Czechoslovak Mathematical Journal

Valter Šeda

Surjectivity of an operator

Czechoslovak Mathematical Journal, Vol. 40 (1990), No. 1, 46–63

Persistent URL: http://dml.cz/dmlcz/102358

Terms of use:

© Institute of Mathematics AS CR, 1990

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 40 (115) 1990, Praha

SURJECTIVITY OF AN OPERATOR

VALTER ŠEDA, Bratislava

(Received February 12, 1988)

A new method of proving surjectivity of an operator f in a Banach space is proposed. For this method the condition of coercivity of f in the form $\lim_{|x|\to\infty} |f(x)| = \infty$ plays an important role. Some results obtained by this method deal with the strict surjective maps and with the quasi-bounded operators. The application of the results to ordinary differential equations is given.

1. PRELIMINARIES

Let $(E, |\cdot|)$ be a linear normed space, $\emptyset \neq X \subset E$ and $F: X \to E$. We recall that F is bounded iff it maps bounded sets into bounded sets. F is open iff it maps open subsets of X onto open sets. F is completely continuous iff it is continuous and maps bounded sets into relatively compact sets. F is proper iff the inverse image of a compact set by the mapping F is compact. If E is a Banach space, then F is said to be *condensing* iff it is continuous, bounded and for every bounded set $A \subset X$ which is not relatively compact we have $\alpha(F(A)) < \alpha(A)$ where α is the Kuratowski measure of non-compactness. A simple example of a condensing map is that of the form $G + H$ where $G: X \to E$ is a strict contraction and $H: X \to E$ is a completely continuous map. $I: E \to E$ will denote the identity mapping. If F is completely continuous condensing), then the mapping $f(x) = x - F(x)$, $x \in X$, will be called a *completely* continuous field (a condensing field).

The first method how to obtain surjectivity results is based on the equivalence of surjectivity to a system of fixed point problems given in the following lemma.

Lemma 1. *Let E be a vector space and $F: E \to E$. Then the corresponding field $f = I - F$ maps E onto itself iff for each $y \in E$ the mapping $G_y: E \to E$ defined by*

$$G_y(x) = F(x) + y\,, \quad x \in E\,,$$

has a fixed point.

The proof follows from the equivalence

$$G_y(x_0) = x_0 \quad \text{iff} \quad f(x_0) = y\,.$$

The second method which gives surjectivity results is based on the domain invariance and on the connectedness of the topological vector space.

Lemma 2. *Let E be a topological vector space and $f\colon E \to E$. Let f be an open mapping and let $f(E)$ be a closed subset of E. Then f is surjective. Moreover, if f is continuous and injective, then f is a homeomorphism of E onto itself.*

Proof. Since $f(E)$ is a nonvoid, open and closed subset of E simultaneously, $f(E) = E$. If f is surjective and injective, it is bijective. At the same time both f and the inverse mapping f^{-1} are continuous and the second statement of the lemma follows.

The following lemma is useful when using the second method.

Lemma 3 (Schauder theorem on domain invariance, [2], p. 66, [1], p. 72). *Let $f\colon E \to E$ be a completely continuous field in a linear normed space E (in a real linear normed space E). Let f be injective (locally injective). Then f is an open map.*

From the results based on Lemma 2 we mention only the following one:

Corollary 1. *Let E be a Banach space, let the map $F\colon E \to E$ and the corresponding field $f = I - F$ have the following properties:*
(1) *f is injective;*
(2) *F is a condensing, locally uniformly continuous mapping;*
(3) $\lim\limits_{|x| \to \infty} |f(x)| = \infty$.
Then f is a homeomorphism of E onto itself.

Proof. By the Nussbaum theorem on domain invariance ([6], p. 753), the assumptions (1) and (2) imply that for each nonvoid open subset $G \subset E$, $f(G)$ is open and hence f is open. Suppose now that $y_n \in f(E)$ and $y_n \to y$ as $n \to \infty$. By (1), there exists a unique sequence $\{x_n\}$ such that $f(x_n) = y_n$. The assumption (3) implies that there is an $r > 0$ such that $|x_n| \leqq r$. As $\overline{U(r)} = \{x \in E\colon |x| \leqq r\}$ is closed and bounded and F is condensing, by Theorem 11.4 in [9], p. 129, $f(\overline{U(r)})$ is a closed set which contains the sequence $\{y_n\}$. Hence $y \in f(\overline{U(r)}) \subset f(E)$ and $f(E)$ is a closed set. This implies that $f(E) = E$. Clearly f is continuous, too. The corollary now follows from Lemma 2.

If there is a q, $0 < q < 1$, such that $|F(x) - F(y)| \leqq q|x - y|$ for all $x, y \in E$, then for the corresponding field $f = I - F$ we have $|f(x) - f(y)| \geqq (1 - q)\,|x - y|$, $|f(x)| + |f(0)| \geqq |f(x) - f(0)| \geqq (1 - q)\,|x|$ and thus, all assumptions of Corollary 1 are satisfied. Hence the following statement is true.

Corollary 2 ([2], p. 11). *Lei E be a Banach space and let $F\colon E \to E$ be strictly contractive. Then the corresponding field $f = I - F$ is a homeomorphism of E onto itself.*

For a completely continuous field the following theorem holds.

Theorem 1. *Let $f\colon E \to E$ be an injective, completely continuous field in a linear* *normed space E. Then*

(4) *f is a homeomorphism of E onto itself, the inverse mapping f^{-1} is a completely continuous field and*

$$\lim_{|x| \to \infty} |f^{-1}(x)| = \infty$$

iff the condition (3) *is satisfied.*

Proof. On the basis of Lemma 3, the conditions (1), (2′) $F = I - f$ is completely continuous on E imply that f is an open mapping and a homeomorphism of E onto $f(E)$. Suppose that the assumption (3) is satisfied. We prove that $f(E)$ is a closed subset of E and thus, $f(E) = E$. Let $\{y_n\} \subset f(E)$ be a convergent sequence and $y_0 = \lim_{n \to \infty} y_n$. Then there is a sequence $\{x_n\} \subset E$ such that $f(x_n) = y_n$, $n = 1, 2, \dots$. Since $\{y_n\}$ is a bounded sequence, by (3) the sequence $\{x_n\}$ is bounded, too. The condition (2′) implies that there is a subsequence $\{x_{n(k)}\}$ of the sequence $\{x_n\}$ and a point $x_0 \in E$ such that

$$x_{n(k)} - y_{n(k)} = x_{n(k)} - f(x_{n(k)}) \to x_0 \quad \text{as} \quad k \to \infty\,.$$

Then $\lim_{k \to \infty} x_{n(k)} = y_0 + x_0$, and by the continuity of f, $f(x_0 + y_0) = y_0$. Thus $y_0 \in f(E)$ and $f(E)$ is closed. Since f is bounded, we have that $\lim_{|x| \to \infty} |f^{-1}(x)| = \infty$. Further, we can write $I - f^{-1} = -(I - f) \circ f^{-1}$ and as by (3) f^{-1} is bounded and, in view of (2′), $I - f$ is completely continuous, $F_1 = I - f^{-1}$ is completely continuous.

