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COMPLETELY MEET-IRREDUCIBLE TOLERANCES IN DISTRIBUTIVE NOETHERIAN LATTICES

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In [3], it has been shown that the poset of all meet-irreducible tolerances in any finite distributive lattice L is order-isomorphic to the set of all intervals in the poset of all join-irreducible elements of L ordered by inclusion. Note that in the finite case the poset of all join-irreducible elements is order-isomorphic to the poset of all meet-irreducible elements. In this paper, the above result will be generalized to distributive Noetherian lattices.

Recall that a lattice L is said to be *Noetherian* if it satisfies the Ascending Chain Condition (ACC): each non-empty subset (or, equivalently, subchain) in L contains a maximal element (see [1] for details). A lattice is said to be *upper bounded* if it contains a greatest element, *conditionally complete* if any subset of L with an upper (lower) bound in L has also a least upper (greatest lower) bound in L, and *compact* if each element of L is compact, i.e. each subset X of L includes a finite subset Y such that $\bigvee X = \bigvee Y$ whenever $\bigvee X$ exists.

Proposition 1. *A lattice is Noetherian if and only if it is upper bounded, conditionally complete and compact.*

Proof. Let L be a Noetherian lattice. As L contains a maximal element, this is the greatest element of L. Let X be any nonempty subset of L. Define $X^\vee = \{x \in L \mid \exists_{n \in N} \exists_{x_0,\ldots,x_n \in X} (x = x_0 \vee \ldots \vee x_n)\}$. This subset $X^\vee$ contains a maximal element; denote it by m. Since $m \vee x \in X^\vee$ for any $x \in X$, it follows that m is an upper bound of X in L. If z is also an upper bound of X in L, then we have $m = x_0 \vee \ldots \vee x_n \leqq z$ for suitable elements $x_0, \ldots, x_n \in X$, hence $m = \bigvee X$ in L. Consequently, L is conditionally complete. We have just shown that $\bigvee X \in X^\vee$, and so there exists a finite subset $Y \subseteq X$ such that $\bigvee X = \bigvee Y$. Thus the lattice L is compact. Conversely, let L be an upper bounded, conditionally complete and compact lattice. Let X be an arbitrary non-empty subchain in L. Then $\bigvee X \in X^\vee = X$, therefore X contains a maximal element. Q.E.D.

Completely (=strictly) meet-irreducible tolerances in distributive lattices were recognized in [2] and [3] as tolerances formed by a proper prime ideal that is maximal among all ideals not containing a given element b, and by a proper dual prime ideal that is maximal among all dual ideals not containing a given element a, where $a < b$. See [2] and [3] for definitions and basic properties.

Proposition 2. *For a distributive Noetherian lattice L, the set of all completely meetirreducible tolerances in L ordered by inclusion is order-isomorphic to the set of all intervals in the poset of all meet-irreducible elements of Lwith completely meet-irreducible greatest elements, which is ordered by inclusion, formally*

$$\mathbf{CM}(\mathrm{TL}(L)) \cong (\mathbf{M}(L) \times \mathbf{CM}(L)) \cap \leqq .$$

Proof. Let $T \in \mathbf{CM}(\mathrm{TL}(L))$, i.e. $T = (I \times I) \cup (F \times F)$, $a, b \in L$, $a < b$, where I is a proper prime ideal which is maximal among all ideals not containing b, and F is a proper dual prime ideal which is maximal among all dual ideals not containing a. Then $J = L \setminus F$ is a proper prime ideal and $J \subseteq I$ holds. Ideals in Noetherian lattices are principal, and greatest elements of prime ideals are meet-irreducible (cf. [1]). If $\bigvee I = \bigwedge X$ for some $X \subseteq L$, then $\bigvee I \notin X$ would imply, in view of maximality of I with respect to b, that $b \leqq x$ for any $x \in X$. This would yield $b \leqq \bigwedge X = \bigvee I$, which contradicts the assumption. Therefore $\bigvee I$ is completely meet-irreducible. Define $h(T) = [\bigvee J, \bigvee I] \in (\mathbf{M}(L) \times \mathbf{CM}(L)) \cap \leqq$. It is obvious that h is injective and order-preserving. It remains to show that it has an inverse.

For $[v, w] \in (\mathbf{M}(L) \times \mathbf{CM}(L)) \cap \leqq$ define $g([v, w]) = ((L \setminus (v\rangle) \times (L \setminus (v\rangle)) \cup ((w\rangle \times (w\rangle)$. It is clear that $L \setminus (v\rangle$ is a dual prime ideal that is maximal among all dual ideals not containing v. Further, the element w is a lower bound of the set $X = \{x \in L \mid w < x\}$, which is not empty as it contains $\bigvee L$ by Proposition 1. Again by Proposition 1, the lattice L is conditionally complete, and therefore $\bigwedge X$ exists. As w is completely meet-irreducible, $w \prec \bigwedge X$. Denote $b = \bigwedge X$. Let K be an ideal that includes $(w\rangle$ and does not contain b. Then $w \leqq \bigvee K$ but $b \nleqq \bigvee K$, and this implies $\bigvee K = w$. Hence $(w\rangle$ is maximal among all ideals not containing b. Consequently, $g([v, w])$ is a completely meet-irreducible tolerance. The mapping g is obviously injective and order-preserving. It is clear that $g = h^{-1}$. Q.E.D.

References

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