Luděk Zajíček

Porosity, derived numbers and knot points of typical continuous functions

Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 1, 45–52

Persistent URL: `http://dml.cz/dmlcz/102277`

Czechoslovak Mathematical Journal, 39 (114) 1989, Praha

POROSITY, DERIVED NUMBERS AND KNOT POINTS OF TYPICAL CONTINUOUS FUNCTIONS

LUDĚK ZAJÍČEK, Praha

(Received June 23, 1986)

INTRODUCTION

The main result of the present article is Theorem 2 which improves an old result of V. Jarník [8] concerning the behaviour of typical continuous functions. We also improve a Thomson's [9] theorem on Dini derivates of continuous functions.

In what follows we denote by $C([0, 1])$ the space of all continuous functions on $[0, 1]$ with the supremum norm $\|\cdot\|$. We put $\bar{R} = R \cup \{-\infty, \infty\}$. Recall that we say that the typical continuous function on $[0, 1]$ has a *property* P if all functions from a residual subset of $C([0, 1])$ have this property. A point x is said to be a *knot point* of f if $D^+ f(x) = D^- f(x) = \infty$ and $D_+ f(x) = D_- f(x) = -\infty$, where D^+, D^-, D_+, D_- are Dini derivates. An extended real number $d \in \bar{R}$ is called a *right* (*left*) *essential derived number* of f at x if there exists a set E such that the right (left) upper outer density of E at x equals 1 and $\lim_{y \to x, y \in E} (f(y) - f(x))(y - x)^{-1} = d$. We shall say that x is an essential knot point of f if any extended real number $d \in \bar{R}$ is a bilateral (i.e. simultaneously right and left) essential derived number of f at x.

V. Jarník [7] proved that for the typical continuous function f on $[0, 1]$ almost all points $x \in [0, 1]$ are knot points of f. As was pointed out by K. M. Garg [3], the general theorems on non-differentiable functions together with Jarník's result imply that for the typical continuous function on $[0, 1]$ the set of points which are not its knot points is a first category set of measure zero. V. Jarník [8] improved his own result by showing that for the typical continuous function f on $[0, 1]$ almost all points are essential knot points of f. In the present article we show that for the typical continuous function f the set of points which are not essential knot points (or knot points in a much stronger sense which is defined via the notion of a generalized porosity) is not only a first category set of measure zero but, moreover, it is σ-porous in a very strong sense ($\sigma - [g]$-totally porous). The result of Theorem 2 is closely connected with results of [1], [4] and [5], and the validity of similar statements is known. For example, a similar result on ordinary knot points was obtained independently by G. Petruska (a private communication). On the other hand, I believe that the notion of a σ-$[g]$-totally porous set is new and can be important in a search of the best possible results concerning exceptional sets of typical continuous functions.

In the formulation of our results we use the new notion of a $[g]$-derived number which is stronger than the notion of an essential derived number defined by V. Jarník. The notion of a $[g]$-derived number is closely related to the porosity Dini derivates of [2] and to the notions used in [4]. It enables us to give a natural formulation of a theorem (Theorem 1*) which generalizes both Theorem 1 of [12] and Thomson's Theorem 3.2 of [9].

1. DEFINITIONS

In the literature several definitions of generalized porosity are used. In [11] the notions of (g)-*porous sets*, (g, c)-*porous sets*, $\langle g\rangle$-*porous sets* and $\langle H\rangle$-*porous sets* are defined in an arbitrary metric space. Some slightly different notions of generalized porosity were considered in [2], [4], and [10].

In the present article we consider the situation when the relative gaps in a set E near a point x are extremely big. For this situation the notions of (g)-porosity used in [11] and [2] are not convenient, but it would be possible to use the notion of $\langle g\rangle$-porosity from [11] or the notion of (g)-porosity from [10]. In the sequel we shall use another new notion of $[g]$-porosity. The reason is that from the point of view of applications in the present article all the three notions are equivalent (cf. Note 1, a, b) and it is most convenient for us to work with the notion of $[g]$-porosity.

Let $E \subset R$ be a set and let $I \subset R$ be an interval. Then we write $p(E, I)$ for the length of the largest open subinterval of I that contains no point of E.

We shall denote by G the system of all positive increasing functions g on $(0, \infty)$ for which $g(x) > x$ for all x.