On the other hand, if (1), (2′) and (4) hold, then f^{-1} satisfies the conditions (1), (2′) and hence f^{-1} is bounded, which implies that (3) is fulfilled.

Corollary 3. *Let G be the family of all transformations $f: E \to E$ in a linear normed space E enjoying the properties* (1), (2′), (3). *Then G is a group of homeomorphic transformations of the space E.*

Proof. By Theorem 1 each $f \in G$ is a homeomorphic mapping of E onto itself, and f^{-1} belongs to G. Suppose now that $f_1 = I - F_1$, $f_2 = I - F_2$ are two transformations from G where F_1, F_2 are completely continuous. Then $I - f_1 \circ f_2 = F_2 + F_1 + (F_1 \circ (-F_2))$ is completely continuous. Clearly $f_1 \circ f_2$ enjoys the other properties of the transformations from G.

Corollary 4 ([2], p. 67). *Let $f: E \to E$ be a completely continuous field in a linear normed space E. If there exists a $k > 0$ such that*

(5) $$|f(x) - f(y)| \geqq k|x - y| \quad \textit{for all} \quad x, y \in E\,,$$

then f is a homeomorphism of E onto itself.

Proof. By (5), f is injective and for an arbitrary $x \in E$ we have

$$|f(x)| + |f(0)| \geqq |f(x) - f(0)| \geqq k|x|\,,$$

which implies that f also satisfies the condition (3). The result follows from Theorem 1.

A modification of Theorem 1 can be proved in a similar way as the original theorem.

Theorem 1'. *Let $f: E \to E$ be a locally injective, completely continuous field satisfying the condition (3) in a real linear normed space E. Then f is surjective, i.e. $f(E) = E$.*

Remark 1. Since in the case $E = R^n$ the continuity of an operator $F: R^n \to R^n$ implies the boundedness as well as the complete continuity of this operator, in this case we can replace the condition (2') by the equivalent condition

(2'') f is continuous in R^n

and Theorems 1, 1', Corollaries 3 and 4 remain valid. We use the Euclidean norm $|\cdot|$ in R^n and the scalar product in this space will be denoted by $(\cdot, \cdot)$.

2. MAIN RESULTS

Further surjectivity results will be based on the next theorem which is a slight modification of Theorem 1 in [8], pp. 161–162. First we shall introduce some notation.

Let E be a Banach space and B the Banach space of all continuous functions $x: \langle 0, 1\rangle \to E$. The norm in B is defined by $\|x\| = \sup\{|x(t)|: 0 \leqq t \leqq 1\}$ for each $x \in B$. Further, let $U(r) = \{x \in E: |x| < r\}$. The degree will be considered in the sense of Nussbaum, [6], p. 744, [8], p. 161.

Theorem 2. *Let $g: E \to B$ be a continuous mapping. Denote by $g(x, t)$ the value of $g(x) \in B$ at the point $t \in \langle 0, 1\rangle$. Assume that*

(i) $v(x) = \inf\{|g(x, t)|: 0 \leqq t \leqq 1\} \to \infty$ *for* $|x| \to \infty$;

(ii) *the mapping $I - g(\cdot, t)$ is condensing for each $t \in \langle 0, 1\rangle$;*

(iii) *for each $y \in E$ there is an $r_0 > 0$ with $v(x) > |y|$ for all $|x| \geqq r_0$ such that*

$$\deg(g(\cdot, 0) - y, U(r_0), 0) \neq 0 ;$$

(iv) *$g(x, \cdot)$ is continuous in t, uniformly in $x \in \overline{U(r)}$ for each $r > 0$.*

Then $g(E, t) = E$ for each $t \in \langle 0, 1\rangle$.

Proof. Let $y \in E$, $t_0 \in \langle 0, 1\rangle$. By (iii), $y \notin g(\partial U(r_0), t)$ for each $t \in \langle 0, 1\rangle$ ($\partial U(r_0)$ means the boundary of $U(r_0)$). Hence the mapping $G: \overline{U(r_0)} \times \langle 0, 1\rangle \to E$ which is defined by $G(x, t) = x - g(x, t) + y$ is continuous and $G(x, t) \neq x$ for $x \in \partial U(r_0)$, $t \in \langle 0, 1\rangle$. By (ii), $G(\cdot, t)$ is a condensing map for $t \in \langle 0, 1\rangle$ and (iv) implies that $G(x, \cdot)$ is continuous in t uniformly in $x \in \overline{U(r_0)}$. Hence, by Corollary 2 in [6], p. 745, and by (iii) we have

$$(6) \qquad \deg(g(\cdot, t_0) - y, U(r_0), 0) = \deg(I - G(\cdot, t_0), U(r_0), 0) =$$
$$= \deg(I - G(\cdot, 0), U(r_0), 0) = \deg(g(\cdot, 0) - y, U(r_0), 0) \neq 0 .$$

By virtue of Proposition 5 in [6], p. 744, the set $S = \{x \in U(r_0): g(x, t_0) - y = 0\}$ is nonempty and this proves the theorem.

Remark 2. In view of the definition of r_0 we have $S_0 = \{x \in E: g(x, 0) - y = 0\} \subset \subset U(r_0)$ and hence, by Proposition 5 cited above, for each $r > r_0$

$$\deg (g(\cdot, 0) - y, U(r), 0) = \deg (g(\cdot, 0) - y, U(r_0), 0) \neq 0 .$$

Hence the condition (iii) (together with (i)) is equivalent to the condition

(iii′) for each $y \in E$ there is a sequence $\{r_n\} \to \infty$ as $n \to \infty$ such that $\deg (g(\cdot, 0) - y, U(r_n), 0) \neq 0$ for each r_n

(together with (i)).

Remark 3. If $E = R^n$, then Theorem 2 is true without the assumptions (ii) and (iv).

Definition 1. Let $F: E \to E$. We shall say that the field $f = I - F$ is strictly surjective if it is condensing and for each $y \in E$ there is a sequence $\{r_n\} \to \infty$ as $n \to \infty$ such that

$$\deg (f - y, U(r_n), 0) \neq 0 .$$

Clearly each strictly surjective map is surjective.

Now we shall give two sufficient conditions that the completely continuous field f be strictly surjective. The first is based on Theorem 1 while the second is proved by means of the Borsuk theorem (Theorem 1 in [1], p. 72).

Theorem 3. *If either*

(i) *f is an injective, completely continuous field satisfying the condition* (3) *in a real Banach space E,*

or

(ii) *f is an odd, completely continuous field satisfying the condition* (3) *in the real linear normed space E,*

then f is strictly surjective.

Proof. Let $y \in E$ be an arbitrary but fixed element. (i) In view of Theorem 1, Theorem 3 in [1], p. 74, gives that $\deg (f - y, U(r), 0) = \pm 1$ for each sufficiently great $r > 0$.

