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TOLERANCES ON GRAPH ALGEBRAS

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A tolerance on an algebra [5] is defined similarly as a congruence, only the requirement of transitivity is omitted. Here we shall study tolerances on graph algebras which were introduced by G. F. McNulty and C. R. Shallon ([2], [4]), and described by R. Pöschel [3]. Before defining these algebras we introduce the necessary terminology and notation of the graph theory.

All graphs mentioned in this paper are directed. We admit loops, but not multiple edges. An edge going from a vertex x to a vertex y is denoted by xy. The vertex set of the graph G is denoted by $V(G)$, its edge set by $E(G)$. Further (following Berge [1]), we denote $\Gamma(x) = \{y \in V(G) \mid xy \in E(G)\}$, $\Gamma^{-1}(x) = \{y \in V(G) \mid yx \in E(G)\}$.

A quasicomponent of a graph G is a maximal (with respect to inclusion) strongly connected subgraph of G. The family of all quasicomponents of G will be denoted by $\mathcal{Q}(G)$. The family $\mathcal{Q}(G)$ can be partially ordered by the relation $\lesssim$ such that $Q_1 \lesssim Q_2$ for $Q_1 \in \mathcal{Q}(G)$, $Q_2 \in \mathcal{Q}(G)$ means that there exists a directed path in G from a vertex of Q_1 into a vertex of Q_2.

To a graph G the graph algebra $\mathfrak{A}(G)$ of type $\langle 2, 1 \rangle$ is assigned. The support of $\mathfrak{A}(G)$ is the set $V(G) \cup \{\infty\}$, where ∞ is an element not belonging to $V(G)$. The element ∞ is considered as a nullary operation on $\mathfrak{A}(G)$. Further, $\mathfrak{A}(G)$ has a binary operation denoted by a dot. It is defined in such a way that for any two elements x, y of $V(G) \cup \{\infty\}$ the element $x \,.\, y$ is equal to x if and only if $xy \in E(G)$, and is equal to ∞ otherwise. (In particular, $x \,.\, \infty = \infty \,.\, x = \infty$ for each $x \in V(G) \cup \{\infty\}$.)

Hence the graph algebra is a groupoid with one nullary operation added. A tolerance on the graph algebra $\mathfrak{A}(G)$ is a reflexive and symmetric binary relation T on the set $V(G) \cup \{\infty\}$ with the property that $(x_1, x_2) \in T$ and $(y_1, y_2) \in T$ imply $(x_1 \,.\, y_1, x_2 \,.\, y_2) \in T$ for any four elements x_1, x_2, y_1, y_2 of $V(G) \cup \{\infty\}$.

All tolerances on $\mathfrak{A}(G)$ form a lattice with respect to the set inclusion; we denote this lattice by $LT(\mathfrak{A}(G))$. If a tolerance on $\mathfrak{A}(G)$ is transitive, it is a congruence on $\mathfrak{A}(G)$. By the symbol Δ we shall denote the equality relation on $V(G) \cup \{\infty\}$, i.e. the set of all pairs (x, x) for $x \in V(G) \cup \{\infty\}$. As any tolerance T on $\mathfrak{A}(G)$ is a reflexive relation, we always have $\Delta \subseteq T$.

If R is a binary relation on $V(G) \cup \{\infty\}$, then we denote

$$V_0(R) = \{x \in V(G) \mid (x, \infty) \in R\} .$$

Proposition 1. *Let $T \in LT(\mathfrak{A}(G))$. Then there exists a subfamily $\mathcal{Q}_0(T)$ of $\mathcal{Q}(G)$ with the properties that $Q_1 \in \mathcal{Q}_0(T)$, $Q_2 \in \mathcal{Q}(G)$, $Q_2 \leqq Q_1$ imply $Q_2 \in \mathcal{Q}_0(T)$ and $V_0(T) = \bigcup_{Q \in \mathcal{Q}_0(T)} V(Q)$.*

Proof. If $V_0(T) = \emptyset$, then we may put $\mathcal{Q}_0(T) = \emptyset$ and the assertion holds. Now let $V_0(T) \neq \emptyset$ and $a \in V_0(T)$. Let $Q(a)$ be the quasicomponent of G which contains a; let $Q \in \mathcal{Q}(G)$, $Q \leqq Q(a)$, let $b \in Q$. Then there exists a directed path from b into a. Let $b = x_0, x_1, \ldots, x_k = a$ be the vertices of this path and let $x_{i-1}x_i \in E(G)$ for $i = 1, \ldots, k$. As $a \in V(T_0)$, we have $(a, \infty) = (x_k, \infty) \in T$. Suppose that $(x_i, \infty) \in T$ for some i; then $(x_{i-1}, x_{i-1}) \in T$, $(x_i, \infty) \in T$ imply $(x_{i-1} \,.\, x_i, x_{i-1} \,.\, \infty) = (x_{i-1}, \infty) \in T$. Thus, by induction, $(x_i, \infty) \in T$ for each $i = 0, 1, \ldots, k$ and also $(b, \infty) \in T$ and $b \in V_0(T)$. Thus with each of its vertices the set $V_0(T)$ contains the whole vertex set of the quasicomponent which contains this vertex, and the vertex sets of all quasicomponents which precede this quasicomponent in the ordering $\leqq$. This implies the assertion. □

