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AFFINE DIFFERENTIAL GEOMETRY OF SURFACES

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The differential geometry of surfaces in the equiaffine 3-dimensional space has been thoroughly studied. On the other hand, the more general case of the full affine group has got little attention; see [3], [4] and the contributions of S. Gigena and U. Simon in [1]. In what follows, I create a systematic theory of this general case.

HYPERBOLIC CASE

1. Let A^3 be the 3-dimensional affine space, $D \subset \mathbb{R}^2$ a domain and $m: D \to A^3$ a surface. Instead of D, we may consider a 2-dimensional differentiable manifold; nevertheless, our considerations will be of local nature. We suppose that all manifolds and mappings are of class C^∞.

With each point of our surface, let us associate a frame $\{m; v_1, v_2, v_3\}$ such that e_1, e_2 are situated in the tangent plane of the surface. Our frames satisfy

$$(1.1)\qquad dm = \omega^i v_i\,, \quad dv_i = \omega_i^j v_j\,; \quad i, j, \ldots = 1, 2, 3\,,$$

with the integrability conditions

$$(1.2)\qquad d\omega^i = \omega^j \wedge \omega_j^i\,, \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j\,;$$

of course, we have

$$(1.3)\qquad \omega^3 = 0\,.$$

The differential consequence of (1.3) being

$$(1.4)\qquad \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0\,,$$

Cartan's lemma implies the existence of functions $a'_1, a'_2, a'_3: D \to \mathbb{R}$ such that

$$(1.5)\qquad \omega_1^3 = a'_1\omega^1 + a'_2\omega^2\,, \quad \omega_2^3 = a'_2\omega^1 + a'_3\omega^2\,.$$

On D, we may choose local coordinates (u, v) such that

$$(1.6)\qquad \omega^1 = f\,du\,, \quad \omega^2 = g\,dv\,; \quad f = f(u, v) \neq 0\,, \quad g = g(u, v) \neq 0\,;$$
$$\omega_i^j = f_i^j(u, v)\,du + g_i^j(u, v)\,dv\,.$$

Let $p_0 = (u_0, v_0) \in D$ be a fixed point and $\gamma: (-h\; h) \to D$ a curve such that $\gamma(0) =$

$= p_0$; let γ be given by $u = u(t)$ $v = v(t)$. Then

$$(1.7) \qquad \frac{\mathrm{d}m}{\mathrm{d}t} = fu'v_1 + gv'v_2 ,$$

$$\frac{\mathrm{d}^2 m}{\mathrm{d}t^2} = (\cdot)\, v_1 + (\cdot)\, v_2 + (a'_1 f^2 u'^2 + 2a'_2 fgu'v' + a'_3 g^2 v'^2)\, v_3$$

with $u' = \mathrm{d}u/\mathrm{d}t$, $v' = \mathrm{d}v/\mathrm{d}t$. A vector $T \in T_{p_0}(D)$ is called *asymptotic* (with respect to the given surface $m: D - A^3$) if we have $\mathrm{d}^2 m/\mathrm{d}t^2 \in \mathrm{d}m(T_{p_0}(D))$ for each curve γ with the tangent vector T at p_0. From (1.7_2), we see that the vector $T = u_0\, \partial/\partial u + v_0\, \partial/\partial v \in T_{p_0}(D)$ is asymptotic if and only if

$$(1.8) \qquad a'_1 f^2 u_0^2 + 2a'_2 fgu_0 v_0 + a'_3 g^2 v_0^2 = 0 \quad \text{at} \quad p_0 .$$

Of course, $\mathrm{d}m(T_{p_0}(D))$ is the *tangent plane* of our surface m at the point $m_0 = m(p_0)$; let us denote it by $T_{m_0}(m)$. A tangent vector $v \in T_{m_0}(m)$ is called *asymptotic* if there is an asymptotic vector $T \in T_{p_0}(D)$ such that $\mathrm{d}m(T) = v$. Finally, the surface $m: D \to A^3$ is called *hyperbolic* if there are exactly two linearly independent asymptotic vectors at each point $m(p)$, $p \in D$.

Let m be a hyperbolic surface, and let us choose our frames in such a way that e_1, e_2 are asymptotic. Because of

$$(1.9) \qquad \mathrm{d}m(T) = \mathrm{d}m\left(u_0 \frac{\partial}{\partial u} + v_0 \frac{\partial}{\partial v}\right) = u_0 f v_1 + v_0 g v_2 ,$$

the vectors $f^{-1}\, \partial/\partial u$, $g^{-1}\, \partial/\partial v \in T_{p_0}(D)$ must be asymptotic, and (1.8) implies

$$(1.10) \qquad a'_1 = a'_3 = 0 , \quad a'_2 \neq 0 .$$

Thus we have $\mathrm{d}v_1 = \omega_1^1 v_1 + \omega_1^2 v_2 + \omega^2 a'_2 v_3$, $\mathrm{d}v_2 = \omega_2^1 v_1 + \omega_2^2 v_2 + \omega^1 a'_2 v_3$. Hence we see that we are in the position to choose our frames in such a way that

$$(1.11) \qquad \omega_1^3 = \omega^2 , \quad \omega_2^3 = \omega^1 .$$

The integrability conditions of (1.11) are

$$(1.12) \qquad \omega_1^2 \wedge \omega^1 + \tfrac{1}{2}(\omega_1^1 + \omega_2^2 - \omega_3^3) \wedge \omega^2 = 0 ,$$

$$\tfrac{1}{2}(\omega_1^1 + \omega_2^2 - \omega_3^3) \wedge \omega^1 + \omega_2^1 \wedge \omega^2 = 0 ,$$

and there are functions $a_1, \ldots, a_4 : D \to \mathbb{R}$ such that

$$(1.13) \qquad \omega_1^2 = a_1 \omega^1 + a_2 \omega^2 , \quad \tfrac{1}{2}(\omega_1^1 + \omega_2^2 - \omega_3^3) = a_2 \omega^1 + a_3 \omega^2 ,$$