(ii) By (3), there is an $r_0 > 0$ such that

$$|f(x)| > |y| \quad \text{for each} \quad x \in E , \quad |x| \geqq r_0 . \tag{7}$$

Let $r \geqq r_0$. Then $U(r)$ is an open, bounded and symmetric neighbourhood of the origin and $f(x) - y \neq 0$ for each $x \in \partial U(r) = \{x \in E: |x| = r\}$. If there existed an $x \in \partial U(r)$ such that

$$|f(x) - y|^{-1} (f(x) - y) = |f(-x) - y|^{-1} (f(-x) - y) , \tag{8}$$

then by the assumptions on f there would be a $k > 0$ such that either

$$f(x) - y = k(-f(x) - y) \quad \text{or} \quad f(-x) - y = (1/k)(-f(-x) - y) .$$

In the former case $f(x) = y(1 - k)(1 + k)^{-1}$ and $0 < k \leqq 1$ cannot occur in view of (7). In the latter case

$$f(-x) = y\left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right)^{-1}$$

and the case $1 < k < \infty$ cannot occur. This implies that (8) is not possible and the Borsuk theorem implies the statement.

Using Theorem 2, we can enlarge the collection of strictly surjective maps.

Theorem 4. *Let E be a Banach space, let $f = I - F$, $h = I - H$ be two fields in E, i.e. $F, H\colon E \to E$. Suppose that*

(a) *both fields f and h satisfy the condition* (3);

(b) *f is strictly surjective*;

(c) *h is a condensing field*;

(d) *there exists a k, $0 < k < 1$ and an $r \geqq 0$ such that*

$$|(1-t)f(x) + t\,h(x)| \geqq k[(1-t)\,|f(x)| + t|h(x)|] \tag{9}$$

$$\textit{for all} \quad x \in E\,, \quad |x| \geqq r\,, \quad 0 \leqq t \leqq 1\,.$$

Then h is strictly surjective.

Proof. Let B be the Banach space as above. Consider the mapping g which is defined for each $x \in E$, $0 \leqq t \leqq 1$, by

$$g(x, t) = (1-t)f(x) + t\,h(x)\,. \tag{10}$$

Clearly $g\colon E \to B$ and in view of the continuity of f and h, g is continuous. Further, $g(x, 0) = f(x)$, $g(x, 1) = h(x)$ for each $x \in E$. We shall show that g satisfies all assumptions of Theorem 2. By (a), (d) we have that $v(x) = \inf\{|g(x, t)|: 0 \leqq t \leqq \leqq 1\} \to \infty$ for $|x| \to \infty$ and hence (i) is satisfied. The conditions (ii), (iii) of Theorem 2 follow from the strict surjectivity of f, the equality $I - g(\cdot, t) = (1-t)\,F(x) + t\,H(x)$ and the inequality $\alpha[(1-t)\,F(A) + t\,H(A)] \leqq (1-t)\,\alpha[F(A)] + t\,\alpha[H(A)] < \alpha(A)$ for each bounded set A with the Kuratowski measure of noncompactness $\alpha(A) > 0$. Since f and h are bounded, it follows from (10) that $g(x, t)$ is continuous in t, uniformly in $x \in \overline{U(r)}$ for each $r > 0$. Hence all conditions of Theorem 2 are satisfied, and by (6) for $t_0 = 1$, $r_0 = r_n$ we get the theorem.

Remark 4. By virtue of Theorem 3, Theorem 4 remains to be true if the condition (b) is replaced by one of the conditions

(b′) f is an injective, completely continuous field in a real Banach space E;

(b) f is an odd, completely continuous field in a real Banach space E.

Now we extend the notion of the quasi-bounded operator ([2], p. 62).

Definition 2. Let E be a linear normed space, let $f\colon E \to E$ be an injective completely continuous field satisfying (3). An operator $G\colon E \to E$ is called *f-quasibounded* if

$$|G|_f := \limsup_{|x| \to \infty} \frac{|G(x)|}{|f(x)|} = \inf_{\varrho > 0} \sup_{|x| \geqq \varrho} \frac{|G(x)|}{|f(x)|} < \infty\,.$$

Theorem 5. *Let E be a real Banach space, let $f\colon E \to E$ be an injective completely continuous field satisfying* (3) *and $G\colon E \to E$ a condensing* (*completely continuous*) *f-quasibounded operator. Then for each real λ such that*

$$(11) \qquad |\lambda| \leqq 1 \quad \textit{for} \quad |G|_f < 1\,, \quad |\lambda| < \frac{1}{|G|_f} \quad \textit{for} \quad |G|_f \geqq 1$$

$$\left(|\lambda| < \frac{1}{|G|_f} \quad \textit{and for all real } \lambda \textit{ whenever } |G|_f = 0\right)$$

the operator $h = f + \lambda G$ is stricly surjective.

Proof. We shall apply Theorem 4 and Remark 4. If λ is a fixed real number satisfying the inequalities (11), then in both cases (G is condensing or G is completely continuous) we obtain that λG is a condensing map. Then denoting $f = I - F$ where $F\colon E \to E$ is completely continuous, we have that $h_\lambda = I - (F - \lambda G)$ is a condensig field, since the Kuratowski measure of noncompactness satisfies $\alpha((F - \lambda G)\,.\,(A)) \leqq \alpha(F(A)) + \alpha((\lambda G)(A)) < \alpha(A)$ for each bounded set A with $\alpha(A) > 0$. Thus (b′) and (c) from Theorem 4 and Remark 4 are fulfilled.

Further, $|\lambda G|_f = |\lambda|\,|G|_f < 1$ and hence for any q, $|\lambda G|_f < q < 1$, there is an $r > 0$ such that for $x \in E$,

$$(12) \qquad |x| \geqq r \quad \text{implies} \quad \frac{|\lambda\, G(x)|}{|f(x)|} \leqq q\,.$$

Then $|f(x) + \lambda\, G(x)| \geqq |f(x)| - |\lambda\, G(x)| \geqq |f(x)|\,(1 - q)$ for $|x| \geqq r$ which gives that $\lim_{|x| \to \infty} |h_\lambda(x)| = \infty$ and the condition (a) in Theorem 4 is satisfied, too.

Now put $(1 - q)/(1 + q) = k$. Then by (12), for $|x| \geqq r$ and $0 \leqq t \leqq 1$ we have

$$\begin{aligned} |(1 - t)\,f(x) + t\,h_\lambda(x)| &= |f(x) + t\lambda\, G(x)| \geqq |f(x)| - |\lambda\, G(x)| \geqq \\ &\geqq |f(x)|\,(1 - q) \geqq k[|f(x)| + q|f(x)|] \geqq k[|f(x)| + |\lambda\, G(x)|] \geqq \\ &\geqq k[|f(x)| + t|\lambda\, G(x)|] = k[(1 - t)\,|f(x)| + t|f(x)| + t|\lambda\, G(x)|] \geqq \\ &\geqq k[(1 - t)\,|f(x)| + t|h_\lambda(x)|]\,. \end{aligned}$$

Since the last condition in Theorem 4 is satisfied, Theorem 5 follows.

Remark 5. Theorem 5 extends Theorem 5.4 in [2], p. 62, to f-quasibounded condensing operators.

The next corollary brings the third sufficient condition for a field to be strictly surjective (the first two sufficient conditions were given in Theorem 3).