Proposition 2. *Let $T \in LT(\mathfrak{A}(G))$, let $x \in V(G) - V_0(T)$, $y \in V(G) - V_0(T)$, $(x, y) \in T$. Then $\Gamma(x) = \Gamma(y)$, $\Gamma^{-1}(x) - V_0(T) = \Gamma^{-1}(y) - V_0(T)$.*

Proof. Suppose the contrary. If $\Gamma(x) \neq \Gamma(y)$, then either $\Gamma(x) - \Gamma(y) \neq \emptyset$, or $\Gamma(y) - \Gamma(x) \neq \emptyset$. If $\Gamma(x) - \Gamma(y) \neq \emptyset$, let $z \in \Gamma(x) - \Gamma(y)$. Then we have $xz \in E(G)$, $yz \in E(G)$, $x \,.\, z = x$, $y \,.\, z = \infty$. From $(x, y) \in T$, $(z, z) \in T$ we have $(x \,.\, z, y \,.\, z) = (x, \infty) \in T$ and thus $x \in V_0(T)$, which is a contradiction. If $\Gamma(y) - \Gamma(x) \neq \emptyset$, we analogously obtain $y \in V_0(T)$, again a contradiction. Now suppose $\Gamma^{-1}(x) - V_0(T) \neq \Gamma^{-1}(y) - V_0(T)$. If $(\Gamma^{-1}(x) - V_0(T)) - (\Gamma^{-1}(y) - V_0(T)) = \Gamma^{-1}(x) - (\Gamma^{-1}(y) \cup V_0(T)) \neq \emptyset$, let u be an element of this set. We have $ux \in E(G)$, $uy \in E(G)$, $u \,.\, x = u$, $u \,.\, y = \infty$. From $(u, u) \in T$, $(x, y) \in T$ we have $(u \,.\, x, u \,.\, y) = (u, \infty) \in T$, which is a contradiction with the assumption that $u \in \Gamma^{-1}(x) - V_0(T)$. Analogously in the other case. □

Proposition 3. *Let $T \in LT(\mathfrak{A}(G))$, let $x \in V(G) - V_0(T)$, $y \in V_0(T)$, $(x, y) \in T$. Then $\Gamma(x) \subseteq \Gamma(y)$, $\Gamma^{-1}(x) \subseteq V_0(T)$.*

Proof. Suppose that $\Gamma(x)$ is not a subset of $\Gamma(y)$, i.e. that there exists $z \in \Gamma(x) - \Gamma(y)$. Then $x \,.\, z = x$, $y \,.\, z = \infty$. From $(x, y) \in T$, $(z, z) \in T$ we have $(x \,.\, z, y \,.\, z) = (x, \infty) \in T$, which is a contradiction. If $\Gamma^{-1}(x)$ is not a subset of $V_0(T)$, there exists $u \in \Gamma^{-1}(x) - V_0(T)$. Then $u \notin \Gamma^{-1}(y)$, because $y \in V_0(T)$ evidently implies $\Gamma^{-1}(y) \subseteq V_0(T)$. We have $u \,.\, x = u$, $u \,.\, y = \infty$ and from $(u, u) \in T$, $(x, y) \in T$ we have $(u \,.\, x, u \,.\, y) = (u, \infty) \in T$ and $u \in V_0(T)$, which is a contradiction. □

The last two propositions lead us to the definition of certain relations on $V(G)$. For a given relation R on $V(G) \cup \{\infty\}$ we define

$$S(R) = \{(x, y) \in (V(G) - V_0(R)) \times (V(G) - V_0(R)) \mid \Gamma(x) = \Gamma(y),\ \Gamma^{-1}(x) - V_0(R) = \Gamma^{-1}(y) - V_0(R)\},$$

$$S'(R) = \{(x, y) \in V_0(R) \times (V(G) - V_0(R)) | \Gamma(y) \subseteq \Gamma(x),\ \Gamma^{-1}(y) \subseteq V_0(R)\} \cup$$
$$\cup \{(x, y) \in (V(G) - V_0(R)) \times V_0(R) | \Gamma(x) \subseteq \Gamma(y),\ \Gamma^{-1}(x) \subseteq V_0(R)\}\,.$$

Further, let

$$S = \{(x, y) | \Gamma(x) = \Gamma(y),\ \Gamma^{-1}(x) = \Gamma^{-1}(y)\}\,.$$

In the following theorem we shall characterize all tolerances on a graph algebra.