$$\omega_2^1 = a_3 \omega^1 + a_4 \omega^2 .$$

Let $\{m; \tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ be another field of frames associated with our surface, let it satisfy the analogous equations

$$(1.14) \qquad \mathrm{d}m = \tilde{\omega}^i \tilde{v}_i , \quad \mathrm{d}\tilde{v}_i = \tilde{\omega}_i^j \tilde{v}_j$$

and $(\widetilde{1.3}) + (\widetilde{1.11}) + (\widetilde{1.13})$. Then we have

$$(1.15) \qquad \tilde{v}_1 = \alpha v_1 , \quad \tilde{v}_2 = \beta v_2 , \quad \tilde{v}_3 = \varphi v_1 + \psi v_2 + \gamma v_3 ; \quad \alpha\beta\gamma \neq 0 .$$

From (1.1) and (1.14) we easily see that

$$(1.16)\qquad \tilde{\omega}^1 = \alpha^{-1}\omega^1\,, \quad \tilde{\omega}^2 = \beta^{-1}\omega^2\,;$$

$$(1.17)\qquad \gamma = \alpha\beta$$

and

$$(1.18)\qquad \tilde{a}_1 = \alpha^2\beta^{-1}a_1\,, \quad \tilde{a}_4 = \alpha^{-1}\beta^2 a_4\,,$$

$$(1.19)\qquad \tilde{a}_2 = \alpha a_2 - \beta^{-1}\psi\,, \quad \tilde{a}_3 = \beta a_3 - \alpha^{-1}\varphi\,.$$

Thus it is possible to choose the frames in such a way that $a_2 = a_3 = 0$, i.e.,

$$(1.20)\qquad \omega_1^2 = a_1\omega^1\,, \quad \omega_3^3 = \omega_1^1 + \omega_2^2\,, \quad \omega_2^1 = a_4\omega^2\,;$$

the admissible changes of the frames are then

$$(1.21)\qquad \tilde{v}_1 = \alpha v_1\,, \quad \tilde{v}_2 = \beta v_2\,, \quad \tilde{v}_3 = \alpha\beta v_3\,; \quad \alpha\beta \neq 0\,.$$

The differential consequences of (1.20) are

$$(1.22)\qquad \begin{aligned} &\{\mathrm{d}a_1 + a_1(\omega_2^2 - 2\omega_1^1)\} \wedge \omega^1 + \omega_3^2 \wedge \omega^2 = 0\,,\\ &\omega_3^2 \wedge \omega^1 + \omega_3^1 \wedge \omega^2 = 0\,,\\ &\omega_3^1 \wedge \omega^1 + \{\mathrm{d}a_4 + a_4(\omega_1^1 - 2\omega_2^2)\} \wedge \omega^2 = 0\,,\end{aligned}$$

and we get the existence of functions $b_1, \ldots, b_5 \colon D \to \mathbb{R}$ such that

$$(1.23)\qquad \begin{aligned} &\mathrm{d}a_1 + a_1(\omega_2^2 - 2\omega_1^1) = b_1\omega^1 + b_2\omega^2\,, \quad \omega_3^2 = b_2\omega^1 + b_3\omega^2\,,\\ &\omega_3^1 = b_3\omega^1 + b_4\omega^2\,, \quad \mathrm{d}a_4 + a_4(\omega_1^1 - 2\omega_2^2) = b_4\omega^1 + b_5\omega^2\,.\end{aligned}$$

Elementary calculations yield the relations between $b_1, \ldots, b_5$ and $\tilde{b}_1, \ldots, \tilde{b}_5$; we get

$$(1.24)\qquad \begin{aligned} &\tilde{b}_1 = \alpha^3\beta^{-1}b_1\,, \quad \tilde{b}_2 = \alpha^2 b_2\,, \quad \tilde{b}_3 = \alpha\beta b_3\,, \quad \tilde{b}_4 = \beta^2 b_4\,,\\ &\tilde{b}_5 = \alpha^{-1}\beta^3 b_5\,.\end{aligned}$$

The exterior differentiation of (1.23) gives the equalities

$$(1.25)\qquad \begin{aligned} &\{\mathrm{d}b_1 + b_1(\omega_2^2 - 3\omega_1^1)\} \wedge \omega^1 + (\mathrm{d}b_2 - 2b_2\omega_1^1) \wedge \omega^2 =\\ &= 3a_1(b_3 - a_1a_4)\,\omega^1 \wedge \omega^2\,,\\ &(\mathrm{d}b_2 - 2b_2\omega_1^1) \wedge \omega^1 + (\mathrm{d}b_3 - b_3\omega_3^3) \wedge \omega^2 = -a_1b_4\omega^1 \wedge \omega^2\,,\\ &(\mathrm{d}b_3 - b_3\omega_3^3) \wedge \omega^1 + (\mathrm{d}b_4 - 2b_4\omega_2^2) \wedge \omega^2 = \quad a_4b_2\omega^1 \wedge \omega^2\,,\\ &(\mathrm{d}b_4 - 2b_4\omega_2^2) \wedge \omega^1 + \{\mathrm{d}b_5 + b_5(\omega_1^1 - 3\omega_2^2)\} \wedge \omega^2 =\\ &= 3a_4(a_1a_4 - b_3)\,\omega^1 \wedge \omega^2\end{aligned}$$

and the existence of functions $b_{ij} \colon D \to \mathbb{R}$ such that

$$(1.26)\qquad \begin{aligned} &\mathrm{d}b_1 + b_1(\omega_2^2 - 3\omega_1^1) = b_{11}\omega^1 + b_{12}\omega^2\,,\\ &\mathrm{d}b_2 - 2b_2\omega_1^1 = b_{21}\omega^1 + b_{22}\omega^2\,,\end{aligned}$$

$$db_3 - b_3\omega_3^3 = b_{31}\omega^1 + b_{32}\omega^2 ,$$
$$db_4 - 2b_4\omega_2^2 = b_{41}\omega^1 + b_{42}\omega^2 ,$$
$$db_5 + b_5(\omega_1^1 - 3\omega_2^2) = b_{51}\omega^1 + b_{52}\omega^2 ;$$

(1.27) $$b_{21} - b_{12} = 3a_1(b_3 - a_1a_4) , \quad b_{22} - b_{31} = a_1b_4 ,$$
$$b_{41} - b_{32} = a_4b_2 , \quad b_{51} - b_{42} = 3a_4(a_1a_4 - b_3) .$$

Because of (1.24), the functions b_i are the so-called relative invariants, i.e., $b_i = 0$ has a geometrical meaning.