Corollary 5. *Let E be a real Banach space, $g = I - G\colon E \to E$ a condensing field such that there is a q, $0 < q < 1$, and an $r > 0$ with the property*

$$(13) \qquad |x| \geqq r \quad \textit{implies that} \quad |G(x)| \leqq q|x|\,.$$

Then g is strictly surjective.

Proof. If we put $f = I$ and $\lambda = -1$, we see that all assumptions of Theorem 5 are satisfied, because by (13), $|G|_I \leqq q < 1$.

Some surjectivity results in R^n are collected in the following theorem.

Theorem 6. *Let $f: R^n \to R^n$ satisfy the conditions*

(2″) *f is continuous,*

(3) $\lim_{|x| \to \infty} |f(x)| = \infty$,

and one of the conditions:

either

(14) *there is an $x_0 \in R^n$ such that*

$$f(x) - x_0 = k(x - x_0) \quad \textit{implies} \quad k \geqq 0 \quad \textit{for each} \quad x \in R^n,\ x \neq x_0\,,$$

or

(14′) *there is an $x_0 \in R^n$ such that*

$$f(x) - x_0 = k(x - x_0) \quad \textit{implies} \quad k \leqq 0 \quad \textit{for each} \quad x \in R^n\,, \quad x \neq x_0\,,$$

or

(15) *there is an $x_0 \in R^n$ and an $r > 0$ such that the scalar product satisfies*

$$(f(x) - x_0, x - x_0) \geqq 0 \quad \textit{for al} \quad x \in R^n, \quad |x - x_0| \geqq r$$

or

(15′) *there is an $x_0 \in R^n$ and an $r > 0$ such that*

$$(f(x) - x_0,\ x - x_0) \leqq 0 \quad \textit{for all} \quad x \in R^n\,, \quad |x - x_0| \geqq r\,.$$

Then $f(R^n) = R^n$.

Proof. If we introduce $y = x - x_0$ and the mapping $h: R^n \to R^n$ by $h(y) = = f(x) - x_0 = f(y + x_0) - x_0$, we see that h satisfies (2″), (3) and one of the conditions (14), (14′), (15), (15′) with $x_0 = 0$. If h maps R^n onto itself, then f does the same. Hence it suffices to show the surjectivity of f in the special case that $x_0 = 0$. Under the condition (14) or the condition (15) the statement of the theorem was proved in Corollary 2, [8], pp. 163–165. In the case (14′) or (15′) we consider the mapping $-f$. This mapping already satisfies (in addition to (2″), (3)) either (14) or (15) and hence, it is surjective. Then f is surjective, too.

3. GENERALIZED BOUNDARY VALUE PROBLEM FOR DIFFERENTIAL SYSTEMS

Similarly as in [8], by the generalized boundary value problem for the system of differential equations

$$(16) \qquad x' = f(t, x)\,, \quad t \in i\,, \quad x \in R^n\,,$$

and the given continuous mapping F (not necessarily linear) of the space $C(i, R^n)$ of all continuous n-dimensional vector functions defined on the interval i into R^n

we understand the problem to find a solution $x(t)$ of the system (16) on the interval i for which $F(x)$ is a given vector r in R^n, i.e.

(17) $\qquad F(x) = r\,.$

The topology in $C(i, R^n)$ depends on whether i is compact or not. If $i = \langle a, b\rangle$ is a compact interval, then we consider the topology of uniform convergence determined by the sup-norm, while in the case that i is a noncompact interval, e.g. $i = \langle a, \infty)$, then we use the topology of locally uniform convergence. This topology can be introduced by a countable system of seminorms.

In the proof of the next theorem we shall use the Kamke convergence lemma ([3], Theorem 3.2, pp. 26–27) which has been formulated for f defined on an open set. In the case that a boundary point of i belongs to i, f can be extended (e.g. by linear extrapolation) to an open set which contains $i \times R^n$ and again this lemma can be applied.

Theorem 7. *Let $f = f(t, x) \in C(i \times R^n, R^n)$ and let the equation (16) have the following properties*:

(a) *there is a point $t_0 \in i$ such that for each vector $x_0 \in R^n$ there exists a unique solution $x(t)$ on i to the initial-value problem* (16),

(18) $\qquad x(t_0) = x_0$

(*this solution will be denoted by $x(t, x_0)$*) *and*
either

(b) *the problem* (16), (17) *has at most one solution for each $r \in R^n$*,
or

(c) *for each solution $x(t)$ or* (16), (18) *the following implication is true*:

if $F(x) = k\,x(t_0)$, $x(t_0) \neq 0$, then $k \geqq 0$,
or

(d) *for each solution $x(t)$ of* (16), (18) *the following implication holds*:

if $F(x) = k\,x(t_0)$, $x(t_0) \neq 0$, then $k \leqq 0$.

Then in the case (a), (b) *a necessary and sufficient condition, and in the case* (a), (c) *or in the case* (a), (d) *a sufficient condition that there exist at least one solution of the problem* (16), (17) *for each $r \in R^n$ is that the following compactness condition be satisfied*:

(e) *if $\{x_k\}$ is an arbitrary sequence of solutions of* (16) *on the interval i such that $\{F(x_k)\}$ is bounded, then there is a subsequence $\{x_{k(l)}\}$ of $\{x_k\}$ such that $\{x_{k(l)}\}$ is convergent in $C(i, R^n)$.*

Proof. First we consider the mapping $G: R^n \to C(i, R^n)$ such that $G(x_0) = x(\cdot, x_0)$ for each $x_0 \in R^n$. By the Kamke convergence lemma, G is continuous. Then the composite mapping $f = F \circ G$ is, on the basis of the assumption on F, a continuous mapping from R^n into R^n. The problem (16), (17) has a solution for each $r \in R^n$ iff f is surjective. To show the surjectivity of f we shall use Theorem 6.

The condition $f(x_0) = kx_0$ is equivalent to the equality $F(x) = k\,x(t_0)$ and hence the conditions (c), (d) imply that the conditions (14), (14') with $x_0 = 0$ of Theorem 6 are satisfied. The condition (3) of that theorem means that the f-preimage of each bounded set in R^n is bounded in R^n.

Suppose that $\{x_{k0}\}$ is a sequence in R^n. As $f(x_{k0}) = F(x_k)$ where $x_k = x(\cdot, x_{k0})$, the condition (3) is equivalent to the condition

(f) if $x_k = x(\cdot, x_{k0})$, $k = 1, 2, \ldots$, is an arbitrary sequence of solutions of (16) such that $\{F(x_k)\}$ is bounded, then the sequence $\{x_{k0}\}$ is bounded.

This in turn is equivalent to the condition

(g) if $x_k = x(\cdot, x_{k0})$, $k = 1, 2, \ldots$, is an arbitrary sequence of solutions of (16) such that $\{F(x_k)\}$ is bounded, then there is a convergent subsequence $\{x_{k(m)0}\}$ of the sequence $\{x_{k0}\}$.

By the Kamke convergence lemma the condition (g) is equivalent to the compactness condition (e). Hence we have that in the case (a), (c), (e) or in the case (a), (d), (e), f satisfies the conditions of Theorem 6. By this theorem $f(R^n) = R^n$ and therefore (16), (17) has a solution for each $r \in R^n$.

