Gábor Révész

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A SIMPLE PROOF OF VINOGRADOV'S THEOREM ON THE ORDERABILITY OF THE FREE PRODUCT OF o-GROUPS

GABOR REVESZ, Lawrence

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Let H and G be groups and let k be a field of characteristics not equal to 2. Denote by $\mathrm{Tr}_2(k[H \times G])$ the group of 2×2 upper triangular, invertible matrices over the group algebra $k[H \times G]$ of the direct product $H \times G$ of H and G. We prove:

Theorem A. *The homomorphism of the free product $H * G$ into $\mathrm{Tr}_2([H \times G])$, induced by*

$$h \mapsto \begin{pmatrix} 1 & 1-h \\ 0 & h \end{pmatrix} (h \in H) \quad \textit{and} \quad g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} (g \in G)\,,$$

is an embedding.

This embedding is then used to give a simple proof of

Theorem B. ([2], [3], [4]) *The free product of o-groups is an o-group.*

Essentially, what we show is that the construction on $\omega \times \omega$ upper triangular matrices used by Johnson in [2] in proving Vinogradov's theorem, can be adapted in the context of 2×2 matrices. Once Theorem A is established, Theorem B follows by simply restricting Johnson's ordering of infinite matrices to an ordering of the corresponding top-left-corner 2×2 ones.

1. Proof of Theorem A. Denote the homomorphism in question by α. In proving that α is injective we may assume that both H and G contain non-torsion elements. For if $H \subset H'$ and $G \subset G'$, then $\alpha' : H' * G' \to \mathrm{Tr}_2(k[H' \times G'])$ extends α and hence if α' is injective, so is α. We claim that $\ker \alpha = 1$. Suppose to the contrary that $1 \neq a \in \ker \alpha$. Every element of $H * G \setminus 1$ can be written as an alternating product of elements of H and G. By taking a suitable conjugate of a is necessary, we may assume that

$$a = h_{n+1} g_n h_n \dots g_2 h_2 g_1 h_1$$

where $h_i \neq 1$ and $g_i \neq 1$. Further we may choose a so that n, which we shall call the length of a, is minimal. The minimality of n now implies that $h_{n+1} \neq h_1^{-1}$. For otherwise

$$h^{-1} g_1 h_1 a h_1^{-1} g_1^{-1} h$$

is also in ker α and is of length $n - 1$ for any $1 \neq h \in H$. A straightforward induction on n shows that

$$a^\alpha = \begin{pmatrix} 1 & f(a) \\ 0 & h_{n+1}h_n \dots h_1 g_n \dots g_1 \end{pmatrix}'$$

where

$$f(a) = 1 - \sum_{r=0}^{n-1} h_{r+1}h_r \dots h_0 g_r g_{r-1} \dots g_0 +$$

$$+ \sum_{r=0}^{n-1} h_{r+1}h_r \dots h_0 g_{r+1} g_r \dots g_0 - h_{n+1}h_n \dots h_1 g_n g_{n-1} \dots g_1$$

($h_0 = 1$ and $g_0 = 1$ are introduced merely to simplify presentation). This is in fact true for any alternating product of $H * G$ bordered by elements of $H \setminus 1$. Since $a \in \ker \alpha$, the (2,2) entry of a^α is 1 and hence the first and the last summands of $f(a)$ cancel out. Moreover, since the remaining summands are $\pm$ basis elements of the k-space $k[H \times G]$, they pairwise coincide. In particular, for each r between 0 and $n - 1$

$$h_{r+1}h_r \dots h_0 g_r g_{r-1} \dots g_0 =$$

$$= h_{s+1}h_s \dots h_0 g_{s+1} g_s \dots g_0 \quad \text{for some} \quad 0 \leqq s \leqq n - 1 .$$