Definition. The surfaces with $b_3 \neq 0$ are called *non-maximal.* (The geometrical meaning of this condition will be presented in the next section.)

Let us consider a non-maximal surface $m: D \to A^3$; we may choose its moving frames in such a way, see (1.24_3), that

(1.28) $$b_3 = \varepsilon' \in \mathbb{R} , \quad \varepsilon' \neq 0 .$$

The equation (1.26_3) reduces then to $-\varepsilon'\omega_3^3 = b_{31}\omega^1 + b_{32}\omega^2$, which may be written, see (1.20_2), as

(1.29) $$\omega_1^1 + \omega_2^2 = \omega_3^3 = c_1\omega^1 + c_2\omega^2 .$$

According to (1.24_3), the admissible changes of the frames are given by (1.21) with

(1.30) $$\alpha\beta = 1 .$$

Because of (1.16), we get

(1.31) $$dS^2 := \omega^1\omega^2 = \tilde{\omega}^1\tilde{\omega}^2 = d\tilde{S}^2 .$$

Thus our surface m induces, on D, a hyperbolic metric. Its Gauss curvature may be calculated as follows: there is exactly one 1-form ω such that

(1.32) $$d\omega^1 = \omega^1 \wedge \omega , \quad d\omega^2 = \omega \wedge \omega^2 ;$$

the Gauss curvature $\varkappa$ is then given by

(1.33) $$d\omega = \tfrac{1}{2}\varkappa\omega^1 \wedge \omega^2 .$$

Let us calculate $\varkappa$ in our case. It is easy to see that the 1-form

(1.34) $$\omega = \omega_1^1 - c_1\omega^1 = -\omega_2^2 + c_2\omega^2$$

satisfies (1.32). The differential consequence of (1.29) being

(1.35) $$(dc_1 - c_1\omega) \wedge \omega^1 + (dc_2 + c_2\omega) \wedge \omega^2 = 0 ,$$

we get the existence of functions $c_{ij}: D \to \mathbb{R}$ such that

(1.36) $$dc_1 - c_1\omega = c_{11}\omega^1 + c_{12}\omega^2 , \quad dc_2 + c_2\omega = c_{12}\omega^1 + c_{22}\omega^2 .$$

From (1.1), (1.14), (1.21) and (1.30),

(1.37) $$\tilde{\omega}_3^3 = \omega_3^3 , \quad \tilde{\omega}_1^1 = \omega_1^1 + \alpha^{-1} d\alpha ,$$

and we get

$$(1.38) \qquad \tilde{c}_1 = \alpha c_1 , \quad \tilde{c}_2 = \beta c_2 ,$$

$$(1.39) \qquad \tilde{\omega} = \omega + \alpha^{-1} \, d\alpha$$

and

$$(1.40) \qquad \tilde{c}_{11} = \alpha^2 c_{11} , \quad \tilde{c}_{12} = c_{12} , \quad \tilde{c}_{22} = \beta^2 c_{22} .$$

Finally, from (1.34) and (1.36),

$$(1.41) \qquad d\omega = (a_1 a_4 + c_{12} - \varepsilon') \, \omega^1 \wedge \omega^2 ;$$

the functions $a_1 a_4$, c_{12}: $D \to R$ are, see (1.18) + (1.30) and (1.40_2), invariants of our surface. Let us summarize:

Proposition 1. *Let* $m: D \to A^3$ *be a hyperbolic non-maximal surface. We may choose its frames in such a way that we have* (1.1) *with* (1.3), (1.11), (1.20), (1.29), (1.28), (1.23), (1.26) *and* (1.36), *the form* ω *being given by* (1.34) *and satisfying* (1.32). *The possible changes of the frames are given by* (1.21) + (1.30); *we get* (1.18), (1.24), (1.38) *and* (1.40). *The hyperbolic metric* (1.31) *is invariant, and its Gauss curvature is given by*

$$(1.42) \qquad 2\varkappa = a_1 a_4 + c_{12} - \varepsilon'$$

(theorema egregium).

2. Consider a general hyperbolic surface $m: D \to A^3$. The functions a_i, b_i are relative invariants, see (1.18) and (1.24), and using them, we may construct absolute invariants of our surface. In this section I am going to explain the geometrical signification of the conditions $a_i = 0$ and $b_i = 0$, and to define the fundamental invariants of our surface. I am going to restrict myself to the case in which all the relative invariants a_i, b_i are different from zero; the investigation of special cases is quite similar and it is left to the reader.

Let $V^3 \equiv V^3(R)$ be the vector space of our affine space A^3, and let $P^2 \equiv P^2(R)$ be the improper plane (i.e., the plane at infinity) of A^3. The plane P^2 consists of the points $\mathfrak{v} = \{\varrho v;\ O \neq v \in V^3,\ 0 \neq \varrho \in R\}$; we write simply $\mathfrak{v} = (v)$. Let $\mathfrak{v}, \mathfrak{w} \in P^2$, $\mathfrak{v} \neq \mathfrak{w}$; by $\{\mathfrak{v}, \mathfrak{w}\}$ we denote the straight line through $\mathfrak{v}$ and $\mathfrak{w}$. Further, let the anharmonic ratio of the points $\mathfrak{v} = (v)$, $\mathfrak{w} = (w)$, $\mathfrak{x} = (av + bw)$, $\mathfrak{y} = (cv + dw)$; $a, \ldots, d \in R$; be define by

$$(2.1) \qquad (\mathfrak{v}, \mathfrak{w}; \mathfrak{x}, \mathfrak{y}) = a^{-1} b c d^{-1} \quad \text{for} \quad ad \neq 0 ;$$

for $ad = 0$ it is not defined.