Further, if each problem (16), (17) has at most one solution, then both mappings $F|_{G(R^n)}$ and G are injective which implies that f is an injective, continuous mapping from R^n into R^n. With respect to Remark 1, Theorem 1 gives that in this case $f(R^n) = R^n$ iff f satisfies the condition (3). Hence, if (a), (b) are supposed, the problem (16), (17) has a unique solution for each $r \in R^n$ iff the compactness condition (e) is fulfilled. The proof of the theorem is complete.

Remark 6. Under the conditions (a), (b), (e) f is not only surjective, but even homeomorphic. Since G is also homeomorphic, $F|_{G(R^n)}$ as well as its inverse function $(F|_{G(R^n)})^{-1}$ is homeomorphic, too. Hence, in this case the problem (16), (17) is well posed, since the existence, uniqueness and continuous dependence of the solution to this problem on r is guaranteed.

Remark 7. From the proof of Theorem 7 we see that the condition (e) in this theorem can be replaced by the „apriori estimate" (f) or by the condition (g).

Now consider a linear nonhomogeneous system of differential equations

$$(19) \qquad x' = A(t)\,x + b(t)\,,$$

where $A(t)$ is an $n \times n$ real continuous matrix function on i, $b(t)$ is an n-dimensional continuous vector function on i, and suppose that

$$(20) \qquad F \text{ is linear}\,.$$

It is known (see [7], p. 585, for the case of a compact interval i, but in the case of a noncompact interval the same considerations yield the result) that the problem (19), (17) under the hypothesis (20) has a unique solution for any $r \in R^n$ iff the corresponding homogeneous linear differential system

$$(21) \qquad x' = A(t)\,x$$

with the generalized boundary-value condition

(22) $\qquad F(x) = 0$

has only the trivial solution. Since for (19) the assumption (a) in Theorem 6 is fulfilled at each point $t_0 \in i$, the following corollary to Theorem 7 is true.

Corollary 6. *If under the condition* (20) *the problem* (21), (22) *has only the trivial solution, then for the differential system* (19) *the following compactness condition is true*:

If $\{x_k\}$ *is an arbitrary sequence of solutions of* (19) (*on* i) *such that* $\{F(x_k)\}$ *is bounded, then there is a subsequence* $\{x_{k(l)}\}$ *of the sequence* $\{x_k\}$ *such that* $\{x_{k(l)}\}$ *is convergent in* $C(i, R^n)$.

The next result follows from the preceding corollary.

Corollary 7. *Suppose that* F_1: $C(i, R^n) \to R^n$ *is a continuous functional such that*

1. *there exists a linear continuous functional* F: $C(i, R^n) \to R^n$ *with the following properties*:

(i) $|F_1(x)| \geqq |F(x)|$ *for each* $x \in C(i, R^n)$,

(ii) *the problem* (21), (22) *has only the trivial solution*;

2. *either the problem* (19),

(23) $\qquad F_1(x) = r$

has at most one solution for each $r \in R^n$,

or

there is a point $t_0 \in i$ *such that for each solution* x *of* (19), (23) *the following implication is true*:

if $F_1(x) = k\, x(t_0)$, $x(t_0) \neq 0$, *then* $k \geqq 0$;

or

there is a point $t_0 \in i$ *such that for each solution* x *of* (19), (23) *the following implication holds*:

If $F_1(x) = k\, x(t_0)$, $x(t_0) \neq 0$, *then* $k \leqq 0$.

Then for each $r \in R^n$ *there exists at least one solution of the problem* (19), (23).

Proof. Clearly it suffices to check the compactness condition (e) in Theorem 7. In view of the assumption (i), if $\{F_1(x_k)\}$ is bounded, then $\{F(x_k)\}$ is bounded, too and by Corollary 6 it follows that (e) is satisfied. Then the result is a consequence of Theorem 7.

Remark 8. By looking through the proof of Theorem 7, we see that this theorem remains valid if F is supposed to be continuous only on the set $G(R^n)$ of the solutions $x(\cdot, x_0)$ of (16) for all $x_0 \in R^n$. The same remark applies to Corollaries 6 and 7.

Example. Suppose that $f = f(t, x)$: $\langle a, b \rangle \times R \to R$ is a scalar continuous function such that

(i) each complete (i.e. noncontinuable) solution of the equation (16) exists on $\langle a, b\rangle$ and there is a $t_0 \in \langle a, b\rangle$ such that for each $x_0 \in R$ there exists a unique solution $x(t)$ on $\langle a, b\rangle$ of the problem (16), (18),

(ii) $f(t, 0) \equiv 0$ in $\langle a, b\rangle$.

Then for each $r \geqq 0$ there exists a pair of solutions x_1, x_2 of (16) with $x_1(t_0) \geqq 0$, $x_2(t_0) \leqq 0$ such that

$$\int_a^b |x_i(t)| \, dt = r\,, \quad i = 1, 2\,.$$

Indeed, consider the mapping $F: C(\langle a, b\rangle, R) \to R$ defined by

$$F(x) = \operatorname{sgn} x(t_0) \int_a^b |x(t)| \, dt\,, \tag{24}$$

where as usual $\operatorname{sgn} u = 1$ for $u > 0$, $\operatorname{sgn} 0 = 0$, $\operatorname{sgn} u = -1$ for $u < 0$. Then F is continuous at each function $x \in C(\langle a, b\rangle, R)$ with $x(t_0) \neq 0$ or $x(t_0) = 0$, $x(t) \equiv 0$ in $\langle a, b\rangle$, and it is discontinuous at any function x such that $x(t_0) = 0$, $x(t) \not\equiv 0$ in $\langle a, b\rangle$. In view of the assumptions (i), (ii), there is no nontrivial solution of (16) satisfying $x(t_0) = 0$, and hence the restriction of F to the set of all complete solutions of (16) is continuous. The assumptions (a) and (c) of Theorem 7 are clearly satisfied. As for the assumption (e), suppose that $\{x_k\}$ is a sequence of solutions of (16) on $\langle a, b\rangle$ such that $\int_a^b |x_k(t)| \, dt \leqq M$ for an $M > 0$. Consider the sequence $\{x_k(t_0)\}$. Two cases may occur. In the first, the sequence $\{x_k(t_0)\}$ contains a bounded subsequence and hence also a convergent subsequence $\{x_{k(l)}(t_0)\}$. By the Kamke convergence lemma there is a subsequence $\{x_{k(m)}\}$ of the sequence $\{x_k\}$ which is uniformly convergent in $\langle a, b\rangle$ and the compactness condition is satisfied.