Definition 1. Let $g \in G$. We say that $E \subset R$ is $[g]$-*porous from the right* (*from the left*) *at a point* $x \in R$ if there is a sequence of positive numbers (h_n) such that $h_n \searrow 0$ and $g(p(E, (x, x + h_n))) > h_n$ (or $g(p(E, (x - h_n, x))) > h_n$) for all n.

The system of the form $D = \{[a + nh, a + (n + 1) h]; n \text{ is an integer}\}$ where $a \in R$ and $h > 0$ will be called an *equidistant division of* R with the norm h.

Definition 2. Let $g \in G$. We say that $E \subset R$ is $[g]$-*totally porous* if for any $\varepsilon > 0$ there exists an equidistant division D of R with the norm less than ε such that $g(p(E, I)) >$ length I for any $I \in D$.

Note 1. (a) Let h be increasing on $[0, \infty)$ and let $h(0) = 0$ (i.e. h is an admissible porosity function in the sense of [10], p. 197). Put $g(x) = x + h(x)$. Then it is easy to see that $g \in G$ and that E is (h)-porous at x in the sense of [10] whenever E is $[g]$-porous at x.

(b) Let $h \in G$. Put $g(x) = x/2 + h(x/2)$. Then $g \in G$ and if E is $[g]$-porous at x then E is $\langle h\rangle$-porous at x in the sense of [11].

(c) Let $h \in G$. Put $g(x) = (h(x) + x)/2$. Then $g \in G$ and it is easy to prove that E is bilaterally $[h]$-porous at all its points whenever E is $[g]$-totally porous.

(d) If $g(x) = cx$, $c > 1$, then any $[g]$-totally porous set is globally porous in the sense of [6].

The following definition is very similar to the definitions from [2].

Definition 3. Let $g \in G$ and let f be a real function on R. We say that an extended real number $y \in \bar{R}$ is a *right (left) $[g]$-derived number of f at a point $x \in R$* if there is a set $E \subset R$ such that $\lim_{z \to x, z \in E} (f(z) - f(x))(z - x)^{-1}$ and $R \setminus E$ is $[g]$-porous from the right (from the left) at x.

Note 2. It is easy to see that $y \in \bar{R}$ is a right (left) $[g]$-derived number of f at x iff for any neighbourhood V of y the set $\{z; (f(z) - f(x))(z - x)^{-1} \notin V\}$ is $[g]$-porous from the right (from the left) at x.

Definition 4. Let $g \in G$ and let f be a real function on R. We say that $x \in R$ is a *$[g]$-knot point of f* if any $y \in \bar{R}$ is a bilateral $[g]$-derived number of f at x.

Note 3. Note 2 easily implies that x is a $[g]$-knot point of f iff for all real numbers $a < b$ the set $\{z; (f(z) - f(x))(z - x)^{-1} \notin (a, b)\}$ is bilaterally $[g]$-porous at x.

2. A GENERALIZATION OF THOMSON'S THEOREM

In this section we shall use without an explanation the terminology from Thomson's monograph [10]. In [9] B. S. Thomson proved the following theorem (see also [10], p. 153).

Theorem T. *Let S be a local system of sets that has the following property. For each x and for each set $M \in S(x)$, M has porosity at x less than 1, at least from one side. Then, for any continuous function f, the set of points*

$$\{x; (S) - \underline{D} f(x) \neq \underline{D}^{\#} f(x) \text{ or } (S) - \bar{D} f(x) \neq \bar{D}^{\#} f(x)\}$$

is of the first category.

This theorem generalizes Corollary 1 from [12] but does not generalize Theorem 1 from [12]. Therefore the answer to the question from [9] whether the assumptions concerning S in Theorem T can be relaxed is affirmative. In fact, it is not difficult to rewrite the proof of Theorem T in a more general context and obtain the following theorem.

Theorem 1. *Let $g \in G$ and let S be a local system of sets that has the property that, for each x and for each set $M \in S(x)$, M is not bilaterally $[g]$-porous at x. Then, for any continuous function f, the set of points*

$$\{x; (S) - \underline{D} f(x) \neq \underline{D}^{\#} f(x) \text{ or } (S) - \bar{D} f(x) \neq \bar{D}^{\#} f(x)\}$$

is of the first category.

Denote by $S^{+}(x)$ ($S^{-}(x)$) the system of all sets $M \subset R$ for which $x \in M$ and $R \setminus M$ is $[g]$-potous at x from the right (from the left). We suppose that $g \in G$ is fixed.