Taking into consideration our hyperbolic surface $m: D \to A^3$, we get the associated mapping $\mu: D \to P^2 \times P^2$ with $\mu(p) = (\mathfrak{v}_1(p), \mathfrak{v}_2(p))$; $p \in D$; $\mathfrak{v}_1(p) = (v_1(p))$ and $\mathfrak{v}_2(p) = (v_2(p))$ are the improper points of the asymptotic tangents at $m(p)$. From

(1.11), (1.20) and (1.23), we get

$$(2.2)\qquad dv_1 = \omega_1^1 v_1 + a_1\omega^1 v_2 + \omega^2 v_3\,,\quad dv_2 = a_4\omega^2 v_1 + \omega_2^2 v_2 + \omega^1 v_3\,,$$
$$dv_3 = (b_3\omega^1 + b_4\omega^2)\,v_1 + (b_2\omega^1 + b_3\omega^2)\,v_2 + (\omega_1^1 + \omega_2^2)\,v_3\,.$$

Let t_1, t_2 be the tangent vector fields on D such that $dm(t_i) = v_i$, i.e.,

$$(2.3)\qquad \omega^1(t_1) = \omega^2(t_2) = 1\,,\quad \omega^1(t_2) = \omega^2(t_1) = 0\,.$$

The vector field t_α; $\alpha = 1, 2$, generates a layer $\mathscr{L}_\alpha$ of curves on D; $\mathscr{L}_1$ and $\mathscr{L}_2$ form a net $\mathscr{N}$ of curves on D.

From (2.2) we see that $a_1 \neq 0$ (at a point $p \in D$) if and only if the mapping $p \mapsto \mathfrak{v}_1(p)$ is a local diffeomorphism; similarly for $a_4 \neq 0$. This explains the geometrical meaning of the conditions $a_1 = 0$ and $a_4 = 0$.

Given a mapping $\mathfrak{v}: D \to P^2$ and a tangent vector field t on D, let $p \in D$. For $\mathfrak{v} = (v) = (\varrho v)$ we see that $t_p(\varrho v) = t_p\varrho v + \varrho t_p v$, i.e., $(t_p(\varrho v)) \in \{(v), (t_p(v))\}$. Denote by $t_p\mathfrak{v}$ the straight line $\{(v), (t_p v)\}$ in the case $(v) \neq (t_p(v))$; in the case $(t_p v) = (v)$ or in the case $t_p v = O \in V^3$, let $t_p\mathfrak{v} = \mathfrak{v}$.

We see that $t_2\mathfrak{v}_1$ and $t_1\mathfrak{v}_2$ are straight lines and

$$(2.4)\qquad \mathfrak{v}_3 = t_2\mathfrak{v}_1 \cap t_1\mathfrak{v}_2$$

is an invariant of our surface; it is the improper point of the *affine normal* of our surface.

Suppose $a_1a_4 \neq 0$ on D; in this case the mappings $\mathfrak{v}_1, \mathfrak{v}_2: D \to P^2$ map the net $\mathscr{N}$ into two nets $\mathscr{V}_1$ and $\mathscr{V}_2$ on P^2. It is easy to see that $\mathscr{V}_2$ is the Laplace transform of $\mathscr{V}_1$ and vice versa; indeed,

$$(2.5)\qquad (t_1v_1) = (\omega_1^1(t_1)\,v_1 + a_1v_2) \in \{\mathfrak{v}_1, \mathfrak{v}_2\}\,,$$
$$(t_2v_2) = (a_4v_1 + \omega_2^2(t_2)\,v_2) \in \{\mathfrak{v}_1, \mathfrak{v}_2\}\,.$$

From (2.2),

$$(2.6)\qquad t_1v_3 = b_3v_1 + b_2v_2 + \omega_3^3(t_1)\,v_3\,,\quad t_2v_3 = b_4v_1 + b_3v_2 + \omega_3^3(t_2)\,v_3\,,$$

and *the points*

$$(2.7)\qquad \mathfrak{P}_1 := t_1\mathfrak{v}_3 \cap \{\mathfrak{v}_1, \mathfrak{v}_2\} = (b_3v_1 + b_2v_2)\,,$$
$$\mathfrak{P}_2 := t_2\mathfrak{v}_3 \cap \{\mathfrak{v}_1, \mathfrak{v}_2\} = (b_4v_1 + b_3v_2)$$

are invariants of our surface; we have

$$(2.8)\qquad I_1 := (\mathfrak{v}_1, \mathfrak{v}_2; \mathfrak{P}_1, \mathfrak{P}_2) = b_2b_4b_3^{-2}\,.$$

I claim that *the points*

$$(2.9)\qquad \mathfrak{R}_1 := (b_2v_1 - a_1v_3)\,,\quad \mathfrak{R}_2 := (b_4v_2 - a_4v_3)$$

describe the Laplace transforms of the nets $\mathscr{V}_1$ *and* $\mathscr{V}_2$, *respectively*.