In the second case, $\lim\limits_{k\to\infty} |x_k(t_0)| = \infty$. Hence either there is a subsequence $\{x_{k(l)}(t_0)\}$ of $\{x_k(t_0)\}$ such that $\lim\limits_{l\to\infty} x_{k(l)}(t_0) = \infty$, or a subsequence tending to $-\infty$ as $l \to \infty$. Let us consider only the first possibility, since the second can be dealt with in a similar way. Without loss of generality we may assume that $\{x_{k(l)}(t_0)\}$ is an increasing sequence. By the uniqueness of the initial-value problem at t_0 and by (ii), if $0 < c_1 < c_2$ and $x(\cdot, c_i)$ is the solution of (16) satisfying $x(t_0) = c_i$, $i = 1, 2$, then $0 \leqq x(t, c_1) \leqq$ $\leqq x(t, c_2)$ for $a \leqq t \leqq b$. Hence the sequence $\{x_{k(l)}\}$ of solutions is nondecreasing and by the Beppo Levi theorem the finite $\lim\limits_{l\to\infty} x_{k(l)}(t) = y(t)$ exists a.e. in $\langle a, b\rangle$. Choose a point $t_1 \in \langle a, b\rangle$, $t_1 \neq t_0$, at which $y(t)$ is finite. Then with respect to (i) again by the Kamke convergence lemma $\lim\limits_{l\to\infty} x_{k(l)}(t) = x(t)$ uniformly in $\langle a, b\rangle$, where x is the solution of (16) satisfying $x(t_1) = y(t_1)$. Hence $\lim\limits_{l\to\infty} x_{k(l)}(t_0) = x(t_0)$ which contradicts the fact that this limit is infinite. By this contradiction the latter case cannot occur and thus the assumptions of Theorem 7 are satisfied. By this theorem, which can be applied in view of the last remark, for each $r \in R$ there is a solution of (16) satisfying (17) where F is given by (24). When $r = 0$, this condition is satisfied by the trivial solution. For $r > 0$ ($r < 0$) there exists a solution x of (16), (17) with $x(t_0) > 0$ ($x(t_0) < 0$) and this implies the statement given above.

Now we shall compare the boundary value problem (16), (17) with a scalar linear problem. The existence of all solutions to the linear problem implies the existence of a solution to the problem (16), (17) for each $r \in R^n$.

Suppose that the interval i has the form $\langle t_0, b\rangle$ or $\langle t_0, b)$ with $-\infty < t_0 < b \leqq \leqq \infty$. Then the following theorem is true.

Theorem 8. *Suppose that the differential equation* (16) *satisfies the condition* (a), *and the problem* (16), (17) *fulfils the condition* (b) *or the condition* (c) *or the condition* (d) *of Theorem* 7. *Further, let there exist functions $a_1 \in C(i, R)$, $b_1 \in C(i, R)$ such that $a_1(t) > 0$ in i, $b_1(t) \geqq 0$ in i, and*

$$(25) \qquad |f(t, x)| \leqq a_1(t)\,|x| + b_1(t)\,, \quad t \in i\,, \quad x \in R^n\,.$$

Let there exist a linear, continuous and positive functional $F_1\colon C(i, R) \to R$ such that

$$(26) \qquad |F(x)| \geqq F_1(|x|) \quad \text{for all} \quad x \in C(i, R^n)$$

(*the positivity means that $F_1(y) \geqq 0$ for all $y \geqq 0$, $y \in C(i, R)$*). *Finally, let the problem*

$$(27) \qquad y' = -a_1(t)\,y\,, \quad F_1(y) = 0\,,$$

have only the trivial solution.

Then for each $r \in R^n$ there exists a solution of the problem (16), (17).

Proof. It suffices to show that the compactness condition (e) in Theorem 7 is satisfied. Hence, let $\{x_k\}$ be a sequence of solutions of (16) on the interval i for which the sequence $\{F(x_k)\}$ is bounded in R^n. Two cases may occur.

Either there is a bounded subsequence of the sequence $\{x_k(t_0)\}$ and then, by means of the Kamke convergence lemma, we get that there exists a subsequence $\{x_{k(l)}\}$ of $\{x_k\}$ which is convergent in the space $C(i, R^n)$ and thus, the condition (e) being satisfied, Theorem 7 guarantees the result. Or, $\{x_k(t_0\}$ contains no bounded subsequence and hence we may assume that $\lim\limits_{k\to\infty} |x_k(t_0)| = \infty$. In this case we proceed as follows.

Let $v_k(t) = |x_k(t)|$, $t \in i$, $k = 1, 2, \ldots$. Then for each $t \in i$ where $v_k(t) > 0$, on the basis of (25) and the Schwarz inequality we have

$$v_k'(t) = \frac{(x_k(t), f(t, x_k(t))}{v_k(t)} \geqq -\frac{v_k(t)\,|f(t, x_k(t)|}{v_k(t)} =$$
$$= -|f(t, x_k(t))| \geqq -a_1(t)\,v_k(t) - b_1(t)\,,$$

where $(\cdot, \cdot)$ means the scalar product in R^n. Hence the functions $v_k(t)$ satisfy the inequality

$$v' \geqq -a_1(t)\,v - b_1(t)$$

at each $t \in i$ where $v_k(t) > 0$. Let y_k be the solution of the problem

$$(28) \qquad y' = -a_1(t)\, y - b_1(t)\,, \quad y(t_0) = v_k(t_0)\,.$$

Then we assert that

$$(29) \qquad v_k(t) \geqq y_k(t) \quad \text{for all} \quad t \in i\,.$$

Indeed, this is true on the interval $\langle t_0, t_1) \subset i$ where $v_k(t) > 0$. If $v_k(t_1) = 0$, then $y_k(t_1) \leqq 0$ and by the direction field of (28), we have that $y_k(t) \leqq 0$ for all $t \in i$, $t \geqq t_1$. On the other hand, $v_k(t) \geqq 0$ in i and thus (29) is true.

By virtue of (26), (29) and the positivity of F_1 we have that

$$(30) \qquad |F(x_k)| \geqq F_1(v_k) \geqq F_1(y_k)\,, \quad k = 1, 2, 3, \ldots,$$

and hence, the sequence $\{F_1(y_k)\}$ is bounded from above. As $v_k(t_0) \to \infty$ for $k \to \infty$, we can extract an increasing subsequence $v_{k(l)}(t_0)$ tending to ∞ which we denote again as $v_k(t_0)$. Thus $y_1(t_0) < y_2(t_0) < \ldots \to \infty$ and by the uniqueness of the initial-value problem for (28) we also have $y_1(t) < y_2(t) < y_3(t) < \ldots$ for all $t \in i$. Thus

$$(31) \qquad F_1(y_k) \geqq F_1(y_1)\,, \quad k = 1, 2, 3, \ldots,$$

and (30) together with (31) imply that the sequence $\{F_1(y_k)\}$ is bounded. Corollary 6 then gives that there is a subsequence $y_{k(l)}$ which is convergent in $C(i, R)$. But this implies that $\{y_{k(l)}(t_0)\}$ as well as $\{v_{k(l)}(t_0)\}$ are convergent. Hence there is a subsequence $\{x_{k(m)}(t_0)\}$ of $\{x_{k(l)}(t_0)\}$ which is convergent, but this contradicts the assumption that $\{x_k(t_0)\}$ contains no bounded subsequence. Therefore the second case cannot occur and the proof of the theorem is complete.