Put $S^+ = \{S^+(x);\ x \in R\}$, $S^- = \{S^-(x);\ x \in R\}$ and $S_b = S^+ \wedge S^-$. Clearly S^+, S^-, S_b are local systems and

$$(S_b) - \lim_{y \to x} (f(y) - f(x))(y - x)^{-1} = c$$

iff c is a bilateral $[g]$-derived number of f at x. Thus Note 2 can be considered as a consequence of the obvious fact that S_b satisfies the property $[J_2]$ from [10].

Let S_b^* be the dual of S_b. The assumption concerning S in Theorem 1 is clearly equivalent to the condition $S \ll S_b^*$. Consequently, we have

$$(S_b^*) - \bar{D} f(x) \leqq (S) - \bar{D} f(x) \leqq (S_0) - \bar{D} f(x) = \bar{D} f(x) \leqq \bar{D}^{\#} f(x) .$$

Thus we see that Theorem 1 says nothing else than that for any continuous function f at every point x with the possible exception of a first category set we have $(S_b^*) - \bar{D} f(x) = \bar{D}^{\#} f(x)$ and $(S_b^*) - \underline{D} f(x) = \underline{D}_{\#} f(x)$.

Now observe that

$$(S_b^*) - \bar{D} f(x) = (S_b^*) - \limsup_{y \to x} (f(y) - f(x))(y - x)^{-1}$$

and that for any local system T the equality

$$(T^*) - \limsup_{y \to x} h(y) = (S_0) - \limsup_{y \to x} h(y) = c$$

implies

$$(T) - \lim_{y \to x} h(y) = c .$$

Consequently, we can rewrite Theorem 1 in the following equivalent form.

Theorem 1*. *Let $g \in G$ and let f be a continuous function on R. Then at every point x, with the possible exception of a first category set, $\underline{D}^{\#} f(x)$ and $\bar{D}^{\#} f(x)$ are bilateral $[g]$-derived numbers of f at x.*

Proof. It is clearly sufficient to prove that the set

$$A = \{x;\ \bar{D}^{\#} f(x) \text{ is not a right } [g]\text{-derived number of } f \text{ at } x\}$$

is a first category set. It is easy to see that

$$A = \bigcup \{A_{r,s,n};\ r < s \text{ rational, } n \text{ natural}\} ,$$

where $A_{r,s,n}$ is the set of all points for which $\bar{D}^{\#} f(x) > s$ and

$$(1) \quad g(p(\{y;\ f(y) - f(x) < r(y - x)\},\ (x, x + h))) \leqq h \quad \text{for} \quad 0 < h \leqq n^{-1} .$$

Therefore it is sufficient to prove that all sets $A_{r,s,n}$, $r < s$, are nowhere dense. Suppose on the contrary that a set $A_{r,s,n}$ is dense in an interval (a, b). We can suppose without any loss of generality that $r = 0$. (If $r \neq 0$, we can consider the function $f(x) - rx$.) We shall prove that f is nonincreasing on (a, b) which will contradict the fact that $\bar{D}^{\#} f(x) > 0$ for $x \in A_{r,s,n}$. Thus suppose that there are numbers $a < c < d < b$ for which $f(c) < f(d)$. Choose $f(c) < p < f(d)$ and put $s = \sup \{t; f(t) < p,\ t < d\}$. Obviously $f(s) = p$, $c < s < d$ and $f(t) \geqq p$ for $s \leqq$

$\leqq t \leqq d$. Choose $0 < u < n^{-1}$ for which $s + u < d$. Since $g(u) > u$, f is continuous and $A_{r,s,n}$ is dense in (c, d), we can use the definition of the number s and find a point $x \in A_{r,s,n}$ such that $x < s$, $s + u - x < n^{-1}$, $s + u - x < g(u)$ and $f(x) < p$. Putting $h = s + u - x$ we see that (1) is not satisfied, since $f(y) - f(x) \geqq \geqq r(y - x) = 0$ for $y \in (s, s + u)$, $(s, s + u) \subset (x, x + h)$ and $g(u) > h$. This is a contradiction which completes the proof.

Now we can reformulate Thomson's question.

Problem 1. *Let S be a local system such that for any $g \in G$ there exists a second category set $M_g \subset R$ such that for any $x \in M_g$ there is a set $E \in S(x)$ which is bilaterally $[g]$-porous at x. Does there exist a continuous function f such that $\bar{D}^{\#} f(x) > (S) - \bar{D} f(x)$ for all x from a second category set?*

We can, of course, formulate the same problem in the "dual language".