Indeed, we have

$$(2.10)\qquad t_2v_1 = \omega_1^1(t_2)\,v_1 + v_3\,,\quad t_1v_2 = \omega_2^2(t_1)\,v_2 + v_3\,,$$

and the Laplace transform of $\mathfrak{v}_1$ (or $\mathfrak{v}_2$) different from $\mathfrak{v}_2$ (or $\mathfrak{v}_1$) is situated on the straight line $\{\mathfrak{v}_1, \mathfrak{v}_3\}$ (or $\{\mathfrak{v}_2, \mathfrak{v}_3\}$, respectively). Further, using (1.23) and (1.26) we obtain

$$(2.11)\qquad \begin{aligned} t_1(b_2v_1 - a_1v_3) &= (3b_2\omega_1^1(t_1) + b_{21} - a_1b_3)\, v_1 - (3a_1\omega_1^1(t_1) + b_1)\, v_3\,,\\ t_2(b_4v_2 - a_4v_3) &= (3b_4\omega_2^2(t_2) + b_{42} - a_4b_3)\, v_2 - (3a_4\omega_2^2(t_2) + b_5)\, v_3\,, \end{aligned}$$

i.e., $t_1\mathfrak{R}_1 = \{\mathfrak{v}_1, \mathfrak{v}_3\}$, $t_2\mathfrak{R}_2 = \{\mathfrak{v}_2, \mathfrak{v}_3\}$, and the assertion is proved.

On D, consider the tangent vector fields

$$(2.12)\qquad t'_1 := b_4t_1 - b_3t_2\,. \quad t'_2 := -b_3t_1 + b_2t_2$$

with the following geometrical interpretation: we have $t'_1\mathfrak{v}_3 = \{\mathfrak{v}_2, \mathfrak{v}_3\}$, $t'_2\,\mathfrak{v}_3 = \{\mathfrak{v}_1, \mathfrak{v}_3\}$. Indeed, we have

$$(2.13)\qquad t'_1v_3 = (b_2b_4 - b_3^2)\, v_2 + \omega_3^3(t'_1)\, v_3\,, \quad t'_2v_3 = (b_2b_4 - b_3^2)\, v_1 + \omega_3^3(t'_2)\, v_3\,.$$

Now,

$$(2.14)\qquad t'_1v_1 = \omega_1^1(t'_1)\, v_1 + a_1b_4v_2 - b_3v_3\,, \quad t'_2v_2 = a_4b_2v_1 + \omega_2^2(t'_2)\, v_2 - b_3v_3\,.$$

and the points

$$(2.15)\qquad \begin{aligned} \mathfrak{S}_1 &:= (a_4b_2v_1 - b_3v_3) = t'_2\mathfrak{v}_2 \cap \{\mathfrak{v}_1, \mathfrak{v}_3\}\,,\\ \mathfrak{S}_2 &:= (a_1b_4v_2 - b_3v_3) = t'_1\mathfrak{v}_1 \cap \{\mathfrak{v}_2, \mathfrak{v}_3\} \end{aligned}$$

are further invariants associated with our surface. Further,

$$(2.16)\qquad I_2 := (\mathfrak{v}_1, \mathfrak{v}_3; \mathfrak{R}_1, \mathfrak{S}_1) = (\mathfrak{v}_2, \mathfrak{v}_3; \mathfrak{R}_2, \mathfrak{S}_2) = a_1a_4b_3^{-1}\,.$$

Let

$$(2.17)\qquad \mathfrak{S} := \{\mathfrak{S}_1, \mathfrak{v}_2\} \cap \{\mathfrak{S}_2, \mathfrak{v}_1\} = (a_4b_2v_1 + a_1b_4v_2 - b_3v_3)$$

and

$$(2.18)\qquad \mathfrak{S}' := \{\mathfrak{v}_3, \mathfrak{S}\} \cap \{\mathfrak{v}_1, \mathfrak{v}_2\} = (a_4b_2v_1 + a_1b_4v_2)\,;$$

$\mathfrak{S}'$ *is an invariant*, and we have

$$(2.19)\qquad \begin{aligned} I_3 &:= (\mathfrak{v}_1, \mathfrak{v}_2; \mathfrak{P}_1, \mathfrak{S}') = a_4b_2^2(a_1b_3b_4)^{-1}\,,\\ I'_3 &:= (\mathfrak{v}_1, \mathfrak{v}_2; \mathfrak{S}', \mathfrak{P}_2) = a_1b_4^2(a_4b_2b_3)^{-1}\,; \end{aligned}$$

of course,

$$(2.20)\qquad I_3I'_3 = I_1\,.$$

The possible changes of the frames associated with our surface being given by (1.21), we see from (1.18) + (1.24_{2-4}) that each absolute invariant which may be constructed from a_1, a_2, b_2, b_3, b_4 may be constructed from I_1, I_2 and I_3 (or I'_3). Thus it remains to construct absolute invariants containing b_1 and b_5.

The dual projective plane P^{2*} to our plane P^2 consists of all planes $\mathfrak{E} = \{\sigma v \wedge w;\ 0 \neq \sigma \in \mathbb{R};\ v, \omega \in V^3,\ v$ and w linearly independent$\}$. Writing

$$(2.21)\qquad E^3 := v_1 \wedge v_2\,, \quad E^2 := v_3 \wedge v_1\,, \quad E^1 := v_2 \wedge v_3\,,$$

it is easy to see that

$$(2.22)\qquad \begin{aligned} dE^3 &= (\omega_1^1 + \omega_2^2)\,E^3 - \omega^1 E^2 - \omega^2 E^1\,,\\ dE^2 &= -(b_2\omega^1 + b_3\omega^2)\,E^3 + (2\omega_1^1 + \omega_2^2)\,E^2 - a_1\omega^1 E^1\,,\\ dE^1 &= -(b_3\omega^1 + b_4\omega^2)\,E^2 - a_4\omega^2 E^2 + (\omega_1^1 + 2\omega_2^2)\,E^1\,. \end{aligned}$$

Thus

$$(2.3)\qquad \begin{aligned} t_2E^3 &= \omega_3^3(t_2)\,E^3 - E^1,\ t_2\mathfrak{E}^3 = \{\mathfrak{E}^3, \mathfrak{E}^1\}\,;\\ t_1E^3 &= \omega_3^3(t_1)\,E^3 - E^2\,,\quad t_1\mathfrak{E}^3 = \{\mathfrak{E}^3, \mathfrak{E}^2\}\,. \end{aligned}$$

