_1 \ldots C_n \subseteq m_+(C_1 * \ldots * C_n)$.

(2) can be proved similarly.

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