Problem 1*. *Let S be a local system such that for any continuous function f at every point x, with the possible exception of a first category set, $(S) - \lim_{y \to x} (f(y) - f(x))(y - x)^{-1} = \bar{D}^{\#} f(x)$. Does there exist $g \in G$ and a residual set $B \subset R$ such that $T \cup \{x\} \in S(x)$ whenever $R \setminus T$ is bilaterally $[g]$-porous at x and $x \in B$?*

3. KNOT POINTS OF TYPICAL CONTINUOUS FUNCTIONS

Construction 1. Let s be a K-Lipschitz function on $[0, 1]$, $c \in R$, n a natural number and $0 < v < n^{-1}$ a real number. We shall denote by $f = f(s, c, n, v)$ the function on $[0, 1]$ uniquely determined by the following properties:

(a) $f(k/n) = s(k/n)$ for $k = 0, \ldots, n$,

(b) $f(x) = s(x) + c(x - k/n)$ for $x \in [k/n, (k + 1)/n - v]$, $k = 0, \ldots, n - 1$,

(c) f is linear on any interval $[(k + 1)/n - v, (k + 1)/n]$, $k = 0, \ldots, n - 1$.

We shall need the following simple lemma.

Lemma 1. *Let s, K, c, n, v, f be as in Construction 1. Then $\|f - s\| \leqq \leqq (2K + |c|)\, n^{-1}$.*

Proof. If $x \in [k/n, (k + 1)/n - v]$, $k = 0, \ldots, n - 1$, then

$$|f(x) - s(x)| \leqq |f(x) - f(k/n)| + |s(x) - s(k/n)| \leqq (K + |c|)\, n^{-1}.$$

If $x \in [(k + 1)/n - v, (k + 1)/n]$, $k = 0, \ldots, n - 1$, then observe that

$$|f(x) - s(x)| \leqq \max\left(|s(x) - f((k + 1)/n)|,\ |s(x) - f((k + 1)/n - v)|\right).$$

Since

$$|s(x) - f((k + 1)/n)| \leqq Kn^{-1} \quad \text{we have} \quad |s(x) - f((k + 1)/n - v)| \leqq$$
$$\leqq |s(x) - s((k + 1)/n - v)| + |s((k + 1)/n - v) - f((k + 1)/n - v)| \leqq$$
$$\leqq Kn^{-1} + (K + |c|)\, n^{-1};$$

the lemma is proved.

In the sequel, let $g \in G$ be a fixed continuous function.

Construction 2. Suppose that a polynomial s and real numbers $a < b$, $\delta > 0$ are given. Then we put $c = (a + b)/2$, $d = (b - a)/2$ and define numbers $K = K(s)$, $n = n(a, b, s, \delta)$, $v = v(a, b, s, \delta)$, $\varepsilon = \varepsilon(a, b, s, \delta)$ and an open ball $U(a, b, s, \delta) \subset \subset C([0, 1])$ in the following way. We put $K = \sup \{|s'(x)|;\ x \in [0, 1]\}$. Choose a natural number n such that

$$(2) \qquad (2K + |c|)\, n^{-1} < \delta/2 \quad \text{and} \quad n^{-1} < \delta\,.$$

Further, find $v > 0$ and $0 < p < v$, $0 < \varepsilon$ such that

$$(3) \qquad v < 1/2n\,, \quad g(n^{-1} - 2v) > n^{-1}\,,$$

$$(4) \qquad g(v - p) > v$$

and

$$(5) \qquad \varepsilon < \delta/2\,, \quad 2\varepsilon/p < d\,.$$

Finally, let $U(a, b, s, \delta) \subset C([0, 1])$ be the open ball with the center $f = f(s, c, n, v)$ (cf. Construction 1) and the radius ε.

We shall need the following lemma, in which we use the notation from Construction 2.

Lemma 2. (a) *For any* $h \in U(a, b, s, \delta)$ *we have* $\|h - s\| < \delta$.