The induced mapping $D \to P^{2*}$, $p \mapsto \mathfrak{E}^3(p)$, maps the net $\mathcal{N}$ into a net $\mathcal{E}^3$ of P^{2*}, and the Laplace transforms of $\mathfrak{E}^3$ are situated on the lines $\{\mathfrak{E}^3, \mathfrak{E}^1\}$ and $\{\mathfrak{E}^3, \mathfrak{E}^2\}$. But these Laplace transforms are exactly $\mathfrak{E}^1$ and $\mathfrak{E}^2$, respectively; indeed,

$$(2.24)\qquad \begin{aligned} t_1E^1 &= -b_3E^3 + (\omega_1^1(t_1) + 2\omega_2^2(t_1))\,E^1\,,\quad t_1\mathfrak{E}^1 = \{\mathfrak{E}^1, \mathfrak{E}^3\}\,;\\ t_2E^2 &= -b_3E^3 + (2\omega_1^1(t_2) + \omega_2^2(t_2))\,E^2\,,\quad t_2\mathfrak{E}^2 = \{\mathfrak{E}^2, \mathfrak{E}^3\}\,. \end{aligned}$$

Let us study the induced map $v_1\colon P^2 \to P^{2*}$, $\mathfrak{v}_1(p) \to \mathfrak{E}^3(p)$. To this end, take a fixed point $p_0 \in D$ and local coordinates (u, v) around $p_0 = (u_0, v_0)$ such that the net $\mathcal{N}$ is given by the curves $u = \text{const.}$ and $v = \text{const.}$ This means that we may write

$$(2.25)\qquad \begin{aligned} &\omega^1 = f\,du\,,\quad \omega^2 = g\,dv\,,\quad \omega_1^1 = r_1\,du + r_2\,dv\,,\quad \omega_2^2 = s_1\,du + s_2\,dv\,;\\ &f = f(u, v), \ldots, s_2 = s_2(u, v)\,;\quad fg \neq 0\,. \end{aligned}$$

Recall the following definition: Let $P^n \equiv P^n(\mathbb{R})$ be a projective space, $I = (-h, h) \subset \mathbb{R}$, and $\mathfrak{w}_1, \mathfrak{w}_2\colon I \to P^n$ two curves. Let $\mathfrak{w}_1 = (w_1)$, $\mathfrak{w}_2 = (w_2)$, $w_1, w_2\colon I \to V^{n+1}$, V^{n+1} being the vector space generating P^n. We say that the curves $\mathfrak{w}_1$ and $\mathfrak{w}_2$ have the *contact* of order r at $0 \in \mathbb{R}$ if there is a function $\varrho\colon I \to \mathbb{R}$ such that

$$(2.26)\qquad \frac{d^s w_1}{dt^s} = \frac{d^s(\varrho w_2)}{dt^s}\quad \text{for}\quad t = 0\,,\quad s = 0, 1, \ldots, r\,.$$

Consider a projectivity $K\colon P^2 \to P^{2*}$ given by

$$(2.27)\qquad Kv_1 = \alpha_{13}E^3\,,\quad Kv_2 = \alpha_{22}E^2\,,\quad Kv_3 = \alpha_{31}E^1\,;$$

i.e., for $\mathfrak{v} = (a_1v_1 + a_2v_2 + a_3v_3)$ we have $K\mathfrak{v} = (a_1\alpha_{13}E^3 + a_2\alpha_{22}E^2 + a_3\alpha_{31}E^1)$. The points $\mathfrak{v}_i$ and $\mathfrak{E}^j$ being geometrically invariant, the set of all projectivities (2.27) has a geometrical significance. Let us determine the so-called *tangent projectivities* of our map $v_1\colon P^2 \to P^{2*}$. By this we mean the following: take $p_0 \in D$ and a curve $\gamma\colon I \to D$ with $\gamma(0) = p_0$; further, take the curves $K\mathfrak{v}_1(\gamma(t))$ and $\mathfrak{E}^3(\gamma(t))$; they should have the contact of the first order at $0 \in \mathbb{R}$ for each choice of γ. Let the curve γ be given by $u = u(t)$, $v = v(t)$; $p_0 = (u(0), v(0))$. From (2.2), (2.22) and (2.25) we get

$$(2.28)\qquad \frac{\partial v_1}{\partial u} = r_1v_1 + a_1fv_2\,,\quad \frac{\partial v_1}{\partial v} = r_2v_1 + gv_3\,;$$

$$\frac{\partial v_2}{\partial u} = s_1 v_2 + f v_3 , \quad \frac{\partial v_2}{\partial v} = a_4 g v_1 + s_2 v_2 ;$$

$$\frac{\partial v_3}{\partial u} = b_3 f v_1 + b_2 f v_2 + (r_1 + s_1) v_3 ,$$

$$\frac{\partial v_3}{\partial v} = b_4 g v_1 + b_3 g v_2 + (r_2 + s_2) v_3 ;$$

$$\frac{\partial E^3}{\partial u} = (r_1 + s_1) E^3 - f E^2 , \quad \frac{\partial E^3}{\partial v} = (r_2 + s_2) E^3 - g E^1 ; \tag{2.29}$$

$$\frac{\partial E^2}{\partial u} = -b_2 f E^3 + (2r_1 + s_1) E^2 - a_1 f E^1 ,$$

$$\frac{\partial E^2}{\partial v} = -b_3 g E^3 + (2r_2 + s_2) E^2;$$

$$\frac{\partial E^1}{\partial u} = -b_3 f E^3 + (r_1 + 2s_1) E^1 ,$$

$$\frac{\partial E^1}{\partial v} = -b_4 g E^3 - a_4 g E^2 + (r_2 + 2s_2) E^1 .$$

