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Ivan Chajda

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ALGEBRAS WHOSE PRINCIPAL CONGRUENCES FORM A SUBLATTICE OF THE CONGRUENCE LATTICE

IVAN CHAJDA, Přerov

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The problem under which conditions the set of all principal congruences on an algebra A is closed under joins and meets in Con A was investigated separately by a few authors. First, K. A. Baker [1] studied the so called *Principal Intersection Property* (briefly PIP), i.e. the property that for any $a_1, a_2, b_1, b_2 \in A$ the congruence

$$\Theta(a_1, b_1) \wedge \Theta(a_2, b_2)$$

is principal, i.e. it is equal to $\Theta(a, b)$ for some a, b of A.

P. Zlatoš [5] studied conditions under which the congruence

$$\Theta(a_1, b_1) \vee \Theta(a_2, b_2)$$

is principal for any a_1, a_2, b_1, b_2 of A; in such a case, A is said to have *Principal Compact Congruences*, briefly PCC.

Hence, if an algebra A has both PIP and PCC, the set of all principal congruences forms a sublattice of Con A.

Recall that a variety $\mathscr{V}$ is *congruence distributive* if Con A is distributive for each $A \in \mathscr{V}$. $\mathscr{V}$ is *congruence permutable* if $\Theta \cdot \Phi = \Phi \cdot \Theta$ for each $\Theta, \Phi \in$ Con A for any $A \in \mathscr{V}$. $\mathscr{V}$ is *arithmetic* if it is both congruence distributive and congruence permutable.

J. Duda [4] proved some remarkable results in solving the above problem:

Proposition 1 (Theorem 2 in [4]). *In a congruence permutable variety $\mathscr{V}$, the following conditions are equivalent:*

(1) *$\mathscr{V}$ has PCC;*

(2) *there exists a 6-ary polynomial s such that $s(x, u, x, y, u, v) = s(y, v, x, y, u, v)$ implies $x = y$ and $u = v$.*

An algebra $(H; \vee, \wedge, \rightarrow, 0, 1)$ with three binary and two nullary operations is a *Heyting algebra* if it satisfies

(a) $(H; \vee, \wedge, 0, 1)$ is a bounded distributive lattice,

(b) $x \rightarrow x = 1$,

(c) $(x \to y) \wedge y = y,\ x \wedge (x \to y) = x \wedge y$,
(d) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, and $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

Example 1. *Any variety of Heyting algebras has PCC.*

It is well known that such a variety is congruence permutable and we can put

$$s(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 \to x_3) \wedge (x_3 \to x_1) \wedge (x_2 \to x_5) \wedge (x_5 \to x_2).$$

Proposition 2. (Theorem 4 in [4]). *In an arithmetic variety $\mathscr{V}$, the following conditions are equivalent:*

(1) *$\mathscr{V}$ has PIP;*
(2) *there exists a 5-ary polynomial q such that $q(x, x, y, u, v) = q(y, x, y, u, v)$ if and only if $x = y$ or $u = v$ holds on any subdirectly irreducible member of $\mathscr{V}$.*

Example 2. *Any variety of Heyting algebras has PIP.*

It is known that such a variety is arithmetic and we can put

$$q(x_1, x_2, x_3, x_4, x_5) = [(x_1 \to x_3) \wedge (x_3 \to x_1)] \vee [(x_4 \to x_5) \wedge (x_5 \to x_4)].$$

Clearly, $q(x, x, y, u, v) = q(y, x, y, u, v)$ on a subdirectly irreducible Heyting algebra is satisfied if and only if

$$[(x \to y) \wedge (y \to x)] \vee [(u \to v) \wedge (v \to u)] = 1, \quad \text{i.e.}$$
$$(x \to y) \wedge (y \to x) = 1 \quad \text{or} \quad (u \to v) \wedge (v \to u) = 1,$$

which is equivalent to $x = y$ or $u = v$.

Example 3. *Any discriminator variety $\mathscr{V}$ satisfies PCC* (see Example 2 in [4]). $\mathscr{V}$ is clearly arithmetic and we can put $s(x_1, x_2, x_3, x_4, x_5, x_6) = t(x_1, t(x_3, x_1, x_4), x_2)$, where $t(x, y, z)$ is the discriminator on $\mathscr{V}$. Moreover, *$\mathscr{V}$ satisfies PIP* (Example 3 in [4]), since we can put

$$q(x_1, x_2, x_3, x_4, x_5) = t(t(x_1, x_3, x_4), t(x_1, x_3, x_5), x_5).$$

Corollary 1. *Let $\mathscr{V}$ be a discriminator variety and $A \in \mathscr{V}$. The set of all principal congruences on A forms a sublattice of* Con A.

Examples 1 and 2 imply one result also for an algebra of the lattice type:

Corollary 2. *Let H be a Heyting algebra. The set of all principal congruences on H forms a sublattice of* Con H.