5. SOME BOUNDARY VALUE PROBLEMS

Boundary value problems for ordinary differential equations will be now investigated. This requires to consider instead of the space $C(\langle a, b\rangle, R^n)$ the space $E = \{(x(t), x'(t), \ldots, x^{(n-1)}(t)):\ x(t) \in C^{n-1}(\langle a, b\rangle, R)\}$ provided with the norm $|(x(t), x'(t), \ldots, x^{(n-1)}(t))| = \max\,[\sup_{a \leqq t \leqq b} |x(t)|,\ \sup_{a \leqq t \leqq b} |x'(t)|, \ldots, \sup_{a \leqq t \leqq b} |x^{(n-1)}(t)|]$. This space is a Banach space.

First we shall consider the boundary value problem which is similar to the Bitsadze-Samarskij problem ([4], [5])

$$(32) \qquad x'' = f(t, x, x')\,,$$

$$(33) \qquad x(a) = A\,, \quad x(b) - x(t_0) = B\,,$$

where $a < t_0 < b$, A, B are real numbers and $f \in C(\langle a, b\rangle \times R^2, R)$. The following theorem is true.

Theorem 9. *Suppose that*

(i) *each complete solution of the equation* (32) *exists on* $\langle a, b\rangle$, *and each initial value problem for this equation at the point a has a unique solution in* $\langle a, b\rangle$;

(ii) *there is a constant* $M > 0$ *such that*

$$f(t, x, y) > 0 \quad \textit{for} \quad x \geqq M\,, \quad y \geqq M$$

and

$$f(t, x, y) < 0 \quad \textit{for} \quad x \leqq -M\,, \quad y \leqq -M\,;$$

(iii) *either*

for each point $(A, B) \in R^2$ *there exists at most one solution of the boundary value problem* (32), (33),

or

the solution x of (32) *satisfying* $x(a) = 0$, $x'(a) \neq 0$, *fulfils the inequality* $x'(a)\,[x(b) - x(t_0)] \geqq 0$.

Then for each couple $(A, B) \in R^2$ *there exists at least one solution of the problem* (32), (33).

Proof. The assumption (i) implies that the condition (a) in Theorem 7 is satisfied at the point a. By the assumption (iii) either the condition (b) or the condition (c) of the same theorem is fulfilled. Indeed, the first part of (iii) guarantees the condition (b). Let x be a solution of (32) in $\langle a, b\rangle$. Since the equality $F(x) = k\,x(t_0)$ from Theorem 7 now means two equalities

$$x(a) = k\,x(a)\,, \quad x(b) - x(t_0) = k\,x'(a)\,,$$

we have to consider two cases. Either $x(a) \neq 0$ and then $k = 1 \geqq 0$, or $x(a) = 0$. In the latter case, by the second part of the condition (iii), we have $x(b) - x(t_0) \geqq 0$ $(x(b) - x(t_0) \leqq 0)$ if $x'(a) > 0$ $(x'(a) < 0)$ and again $k \geqq 0$.

Consider the condition (e) in Theorem 7. Let $\{x_k\}$ be a sequence of solutions of (32) in $\langle a, b\rangle$ such that both sequences $\{x_k(a)\}$, $\{x_k(b) - x_k(t_0)\}$ are bounded, say by a constant $K > 0$. We may assume that $K \geqq \max\,(M, (b - t_0)\,M)$. Consider the function x_k with k arbitrary but fixed. We shall show that

$$|x_k(t)| \leqq K + \frac{K}{b - t_0}(t - a) \quad \text{for each} \quad t \in \langle a, b\rangle\,. \tag{34}$$

Since

$$|x_k(b) - x_k(t_0)| \leqq K\,, \tag{35}$$

by the Mean value theorem there exists a t_k, $t_0 < t_k < b$, such that $|x_k'(t_k)| \leqq K/(b - t_0)$. If (34) does not hold, then there is an s_k, $a \leqq s_k < b$, such that

$$|x_k(t)| \leqq K + \frac{K}{b - t_0}(t - a)\,, \quad a \leqq t \leqq s_k$$

and

$$|x_k(t)| > K + \frac{K}{b - t_0}(t - a)\,, \quad s_k < t < s_k + \varepsilon\,,$$

where $\varepsilon > 0$ is sufficiently small.

Suppose that

$$(36) \qquad x_k(s_k) = K + \frac{K}{b - t_0}(s_k - a) .$$

(The case $x_k(s_k) = -[K + K(s_k - a)/(b - t_0)]$ would proceed similarly). Then $x'_k(s_k) \geqq K/b - t_0) \geqq M$, $x_k(s_k) \geqq K \geqq M$ and, by the assumption (ii), $x''_k(s_k) > 0$. We can easily show that $x''_k(t) > 0$ in $\langle s_k, b)$ is true and therefore

$$(37) \qquad x'_k(t) > x'_k(s_k) \geqq \frac{K}{b - t_0} \quad \text{in} \quad (s_k, b\rangle ,$$

which implies that $t_0 < t_k \leqq s_k < b$. Thus

$$(38) \qquad |x_k(t_0)| \leqq K + \frac{K}{b - t_0}(t_0 - a) .$$

By (36), (37) we have

$$x_k(b) = x_k(s_k) + \int_{s_k}^{b} x'(t)\,\mathrm{d}t > K + \frac{K}{b - t_0}(s_k - a) +$$

$$+ \frac{K}{b - t_0}(b - s_k) = K + \frac{K}{b - t_0}(b - a) ,$$

and in view of (38),

$$x_k(b) - x_k(t_0) > K + \frac{K}{b - t_0}(b - a) - \left[K + \frac{K}{b - t_0}(t_0 - a)\right] = K .$$

This contradicts the inequality (35) and hence (34) is true. By this inequality the sequence $\{x_k\}$ is uniformly bounded in $\langle a, b\rangle$. So we have three bounded sequences. The sequence $\{t_k\}$ with the meaning given above, $t_0 < t_k < b$, $k = 1, 2, \ldots$, the sequence $\{x_k(t_k)\}$ and the sequence $\{x'_k(t_k)\}$ which is bounded by $K/(b - t_0)$. Therefore there exists a subsequence $\{t_{k(l)}\}$ of the sequence $\{t_k\}$ such that there exist finite $\lim_{l\to\infty} t_{k(l)} = \tau \in \langle t_0, b\rangle$, $\lim_{l\to\infty} x_{k(l)}(t_{k(l)}) = x_1$, $\lim_{l\to\infty} x'_{k(l)}(t_{k(l)}) = x_2$. Since each complete solution of the equation (32) exists on the whole interval $\langle a, b\rangle$, the Kamke convergence lemma guarantees that there is a subsequence $\{x_{k(m)}\}$ of the sequence $\{x_{k(l)}\}$ of solutions of (32) on $\langle a, b\rangle$, which together with the sequence $\{x'_{k(m)}\}$ is uniformly convergent on $\langle a, b\rangle$. Consequently, all assumptions of Theorem 7 are satisfied. Theorem 9 follows.

Remark 9. By Theorem 1.2 in [5], p. 124, if $f(t, \cdot, y)$ is nondecreasing in R for each $(t, y) \in \langle a, b\rangle \times R$, and for each $r > 0$ there is an $L_r > 0$ such that $|f(t, x, y) - f(t, x, z)| \leqq L_r|y - z|$ for any two points $(t, x, y), (t, x, z) \in \langle a, b) \times \langle -r, r\rangle \times \langle -r, r\rangle$, then there exists at most one solution of the problem (32), (33) for every couple $(A, B) \in R^2$.