(b) *For any* $h \in U(a, b, s, \delta)$ *and* $x \in \bigcup_{k=0}^{n-1} [k/n, (k + 1)/n - 2v]$ *the inequality* $g(p(\{y; (h(y) - h(x))(y - x)^{-1} \notin [a, b]\}, [x, x + v])) > v$ *holds.*

Proof. The assertion (a) immediately follows by Lemma 1, (2) and (5). To prove (b) suppose that $x \in [k/n, (k + 1)/n - 2v]$, $k \in \{0, \ldots, n - 1\}$. On account of (4) it is sufficient to observe that

$$\{y; (h(y) - h(x))(y - x)^{-1} \notin [a, b]\} \cap [x + p, x + v] = \emptyset\,.$$

In fact, choose $y \in [x + p, x + v]$ and consider $f = f(s, c, n, v)$. Then by the construction of f, definition of $U(a, b, s, \delta)$ and (5) we conclude

$$\begin{aligned} |(h(y) - h(x))(y - x)^{-1} - c| &\leqq |(f(y) - f(x))(y - x)^{-1} - c| + \\ &+ |(h(y) - f(y))(y - x)^{-1}| + |(f(x) - h(x))(y - x)^{-1}| \leqq \\ &\leqq 0 + \varepsilon/p + \varepsilon/p < d\,. \end{aligned}$$

Theorem 2. *Let* $g \in G$. *Then for the typical continuous function* f *on* $[0, 1]$ *the set of points from* $[0, 1]$ *which are not* $[g]$*-knot points of* f *is* σ-$[g]$*-totally porous.*

Proof. Obviously for any $g \in G$ there exists a continuous function $\bar{g} \in G$, $\bar{g} \leqq g$. Consequently, we can suppose that g is continuous and therefore we can use Lemma 2. Choose a sequence of polynomials $(s_p)_{p=1}^{\infty}$ which is dense in $C([0, 1])$. Further, for $a < b$ put $V(a, b) = \limsup_{p\to\infty} U(a, b, s_p, p^{-1})$ and $P = \bigcap\{V(a, b);\ a < b$ are rational$\}$, where $U(a, b, s_p, p^{-1})$ are as in Construction 2. Lemma 2 (a) easily implies that any set $V(a, b)$ and consequently also P is a residual subset of $C([0, 1])$. Now choose a function $f \in P$ and denote by A the set of all $x \in [0, 1)$ for which there is $y \in \bar{R}$ which is not the right $[g]$-derived number of f at x. It is easy to see that it is sufficient to prove that A is σ-$[g]$-totally porous. Further, denote by $A(a, b, m)$ the set of all $x \in [0, 1)$ for which

$$g(p(\{y; (f(y) - f(x))(y - x)^{-1} \notin [a, b]\}, (x, x + h))) \leqq h$$

whenever $0 < h \leqq \min(m^{-1}, 1 - x)$. It is easy to see (cf. Note 2 and Note 3) that $A \subset \bigcup\{A(a, b, m);\ m$ is natural and $a < b$ are rational$\}$. Thus it is sufficient to prove that all $A(a, b, m)$ are $[g]$-totally porous. Fix a, b, m and choose $\omega > 0$. Since $f \in V(a, b)$ we can choose a positive integer p for which $p^{-1} < \min(\omega, m^{-1})$ and $f \in U(a, b, s_p, p^{-1})$. Since by (2)

$$v = v(a, b, s_p, p^{-1}) < n^{-1} = (n(a, b, s_p, p^{-1}))^{-1} < p^{-1} < m^{-1},$$

we deduce from Lemma 2(b) that

$$A(a, b, m) \cap \bigcup_{k=0}^{n-1} [k/n, (k + 1)/n - 2v] = \emptyset .$$

Consequently, (2) implies $g(p(A(a, b, m), [k/n, (k + 1)/n])) > n^{-1}$ for any $k \in \{0, \ldots, n - 1\}$. Since the norm of the division $\{[k/n, (k + 1)/n]\}_{k=-\infty}^{\infty}$ is $n^{-1} < p^{-1} < \omega$, the proof is complete.

It would be interesting to know whether Theorem 2 can be improved. We shall formulate two yes-no problems in this direction. We start with the following natural definition.

Definition 5. Let S be a local system on R (see [10]). We shall say that $b \in \bar{R}$ is an (S)-*derived number of a function* f at a point x if $(S) - \lim_{y \to x} (f(y) - f(x))(y - x)^{-1} = b$. We shall say that a point $x \in R$ is an (S)-*knot point of* f if any $b \in \bar{R}$ is an (S)-derived number of f at x.