Since Heyting algebras are special cases of distributive lattices, there is a question if Corollary 2 can be formulated also for other lattices. The disadvantage is that Propositions 1 and 2 require the congruence permutability which is not satisfied in lattice varieties. In the sequel we are going to show that Corollary 2 can be "localized" and this local version can be proved also for some other lattices.

The starting points is K. Baker's result:

Proposition 3 (Theorems 2.8, 2.9 in [1]). *In a congruence distributive variety $\mathscr{V}$, the following conditions are equivalent*:

(1) *$\mathscr{V}$ has PIP*;

(2) *there exist 4-ary polynomials d_0, d_1 such that $\Theta(a_1, b_1) \wedge \Theta(a_2, b_2) = \Theta(d_0(a_1, b_1, a_2, b_2), d_1(a_1, b_1, a_2, b_2))$ holds for each $a_1, a_2, b_1, b_2 \in A \in \mathscr{V}$*;

(3) *there exist 4-ary polynomials d_0, d_1 such that $d_0(x, y, u, v) = d_1(x, y, u, v)$ if and only if $x = y$ or $u = v$ holds on any subdirectly irreducible member of $\mathscr{V}$.*

Now, we can define the local property:

Definition. Let $\mathscr{V}$ be a variety with a nullary operation 0. $\mathscr{V}$ satisfies 0-PIP if for each $a_1, a_2 \in A \in \mathscr{V}$ there exists $a \in A$ such that

$$\Theta(a_1, 0) \wedge \Theta(a_2, 0) = \Theta(a, 0).$$

$\mathscr{V}$ satisfies 0-PCC if for each $a_1, a_2 \in A \in \mathscr{V}$ there exists $b \in A$ such that

$$\Theta(a_1, 0) \vee \Theta(a_2, 0) = \Theta(b, 0).$$

Varieties having 0-PCC were characterized in [3]. For 0-PIP, we can simplify Proposition 3 by putting $d_1 = 0$ and assuming $b_1 = 0 = b_2$, i.e., the second and fourth variables in d_0 are equal to 0. Hence, we obtain only one binary polynomial:

Lemma. *Let $\mathscr{V}$ be a congruence distributive variety with a nullary operation 0. The following conditions are equivalent*·

(1) *$\mathscr{V}$ has 0-PIP*;

(2) *there exists a binary polynomial $d(x, y)$ such that $\Theta(a_1, 0) \wedge \Theta(a_2, 0) = \Theta(d(a_1, a_2), 0)$ for each $a_1, a_2 \in A \in \mathscr{V}$*;

(3) *there exists a binary polynomial $d(x, y)$ such that $d(x, y) = 0$ if and only if $x = 0$ or $y = 0$ holds on any subdirectly irreducible member of $\mathscr{V}$.*

The proof is a word-for-word analogue of that of K. A. Baker [1], and hence omitted.

Theorem 1. *Let D be a distributive lattice with the least element 0 (or the greatest element 1). The set of all principal congruences of the form $\Theta(x, 0)$ (or $\Theta(x, 1)$) forms a sublattice of* Con *D*.

Proof. Let $\mathscr{V}$ be a variety of all distributive lattices with the least element 0. By Theorem 5 in [3], $\mathscr{V}$ has 0-PCC. It is well known that $\mathscr{V}$ is congruence distributive. A distributive lattice is subdirectly irreducible if and only if it is either one element or a two element chain. Thus the polynomial $d(x, y) = x \wedge y$ satisfies (3) of Lemma, i.e. $\mathscr{V}$ has 0-PIP. For a variety of all distributive lattices with 1, the proof is dual.

An algebra A with a nullary operation 0 is weakly regular (see e.g. [3]) if each two congruences $\Phi, \Theta \in \mathrm{Con}\, A$ coincide whenever $[0]_\Phi = [0]_\Theta$.

Theorem 2. *Let D be a weakly regular distributive lattice with the least element* 0. *The set of all principal congruences of D forms a sublattice of* Con D.

Proof. By Theorem 1 in [2], D is weakly regular if and only if for each a, b of D there exists $c \in D$ such that $\Theta(a, b) = \Theta(c, 0)$. Hence, the set of all principal congruences in D coincides with the set of all principal congruences of the form $\Theta(x, 0)$. By Theorem 1, we obtain the assertion.

References

[1] *Baker K. A.:* Primitive satisfaction and equational problems for lattices and other algebras, Trans. Amer. Math. Soc. *190* (1974), 125—150.

[2] *Chajda I.:* Weakly regular lattices, Math. Slovaca *35* (1985), 387—391.

[3] *Chajda I.:* Forms of the congruence principality, Period. Mathem. (Budapest), to appear.

[4] *Duda J.:* Unified polynomials for congruence principality, Math. Slovaca, to appear.

[5] *Zlatoš P.:* A Mal'cev condition for compact congruences to be principal, Acta Sci. Math. (Szeged), *43* (1981), 383--387.

Author's address: 750 00 Přerov, třída Lidových milicí 22, Czechoslovakia.