Now we shall consider the boundary value problem

(39) $$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}),$$

(40) $$\mathscr{C}_j(x^{(j)}(t_j), x^{(j+1)}(t_j), \ldots, x^{(n-1)}(t_j)) = a_j, \quad j = 0, 1, \ldots, n-1,$$

where $a = t_0 < t_1 < \ldots < t_{n-1} = b$ are real numbers, $f \in C(\langle a, b\rangle \times R^n, R)$.

Theorem 10. *Suppose that*

(i) *there is a point $t_0 \in \langle a, b\rangle$ such that each initial value problem for the equation* (39) *at that point has a unique solution in $\langle a, b\rangle$,*

(ii) $\lim_{|x|\to\infty} |\mathscr{C}_j(x, a_{j+1}, \ldots, a_{n-1})| = \infty$ *uniformly in $(a_{j+1}, \ldots, a_{n-1})$ on compact subsets of R^{n-1-j}, $j = 0, 1, \ldots, n-2$,*
and

$$\lim_{|x|\to\infty} |\mathscr{C}_{n-1}(x)| = \infty,$$

(iii) *the boundary value problem* (39), (40) *for each n-tuple $(a_0, a_1, \ldots, a_{n-1})$ has at most one solution,*
and

(iv) *there is a $k_1 > 0$ such that*

$$x_{n-1} f(t, x_0, x_1, \ldots, x_{n-1}) \geqq 0$$

for each $t \in \langle a, b\rangle$, $(x_0, x_1, \ldots, x_{n-2}) \in R^{n-2}$ and $|x_{n-1}| > k_1$.

Then for each $(a_0, a_1, \ldots, a_{n-1}) \in R^n$ there exists a solution of the boundary value problem (39), (40).

Proof. By the assumptions (i), (iii), the conditions (a) and (b) of Theorem 7 are fulfilled. Suppose now that $\{x_k\}$ is a sequence of solutions of (39) such that $\mathscr{C}_j(x_k^{(j)}(t_j), x_k^{(j+1)}(t_j), \ldots, x_k^{(n-1)}(t_j)) = a_{j,k}$, $j = 0, 1, \ldots, n-1$, $k = 1, 2, \ldots$, and the sequences $\{a_{j,k}\}_{k=1}^{\infty}$, $j = 0, 1, \ldots, n-1$, are bounded. Then the last condition in (ii) implies that there is an N_{n-1} such that $|x_k^{(n-1)}(t_{n-1})| \leqq N_{n-1}$ for all $k = 1, 2, \ldots$. By the assumption (iv) for each solution x_k the following alternative holds: either $|x_k^{(n-1)}(t)| < k_1$ in $\langle a, b\rangle$ or there is the first point (the smallest point) $\tau \in \langle a, b\rangle$ at which $|x_k^{(n-1)}(\tau)| \geqq k_1$. Then for all t, $\tau \leqq t \leqq b$, the function $|x_k^{(n-1)}(t)|$ is nondecreasing and hence $|x_k^{(n-1)}(t)| \leqq |x_k^{(n-1)}(t_{n-1})| \leqq N_{n-1}$. Thus in both cases

(41) $$|x_k^{(n-1)}(t)| \leqq \max(k_1, N_{n-1}) \quad \text{for each} \quad t \in \langle a, b\rangle \quad \text{and} \quad k = 1, 2, \ldots.$$

The sequence $\{x_k^{(n-1)}\}_{k=1}^{\infty}$ is uniformly bounded in $\langle a, b\rangle$. Now by the condition $\lim_{|x|\to\infty} |\mathscr{C}_{n-2}(x, a_{n-1})| = \infty$ uniformly in a_{n-1} on compact subsets of R and in view of (37), we have that there is an $N_{n-2} > 0$ such that

(42) $$|x_k^{(n-2)}(t_{n-2})| \leqq N_{n-2} \quad \text{for all} \quad k = 1, 2, \ldots.$$

By (41), (42) we get that

$$|x_k^{(n-2)}(t)| \leqq N_{n-2} + (b-a)\max(k_1, N_{n-1}) = N_{n-2}^*, \quad t \in \langle a, b\rangle;$$
$$k = 1, 2, \ldots.$$

Proceeding in this way we obtain that all sequences $\{x_k^{(j)}\}_{k=1}^{\infty}, j = n - 1, n - 2, \ldots$ $\ldots, 1, 0$, are uniformly bounded on $\langle a, b\rangle$. By the continuity of f, this implies that $\{x_k^{(n)}\}_{k=1}^{\infty}$ is uniformly bounded on $\langle a, b\rangle$, too. Hence $\{x_k^{(j)}\}$, $j = 0, 1, \ldots, n - 1$, are equicontinuous on $\langle a, b\rangle$. Therefore, on the basis of the Ascoli lemma, there is a subsequence $\{x_{k(l)}\}$ of the sequence $\{x_k\}$ which is convergent in the space $C^{(n-1)}(\langle a, b\rangle, R)$ provided with the norm $|x|_{C^{(n-1)}} = \max_{j=0,1,\ldots,n-1} \{\max_{a \leqq t \leqq b} |x^{(j)}(t)|\}$. The condition (e) in Theorem 7 is satisfied, and the result follows.

Literature

[1] *Deimling, K.:* Nichtlineare Gleichungen und Abbildungsgrade. Springer-Verlag, Berlin, 1974.

[2] *Dugundji, J.* and *Granas, A.:* Fixed Point Theory. PWN, Warszawa, 1982.

[3] *Hartman, Ph.:* Ordinary Differential Equations. John Wiley, New York–London–Sydney, 1964 (Russian translation: Izdat. Mir, Moskva, 1976).

[4] *Kiguradze, I. T.* and *Lomtatidze, A. G.:* On Certain Boundary Value Problems for Second-Order Linear Ordinary Differential Equations with Singularities, J. Math. Anal. Appl. *101* (1984), 325—347.

[5] *Ломтатидзе, А. Г.:* Об одной сингулярной трехточечной краевой задаче, Груды Инст. Прикл. Мат. им. И. Н. Векуа *17* (1986), 122—133.

[6] *Nussbaum, R. D.:* Degree Theory for Local Condesing Maps, J. Math. Anal. Appl. *37* (1972), 741—766.

[7] *Opial, Z.:* Linear Problems for Systems of Nonlinear Differential Equations, J. Differential Equations *3* (1967), 580—594.

[8] *Šeda, V.:* Surjectivity and Boundary Value Problems, In: Proc. of the Conference Equadiff 6, Brno (1986), 161—170.

[9] *Zeidler, E.:* Vorlesungen über nichtlineare Funktionalanalysis I — Fixpunktsätze, Teubner Verlagsgesellschaft, Leipzig, 1976.

Author's address: 842 15 Bratislava, Mlynská dolina, Czechoslovakia (Matematicko-fyzikálna fakulta KU).