Theorem 1. *Let G be a directed graph, let $\mathfrak{A}(G)$ be the graph algebra on G, let T be a reflexive and symmetrtc binary relation on $V(G) \cup \{\infty\}$. The relation T is a tolerance on $\mathfrak{A}(G)$ if and only if $T = \Delta \cup T_0 \cup T_1 \cup T_2 \cup T_3$, where T_0, T_1, T_2, T_3 are binary relations on $V(G) \cup \{\infty\}$ described as follows:*

$T_0 = (V_0(T) \times \{\infty\}) \cup (\{\infty\} \cup V_0(T))$;

T_1 is an arbitrary symmetric relation on $V_0(T)$;

T_2 is an arbitrary symmetric relation contained in $S'(T)$;

T_3 is an arbitrary symmetric relation contained in $S(T)$.

Proof. The necessity of the condition follows from Propositions 1, 2, 3. We shall prove its sufficiency. The relation T is a tolerance on $\mathfrak{A}(G)$, if $(x_1, x_2) \in T$, $(y_1, y_2) \in T$ imply $(x_1 . y_1, x_2 . y_2) \in T$ for arbitrary elements x_1, x_2, y_1, y_2 of $V(G) \cup \{\infty\}$. Thus let $(x_1, x_2) \in T$, $(y_1, y_2) \in T$. There are four possibilities for the pair $(x_1 . y_1, x_2 . y_2)$; it is equal to (x_1, x_2) or to (x_1, ∞) or to (∞, x_2) or to (∞, ∞). In the first and the last cases always $(x_1 . y_1, x_2 . y_2) \in T$. Thus we shall study only the cases $(x_1 . y_1, x_2 . y_2) = (x_1, \infty)$ and $(x_1 . y_1, x_2 . y_2) = (\infty, x_2)$.

First, let $(x_1, x_2) \in \Delta$, i.e. $x_1 = x_2$. If $x_1 \in V_0(T) \cup \{\infty\}$, then $(x_1, \infty) \in T$, $(\infty, x_2) \in T$, thus always $(x_1 . y_1, x_2 . y_2) \in T$. If $x_1 \in V(G) - V_0(T)$ and $x_1 . y_1 = x_1$, then also $y_1 \in V(G) - V_0(T)$; otherwise x_1 would be in $V_0(T)$. If $x_1 . y_1 = x_1$, $x_1 . y_2 = \infty$, then $y_1 \in V(G) - V_0(T)$ and thus $(y_1, y_2) \in T_2 \cup T_3$. In both these cases $x_1 \in \Gamma^{-1}(y_1)$ implies $x_1 \in \Gamma^{-1}(y_2)$ (Propositions 2, 3) and thus $x_1 . y_2 = x_2 . y_2 = x_2$; the case $(x_1 . y_1, x_2 . y_2) = (x_1, \infty)$ cannot occur. Analogously, the case $(x_1 . y_1, x_2 . y_2) = (\infty, x_2)$ cannot occur, either.

Now, let $(x_1, x_2) \in T_0 \cup T_1$. Then $(x_1, \infty) \in T$, $(\infty, x_2) \in T$ and thus $(x_1 . y_1, x_2 . y_2) \in T$ for any $(y_1, y_2) \in T$.

Further, let $(x_1, x_2) \in T_2$. Then one of the elements x_1, x_2 is in $V_0(T)$; without loss of generality suppose that $x_2 \in V_0(T)$ and hence $x_1 \in V(G) - V_0(T)$. Then $(\infty, x_2) \in T$ and it remains to consider the pair (x_1, ∞). If $x_1 . y_1 = x_1$, then $y_1 \in \Gamma(x_1)$ and, as $T_2 \subseteq S'(T)$, we have $\Gamma(x_1) \subseteq \Gamma(x_2)$ and $y_1 \in \Gamma(x_2)$, which implies $x_2 y_1 \in E(G)$, $x_2 . y_1 = x_2$. If $y_2 \in V_0(T)$, then $(y_1, y_2) \in T_2$ and $\Gamma^{-1}(y_1) \subseteq V_0(T)$, which is a contradiction with $x_1 \in V(G) - V_0(T)$, $x_1 . y_1 = x_1$. If $y_2 \in V(G) - V_0(T)$ then, by virtue of the evident relation $y_1 \in V(G) - V_0(T)$, we have $(y_1, y_2) \in T_3 \subseteq S(T)$. As $x_1 \in \Gamma^{-1}(y_1)$, we also have $x_1 \in \Gamma^{-1}(y_2)$ and thus $y_2 \in \Gamma(x_1) \subseteq \Gamma(x_2)$ and $x_2 . y_2 = x_2$. Hence the case $(x_1 . x_2, y_1 . y_2) = (x_1, \infty)$ is impossible.

