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DOMATIC NUMBERS OF LATTICE GRAPHS

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The domatic number of an undirected graph was introduced by E. J. Cockayne and S. T. Hedetniemi in [1] and [2], the total domatic number by the same authors and R. M. Dawes in [3].

A subset D of the vertex set $V(G)$ of an undirected graph G is called dominating (totally dominating), if for each vertex $x \in V(G) - D$ ($x \in V(G)$) respectively) there exists a vertex $y \in D$ adjacent to x. A partition of $V(G)$, all of whose classes are dominating (total dominating) sets in G, is called *domatic* (*total domatic*, respectively). The maximum number of classes of a domatic (total domatic) partition of G is called the *domatic* (*total domatic*, respectively) *number of* G. The domatic number of G is denoted by $d(G)$, its total domatic number by $d_t(G)$.

These definitions were originally formulated for finite graphs, but they may be used without change also for infinite graphs which are locally finite. (In the case of other infinite graphs the maximum must be replaced by the supremum.)

For some purposes it is more convenient to consider domatic colourings instead of domatic partitions. A colouring of vertices of a graph G is called domatic, if each vertex of G is adjacent to vertices of all colours different from its own. (Two vertices of the same colour may be adjacent.) Then the domatic number of G is the maximum number of colours of a domatic colouring of G. Analogously, a total domatic colouring of G may be introduced. We shall use these concepts in the proofs.

The lattice graph L_n of dimension n (where n is a positive integer) is the graph whose vertex set is the set of all n-dimensional vectors whose coordinates are integers, and in which two vectors $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_n)$ are adjacent if and only if there exists an integer i such that $1 \leqq i \leqq n$, $|v_i - w_i| = 1$ and $v_j = w_j$ for all $j \in \{1, \ldots, n\} - \{i\}$.

Theorem 1. *Let L_n be the lattice graph of dimension n. Then*

$$d(L_n) = 2n + 1 .$$

Proof. As was proved by E. J. Cockayne and S. T. Hedetniemi, $d(G) \leqq \delta(G) + 1$, where $\delta(G)$ is the minimum degree of a vertex of G. In our case $\delta(L_n) = 2n$ and therefore $d(L_n) \leqq 2n + 1$. Now we shall colour the vertices of L_n by the colours

$0, 1, \ldots, 2n$. For each vertex $\mathbf{v} = (v_1, \ldots, v_n)$ of L_n let $c_0(\mathbf{v}) = \sum_{i=1}^{n} iv_i$, and let $c(\mathbf{v})$ be the number from the set $\{0, 1, \ldots, 2n\}$ such that $c(\mathbf{v}) \equiv c_0(\mathbf{v}) \pmod{2n + 1}$. We colour any vertex $\mathbf{v}$ of L_n with the colour $c(\mathbf{v})$. We shall prove that this colouring is domatic.

Let $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{v}' = (v'_1, \ldots, v'_n)$ be two vertices of L_n whose distance is 2. Then $c_0(\mathbf{v}) - 2n \leqq c_0(\mathbf{v}') \leqq c_0(\mathbf{v}) + 2n$. Thus $c_0(\mathbf{v}) \equiv c_0(\mathbf{v}') \pmod{2n + 1}$ (i.e. $c(\mathbf{v}) = c(\mathbf{v}')$) if and only if $c_0(\mathbf{v}) = c_0(\mathbf{v}')$. If $\mathbf{v}$, $\mathbf{v}'$ differ in one coordinate, say the i-th, then $|v_i - v'_i| = 2$ and $v_k = v'_k$ for $k \neq i$. We have $c_0(\mathbf{v}') = c_0(\mathbf{v}) + 2i$ or $c_0(\mathbf{v}') = c_0(\mathbf{v}) - 2i$ and thus $c_0(\mathbf{v}') \neq c_0(\mathbf{v})$. If $\mathbf{v}$, $\mathbf{v}'$ differ in two coordinates, say the i-th and the j-th, then $|v_i - v'_i| = |v_j - v'_j| = 1$ and $v_k = v'_k$ for $k \neq i$, $k \neq j$. We have $c_0(\mathbf{v}') = c_0(\mathbf{v}) + i + j$ or $c_0(\mathbf{v}') = c_0(\mathbf{v}) + i - j$ or $c_0(\mathbf{v}') = c_0(\mathbf{v}) - i + j$ or $c_0(\mathbf{v}') = c_0(\mathbf{v}) - i - j$. As $i \neq j$, $i > 0, j > 0$, again $c_0(\mathbf{v}') \neq c_0(\mathbf{v})$. We have proved that any two vertices of L_n having the distance 2 have different colours. Analogously, any two vertices of L_n having the distance 1 have different colours as well. For any vertex of a graph the set consisting of this vertex and of all vertices adjacent to it has the property that any two vertices of this set have the distance at most 2. Hence in L_n the vertices of such a set are coloured with pairwise different colours. As such a set has the cardinality $2n + 1$, any vertex of L_n is adjacent to vertices of all colours different from its own. We have $d(L_n) = 2n + 1$. □