Problem 2. *Let S be a local system of sets on R such that for the typical continuous function on $[0, 1]$ the set of points from $[0, 1]$ which are not (S)-knot points of f is a first category set of measure zero. Does there exist $g \in G$ and a first category set N of measure zero such that $B \cup \{x\} \in S(x)$ whenever $x \in [0, 1] \setminus N$ and $R \setminus B$ is bilaterally $[g]$-porous at x?*

Problem 3.*) *Let V be a family of subsets of $[0, 1]$ such that for the typical continuous function f on $[0, 1]$ the set of points from $[0, 1]$ which are not knot points of f belongs to V. Does there exist $g \in G$ such that any σ-$[g]$-totally porous set belongs to V?*

Let $g \in G$ be given. The results of [4] and [5] imply that the typical continuous

*) Added in May, 1988. Problem 3 has a simple negative answer. In fact, it is sufficient to put $V = \{A \subset [0, 1]: 1/2 \notin A\}$. A natural generalization of Problem 3 is discussed in "L. Zajíček, The differentiability structure of typical functions in $C[0, 1]$, Real Analysis Exchange *13* (1987—88), pp. 119, 103—106, 93".

function f on $[0, 1]$ has the property that the set $\{x \in [0, 1]; f(x) = h(x)\}$ is bilaterally $[g]$-porous whenever h is respectively a Lipschitz or a monotone function. In the following proposition which easily follows from Theorem 2 we deal with more general functions h, but we obtain a less restrictive sense of the smallness of the set $\{x \in [0, 1]; f(x) = h(x)\}$.

Proposition 1. *Let $g \in G$. Then for the typical continuous function f on $[0, 1]$ the set $\{x; f(x) = h(x)\}$ is σ-$[g]$-porous whenever h is a function on $[0, 1]$ such that the set of knot points of h is σ-$[g]$-porous.*

Proof. Theorem 2 implies that for the typical continuous function f on $[0, 1]$ the set $N(f)$ of points from $[0, 1]$ which are not $[g]$-knot points of f is σ-$[g]$-porous (cf. Note 1, (c)). Let h be a function on $[0, 1]$ for which the set $K(h)$ of knot points of h is σ-$[g]$-porous. It is clearly sufficient to prove that the set $M = \{x; f(x) = h(x)\} \setminus (N(f) \cup K(h))$ is $[g]$-porous. Let $a \in M$ be given. Since $a \notin K(h)$, there exist real numbers $c < d$ such that the set $P := \{a\} \cup \{y; (h(y) - h(a)) (y - a)^{-1} \notin (c, d)\}$ is a unilateral neighbourhood of a. Since $a \notin N(f)$ we have (cf. Note 3) that the set $Q = \{y; (f(y) - f(a)) (y - a)^{-1} \notin (c, d)\}$ is bilaterally $[g]$-porous at x. Therefore $M = (M \setminus P) \cup (M \cap P) \subset (M \setminus P) \cup Q$ is (unilaterally) $[g]$-porous at x.

References

[1] *A. M. Bruckner* and *J. Haussermann:* Strong porosity features of typical continuous functions, Acta Math. Hungar. *45* (1985), 7—13.

[2] *A. M. Bruckner* and *B. S. Thomson:* Porosity estimates for the Dini derivatives, Real Analysis Exchange *9* (1983—84), 508—538.

[3] *K. M. Garg:* On a residual set of continuous functions, Czechoslovak Math. J. *20 (95)* (1970), 537—543.

[4] *J. Haussermann:* Porosity characterizations of intersection sets with the typical continuous function, Real Analysis Exchange *9* (1983—84), 386—389.

[5] *P. D. Humke* and *M. Laczkovich:* Typical continuous functions are virtually nonmonotone, Proc. Amer. Math. Soc. *94*, (1985), 244—248.

[6] *P. D. Humke* and *T. Vessey:* Another note on σ-porous sets, Real Analysis Exchange *8* (1982—83), 261—271.

[7] *V. Jarník:* Über die Differenzierbarkeit stetiger Funktionen, Fund. Math. *21* (1933), 48—58.

[8] *V. Jarník:* Sur les nombres dérivés approximatifs, Fund. Math. *22* (1934), 4—16.

[9] *B. S. Thomson:* Some theorems for extreme derivates, J. London Math. Soc. (2), *27* (1983), 43—50.

[10] *B. S. Thomson:* Real Functions, Lect. Notes in Math. 1170, Springer-Verlag 1985.

[11] *L. Zajíček:* Sets of σ-porosity and sets of σ-porosity (q), Časopis Pěst. Mat. *101* (1976), 350—359.

[12] *L. Zajíček:* On Dini derivatives of continuous and monotone functions, Real Analysis Exchange *7* (1981—82), 233—238.

Author's address: 186 00 Praha 8, Sokolovská 83, Czechoslovakia (Matematicko-fyzikální fakulta Univerzity Karlovy).