At $t = 0$ we have

$$\frac{\mathrm{d}v_1}{\mathrm{d}t} = \left(r_1 \frac{\mathrm{d}u}{\mathrm{d}t} + r_2 \frac{\mathrm{d}v}{\mathrm{d}t}\right) v_1 + a_1 f \frac{\mathrm{d}u}{\mathrm{d}t} v_2 + g \frac{\mathrm{d}v}{\mathrm{d}t} v_3 , \tag{2.30}$$

$$K \frac{\mathrm{d}v_1}{\mathrm{d}t} = \left(r_1 \frac{\mathrm{d}u}{\mathrm{d}t} + r_2 \frac{\mathrm{d}v}{\mathrm{d}t}\right) \alpha_{13} E^3 + a_1 f \frac{\mathrm{d}u}{\mathrm{d}t} \alpha_{22} E^2 + g \frac{\mathrm{d}v}{\mathrm{d}t} \alpha_{31} E^1 , \tag{2.31}$$

$$\frac{\mathrm{d}E^3}{\mathrm{d}t} = \left\{(r_1 + s_1) \frac{\mathrm{d}u}{\mathrm{d}t} + (r_2 + s_2) \frac{\mathrm{d}v}{\mathrm{d}t}\right\} E^3 - f \frac{\mathrm{d}u}{\mathrm{d}t} E^2 - g \frac{\mathrm{d}v}{\mathrm{d}t} E^1 . \tag{2.32}$$

Because of (2.26) we should have

$$K v_1 = \varrho E^3 , \quad K \frac{\mathrm{d}v_1}{\mathrm{d}t} = \varrho \frac{\mathrm{d}E^3}{\mathrm{d}t} + \frac{\mathrm{d}\varrho}{\mathrm{d}t} E^3 \quad \text{at} \quad t = 0 . \tag{2.33}$$

From (2.27_1) we get $\varrho(0) = \alpha_{13}$, and (2.32_2) reads

$$\left(r_1 \frac{\mathrm{d}u}{\mathrm{d}t} + r_2 \frac{\mathrm{d}v}{\mathrm{d}t}\right) \alpha_{13} E^3 + a_1 f \frac{\mathrm{d}u}{\mathrm{d}t} \alpha_{22} E^2 + g \frac{\mathrm{d}v}{\mathrm{d}t} \alpha_{31} E^1 = \tag{2.34}$$

$$= \alpha_{13} \left\{(r_1 + s_1) \frac{\mathrm{d}u}{\mathrm{d}t} + (r_2 + s_2) \frac{\mathrm{d}v}{\mathrm{d}t}\right\} E^3 - \alpha_{13} f \frac{\mathrm{d}u}{\mathrm{d}t} E^2 - \alpha_{13} \frac{\mathrm{d}v}{\mathrm{d}t} E^1 + \frac{\mathrm{d}\varrho}{\mathrm{d}t} E^3$$

at $t = 0$.

This equation being satisfied for each choice of $du(0)/dt$ and $dv(0)/dt$, we have $\alpha_{22}a_1 = -\alpha_{13}$, $\alpha_{31} = -\alpha_{13}$, and the tangent projectivity of our map v_1 is given by

$$(2.35) \qquad Kv_1 = -a_1 E^3 \,, \quad Kv_2 = E^2 \,, \quad Kv_3 = a_1 E^1 \,,$$

we then have

$$(2.36) \qquad K \frac{dv_1}{dt} = -a_1 \frac{dE^3}{dt} + a_1 \left(s_1 \frac{du}{dt} + s_2 \frac{dv}{dt} \right) E^3 \,.$$

Consider now the special curve $\tilde{\gamma}: I \to D$ given by $u = u_0 + t$, $v = v_0$, i.e., one of the curves of the net $\mathcal{N}$ passing through $p_0 = (u_0, v_0) \in D$. Then (always at $t = 0$)

$$(2.37) \qquad Kv_1 = -a_1 E^3 \,, \quad K \frac{\partial v_1}{\partial u} = -a_1 \frac{\partial E^3}{\partial u} + a_1 s_1 E^3 \,.$$

Write down the equation

$$(2.38) \qquad K \frac{\partial^2 v_1}{\partial u^2} = -a_1 \frac{\partial^2 E^3}{\partial u^2} + 2a_1 s_1 \frac{\partial E^3}{\partial u} + \Omega_3 E^3 + \Omega_1 E^1 + \Omega_2 E^2 \,.$$

Inserting (2.25) into (1.23_1), we easily see that

$$(2.39) \qquad \frac{\partial a_1}{\partial u} + a_1(s_1 - 2r_1) = b_1 f \,;$$

using this, we get

$$(2.40) \qquad \Omega_1 E^1 + \Omega_2 E^2 = f^2(2a_1^2 E^1 + b_1 E^2) \,.$$

In P^{2*}, consider the curves $K\mathfrak{v}_1(u_0 + t, v_0)$ and $\mathfrak{E}^3(u_0 + t, v_0)$; by virtue of (2.37) and $(2.26_{s=2})$, *their projections from each point of the line* $\mathfrak{l} = \{\mathfrak{E}^3, (2a_1^2 E^1 + b_1 E^2)\}$ *have the contact of order 2 at $t = 0$*. Each plane of the line $\mathfrak{l}$ goes through the point

$$(2.41) \qquad \mathfrak{T}_1 = (b_1 v_1 - 2a_1^2 v_2) \,.$$

Considering the point $\mathfrak{v}_2$ instead of $\mathfrak{v}_1$, taking the map $v_2: P^2 \to P^{2*}$, $\mathfrak{v}_2(p) \mapsto \mathfrak{E}^3(p)$, and making the other necessary changes, we get the geometrical description of the point

$$(2.42) \qquad \mathfrak{T}_2 = (b_5 v_2 - 2a_4^2 v_1) \,.$$

Finally,

$$(2.43) \qquad \begin{aligned} I_4 &:= (\mathfrak{v}_1, \mathfrak{v}_2; \mathfrak{P}_1, \mathfrak{T}_1) = -\tfrac{1}{2} b_1 b_2 a_1^{-2} b_3^{-1} \,, \\ I_5 &:= (\mathfrak{v}_2, \mathfrak{v}_1; \mathfrak{P}_2, \mathfrak{T}_2) = -\tfrac{1}{2} b_5 b_4 a_4^{-2} b_3^{-1} \,, \end{aligned}$$

which completes the fundamental set of invariants of the fourth order of our surface.