Theorem 2. *Let L_n be the lattice graph of dimension n. Then*

$$d_t(L_n) = 2n\,.$$

Proof. In [3] it is proved that $d_t(G) \leqq \delta(G)$, therefore $d_t(L_n) \leqq \delta(L_n) = 2n$. Now we shall colour the vertices of L_n with the colours $0, 1, \ldots, 2n - 1$. To every vertex $\mathbf{v} = (v_1, \ldots, v_n)$ we assign the number $c_0(\mathbf{v}) = \sum_{i=1}^{n-1} iv_i$. Let $c(\mathbf{v})$ be the number such that $c(\mathbf{v}) \in \{0, 1, \ldots, 2n - 1\}$ and $c(\mathbf{v}) \equiv c_0(\mathbf{v}) \pmod{2n}$. If $v_n \equiv 0 \pmod 4$ or $v_n \equiv 1 \pmod 4$, we colour $\mathbf{v}$ with the colour $c(\mathbf{v})$; if $v_n \equiv 2 \pmod 4$ or $v_n \equiv 3 \pmod 4$, we colour $\mathbf{v}$ with the colour $c(\mathbf{v}) + n$ (if $c(\mathbf{v}) \leqq n - 1$) or $c(\mathbf{v}) - n$ (if $c(\mathbf{v}) \geqq n$). We shall prove that this colouring is total domatic.

Let $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{v}' = (v'_1, \ldots, v'_n)$ be two vertices of L_n having the distance 2. Suppose that $\mathbf{v}$ and $\mathbf{v}'$ have the same colour. This is possible only if either $c(\mathbf{v}) = c(\mathbf{v}')$, or $|c(\mathbf{v}) - c(\mathbf{v}')| = n$. As the distance between $\mathbf{v}$ and $\mathbf{v}'$ is 2, we have $c_0(\mathbf{v}) - 2n + 2 \leqq c_0(\mathbf{v}') \leqq c_0(\mathbf{v}) + 2n - 2$. Thus $c_0(\mathbf{v}) \equiv c_0(\mathbf{v}') \pmod{2n}$ (i.e. $c(\mathbf{v}) = c(\mathbf{v}')$) if and only if $c_0(\mathbf{v}) = c_0(\mathbf{v}')$. Analogously as in the proof of Theorem 1 we prove that this is possible only if $v_i = v'_i$ for $i = 1, \ldots, n - 1$. Then we must have $|v_n - v'_n| = 2$. If one of the numbers v_n, v'_n is congruent with 0 or 1 modulo 4, then the other is congruent with 2 or 3 modulo 4 and vice versa. Thus they have different colours; we have excluded the case $c(\mathbf{v}) = c(\mathbf{v}')$. Suppose that $|c(\mathbf{v}) - c(\mathbf{v}')| = n$; without loss of generality let $c(\mathbf{v}') = c(\mathbf{v}) + n$. Then either there exist two numbers i, j

from $\{1, \ldots, n-1\}$ such that $|v_i - v'_i| = |v_j - v'_j| = 1$, or there exists one such number i such that $|v_i - v'_i| = 2$; otherwise we would have $|c(\mathbf{v}) - c(\mathbf{v}')| \leqq n - 1$. In both cases we have $v_n = v'_n$; otherwise the distance between $\mathbf{v}$ and $\mathbf{v}'$ would be at least 3. If $v_n \equiv 0 \pmod 4$ or $v_n \equiv 1 \pmod 4$, then $\mathbf{v}$ is coloured with $c(\mathbf{v})$ and $\mathbf{v}'$ is coloured with $c(\mathbf{v}) + n$; if $v_n \equiv 2 \pmod 4$ or $v_n \equiv 3 \pmod 4$, then $\mathbf{v}$ is coloured with $c(\mathbf{v}) + n$ and $\mathbf{v}'$ with $c(\mathbf{v})$. We have proved that any two vertices of L_n having the distance 2 have different colours. Analogously as in the proof of Theorem 1 it follows that any vertex of L_n is adjacent to vertices of all colours. We have $d_t(L_n) = 2n$. $\square$

A finite analogue of a lattice graph is a direct product of circuits. A direct product of n graphs $G_1, \ldots, G_n$ is the graph whose vertex set is the Cartesian product of the vertex sets $V(G_1), \ldots, V(G_n)$, and in which two vertices $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_n)$ are adjacent if and only if there exists an integer i such that $1 \leqq i \leqq n$, v_i and w_i are adjacent in G_i and $v_j = w_j$ for all $j \in \{1, \ldots, n\} - \{i\}$.

If all graphs $G_1, \ldots, G_n$ are circuits, we may represent the vertices of their direct product similarly as the vertices of L_n. They are represented by n-dimensional vectors whose coordinates are residue classes; for each $i = 1, \ldots, n$ the i-th coordinate is a residue class modulo the length of the circuit G_i. The adjacency can be described analogously as in L_n. Thus we have two following theorems whose proofs are quite analogous to the proofs of Theorem 1 and Theorem 2.

Theorem 3. *Let G be the direct product of n circuits whose lengths are divisible by $2n + 1$. Then*

$$d(G) = 2n + 1\,.$$

Theorem 4. *Let G be the direct product of n circuits, one of which has a length divisible by 4 while all others have lengths divisible by $2n$. Then*

$$d_t(G) = 2n\,.$$

References

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