Now let $h \in H \setminus \{1, h_1^{-1}\}$ and $g \in G \setminus \{1, g_1^{-1}\}$ and consider

$$a' = (h^{-1}h_1^{-1})\, g^{-1}(h_1 h_{n+1})\, g_n h_n \dots g_2 h_2 (g_1 g)(h_1 h) ;$$

clearly, $a' \in \ker \alpha$ and is of length $n + 1$. Write $h_1' = h_1 h$, $g_1' = g_1 g$; $h_i' = h_i$ and $g_i' = g_i$ for all $2 \leqq i \leqq n$; $h_{n+1}' = h_1 h_{n+1}$, $g_{n+1}' = g^{-1}$ and $h_{n+2}' = (h_1 h)^{-1} = (h_1')^{-1}$. Now for each $2 \leqq r \leqq n$ we have

$$h_r' \dots h_1' = h_r \dots h_1 h \quad \text{and} \quad g_r' \dots g_1' = g_r \dots g_1 g ;$$

in particular: $g_n' \dots g_1' = g_n \dots g_1 g = g$. Furthermore, $h_{n+1}' h_n' \dots h_1' = h_1 h_{n+1} h_n \dots \dots h_1 h = h_1 h = h_1'$ and $h_{n+2}' h_{n+1}' \dots h_1' = 1$, $g_{n+1}' g_n' \dots g_1' = 1$. By assumption H has a non-torsion element so we can pick h so that $h_r' \dots h_1' \neq h_1'$ for all $2 \leqq r \leqq n$ (the non-torsion element of G is needed to make sure that there exists an element outside $\{1, g_1^{-1}\}$). The fact that $a' \in \ker \alpha$ implies that

$$0 = f(a') = -h_1' + h_1' g_1' - \dots - h_{n+1}' \dots h_1' g_n' \dots g_1' + h_{n+1}' \dots h_1' g_{n+1}' \dots g_1' .$$

By what has been said it is clear that for some $2 \leqq r \leqq n + 1$

$$h_1' g_1' = h_r' \dots h_1' g_{r-1}' \dots g_1' ;$$

the choice of h ensures that $r = n + 1$. But then $g_1' = g_1 g = g_n' \dots g_1' = g$ whence $g_1 = 1$, a contradiction.

2. Proof of Theorem B. Let H and G be o-groups, by Theorem A, $H * G$ embeds in $\mathrm{Tr}_2(k[H \times G])$ and the image of $H * G$ is clearly contained in

$$T = \begin{pmatrix} 1 & k[H \times G] \\ 0 & H \times G \end{pmatrix}.$$

It will suffice to verify that if k is an ordered field then T can be ordered. So let k be an ordered field and let $\leqq$ be one of the lexic orders on $H \times G$ induced by the orderings of H and G. Write $\preceq$ for the lexicographic order on $k[H \times G]$ induced by $\leqq$ (cf. [1; p. 108]). The order on T can now be defined setting

$$\begin{pmatrix} 1 & f \\ 0 & hg \end{pmatrix} \geqq' 1 \quad \text{if} \quad \begin{cases} hg > 1 & \text{or} \\ hg = 1 & \text{and} \quad f \succeq 0 \end{cases}$$

In order to prove that $\leqq'$ is preserved by multiplication one has to use the fact that all elements of $H \times G$ are positive with respect to $\preceq$. (As it turns out, every ordering with this property gives rise to an ordering to T.) The resulting order, in fact, extends the orders on H and G we started with.

References

[1] *Fuchs, L.:* Partially ordered algebraic structures, Pergamon Press, Oxford, 1963.

[2] *Johnson, R. E.:* Free products of ordered semigroups, Proc. Amer. Math. Soc. *19* (1968), 697—700.

[3] *Mura, R. B.* and *Rhemtulla, A.:* Orderable groups, Dekker, New York, 1977.

[4] *Vinogradov, A. A.:* On the free product of ordered groups (in Russian), Mat. Sborník, *25* (1949), 163—168.

Author's address: Department of Mathematics, University of Kansas, Lawrence, Ks. 66045, U.S.A.