Finally, let $(x_1, x_2) \in T_3$. If $x_1 = x_1 . y_1$, then $y_1 \in V(G) - V_0(T)$, $(y_1, y_2) \in T_2 \cup T_3$. We have $y_1 \in \Gamma(x_1)$ and, as $T_3 \subseteq S(T)$, also $y_1 \in \Gamma(x_2)$ and $x_2 \in \Gamma^{-1}(y_1)$.

As $\Gamma^{-1}(y_1)$ contains a vertex of $V(G) - V_0(T)$, we have $(y_1, y_2) \notin T_2$ and thus $(y_1, y_2) \in T_3$. As $x_2 \in \Gamma^{-1}(y_1)$, also $x_2 \in \Gamma^{-1}(y_2)$ and $x_2 \cdot y_2 = x_2$. Analogously, if we suppose $x_2 \cdot y_2 = x_2$, we prove $x_1 \cdot y_1 = x_1$. The proof is complete. □

Corollary 1. *Let T be a tolerance on $\mathfrak{A}(G)$ such that $(x, \infty) \in T$ implies $x = \infty$. Then T is a subset of S.*

Corollary 2. *Let G be a strongly connected graph, let T be a reflexive and symmetric binary relation on $V(G) \cup \{\infty\}$. The relation T is a tolerance on $\mathfrak{A}(G)$ if and only if one of the following conditions holds:*

(a) $(x, \infty) \in T$ *for each* $x \in V(G)$;
(b) $(x, \infty) \notin T$ *for any* $x \in V(G)$ *and* $T \subseteq S$.

Theorem 2. *The lattice $LT(\mathfrak{A}(G))$ is a sublattice of the lattice of all binary relations on $V(G) \cup \{\infty\}$.*

Proof. Let T_1, T_2 be two elements of $LT(\mathfrak{A}(G))$. Then evidently $T_1 \cap T_2 \in LT(\mathfrak{A}(G))$; this holds for tolerances on every algebra. As T_1, T_2 fulfil the conditions of Theorem 1, evidently so does $T_1 \cup T_2$ and thus $T_1 \cup T_2 \in LT\mathfrak{A}(G))$. Hence the meet of T_1 and T_2 in $LT(\mathfrak{A}(G))$ is $T_1 \cap T_2$ and their join is $T_1 \cup T_2$, which implies the assertion. □

Now we turn to congruences, i.e. transitive tolerances. The lattice of all congruences on $\mathfrak{A}(G)$ will be denoted by $Con(\mathfrak{A}(G))$.

Theorem 3. *Let G be a directed graph, let $\mathfrak{A}(G)$ be the graph algebra on G, let C be an equivalence relation on $V(G) \cup \{\infty\}$. The relation C is a congruence on $\mathfrak{A}(G)$ if and only if one of its equivalence classes is $V_0(C) \cup \{\infty\}$ and any other of them is a subset of an equivalence class of $S(C)$.*

Proof. If C fulfils the described conditions, then it fulfils the conditions of Theorem 1 and $C \in LT(\mathfrak{A}(G))$; as it is an equivalence, $C \in Con(\mathfrak{A}(G))$. Conversely, suppose that $C \in Con(\mathfrak{A}(G))$. As $(x, \infty) \in C$ for each $x \in V_0(C) \cup \{\infty\}$ and C is transitive, we have $(x, y) \in C$ for any two elements x, y of $V_0(C) \cup \{\infty\}$. If $x \in V_0(C) \cup \{\infty\}$, $(x, y) \in C$, then $y \in V_0(C) \cup \{\infty\}$, because of transitivity. Thus $V_0(C) \cup \{\infty\}$ is one equivalence class of C. If $(x, y) \in C$ and at least one of the elements x, y is in $V(G) - V_0(C)$, then, according to Theorem 1, $(x, y) \in S(C)$ and the assertion is proved. □

Corollary 3. *Let C be a congruence on $\mathfrak{A}(G)$ such that $(x, \infty) \in C$ implies $x = \infty$. Then C is a refinement of S.*

Corollary 4. *Let G be a strongly connected graph, let C be an equivalence relation on $V(G) \cup \{\infty\}$. The relation C is a congruence on $\mathfrak{A}(G)$ if and only if C is either the universal relation on $V(G) \cup \{\infty\}$, or a refinement of S.*

Theorem 4. *The lattice* $Con(\mathfrak{A}(G))$ *is a sublattice of the lattice of all equivalence relations on* $V(G) \cup \{\infty\}$.

Proof is analogous to the proof of Theorem 2. □

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