Let $p_0 = (0, 0) \in D$, and consider the curves γ_1 and γ_2 in P^2 given by $\mathfrak{v}_1 = \mathfrak{v}_1(0, v)$ and $\mathfrak{v}_2 = \mathfrak{v}_2(u, 0)$, respectively. We have

$$(2.44)\qquad v_1(0, v) = v_1(p_0) + \frac{\partial v_1(p_0)}{\partial v} v + \frac{1}{2}\frac{\partial^2 v_1(p_0)}{\partial v^2} v^2 + O(v^3) =$$

$$= v_1 + (r_2 v_1 + g v_3) v + \frac{1}{2}\left\{\left(\frac{\partial r_2}{\partial v} + r_2^2 + b_4 g^2\right) v_1 + b_3 g^2 v_2 + \right.$$

$$\left. + (2r_2 + s_2) g v_3\right\} v^2 + O(v^3),$$

$$v_2(u, 0) = v_2(p_0) + \frac{\partial v_2(p_0)}{\partial u} u + O(u^2) = v_2 + (s_1 v_2 + f v_3) u + O(u^2);$$

all the expressions are to be considered at p_0. Let $v_i \in \mathfrak{v}_i$ and $x \in \mathfrak{x} \in P^2$; the local coordinates of the point $\mathfrak{x}$ are introduced by

$$(2.45)\qquad x = x^1 v_1 + x^2 v_2 + x^3 v_3 ;$$

the curves γ_1 and γ_2 are thus given by

$$(2.46)\qquad x^1 = 1 + r_2 v + \frac{1}{2}\left(\frac{\partial r_2}{\partial v} + r_2^2 + b_4 g^2\right) v^2 + O(v^3).$$

$$x^2 = \tfrac{1}{2} b_3 g^2 v^2 + O(v^3), \quad x^3 = g v + \tfrac{1}{2}(2r_2 + s_2) g v^2 + O(v^3);$$

$$x^1 = O(u^2), \quad x^2 = 1 + s_1 u + O(u^2), \quad x^3 = f u + O(u^2),$$

respectively. It is easy to see that there is exactly one conic section in P^2 having the contact of order 2 with γ at $\mathfrak{v}_1(p_0)$ and the contact of order 1 with γ_2 at $\mathfrak{v}_2(p_0)$; this conic section is given by

$$(2.47)\qquad b_3(x^3)^2 - 2x^1 x^2 = 0,$$

and it is called the *Lie conic*. It follows that the Lie conic has the contact of order 2 with γ_2 at $\mathfrak{v}_2(p_0)$.

From (1.16) (1.18) and (1.24) we see that *the forms*

$$(2.48)\qquad \varphi := a_1 a_4 \omega^1 \omega^2, \quad \psi := b_3 \omega^1 \omega^2, \quad \psi_2 := b_2 (\omega^1)^2, \quad \psi_4 := b_4 (\omega^2)^2$$

are invariant; let us explain their geometrical interpretation. Let three curves $\gamma_{(i)}: (-h, h) \to D$ be given such that $\gamma_{(i)}(0) = p_0 = (u_0, v_0) \in D$; let the curve $\gamma_{(i)}$ be $u = u_{(i)}(t)$, $v = v_{(i)}(t)$. Then

$$(2.49)\qquad v_1(u_{(1)}(t), v_{(1)}(t)) = v_1(p_0) + \frac{\mathrm{d}v_1(p_0)}{\mathrm{d}t} t + O(t^2) =$$

$$= \left\{1 + \left(r_1 \frac{\mathrm{d}u_{(1)}(0)}{\mathrm{d}t} + r_2 \frac{\mathrm{d}v_{(1)}(0)}{\mathrm{d}t}\right) t + O(t^2)\right\} v_1 +$$

$$+ \left(a_1 f \frac{\mathrm{d}u_{(1)}(0)}{\mathrm{d}t} + O(t^2)\right) v_2 + \left(g \frac{\mathrm{d}v_{(1)}(0)}{\mathrm{d}t} t + O(t^2)\right) v_3,$$

$$v_2(u_{(2)}(t), v_{(2)}(t)) = v_2(p_0) + \frac{dv_2(p_0)}{dt} t + O(t^2) =$$

$$= \left(a_4 g \frac{dv_{(2)}(0)}{dt} t + O(t^2)\right) v_1 +$$

$$+ \left\{1 + \left(s_1 \frac{du_{(2)}(0)}{dt} + s_2 \frac{dv_{(2)}(0)}{dt}\right) t + O(t^2)\right\} v_2 +$$

$$+ \left(f \frac{du_{(2)}(0)}{dt} t + O(t^2)\right) v_3 ,$$

$$v_3(u_{(3)}(t), v_{(3)}(t)) = v_3(p_0) + \frac{dv_3(p_0)}{dt} t + O(t^2) =$$

$$= \left\{\left(b_3 f \frac{du_{(3)}(0)}{dt} + b_4 g \frac{dv_{(3)}(0)}{dt}\right) t + O(t^2)\right\} v_1 +$$

$$+ \left\{\left(b_2 f \frac{du_{(3)}(0)}{dt} + b_3 g \frac{dv_{(3)}(0)}{dt}\right) t + O(t^2)\right\} v_2 +$$

$$+ \left\{1 + (r_1 + s_1) \frac{du_{(3)}(0)}{dt} t + (r_2 + s_2) \frac{dv_{(3)}(0)}{dt} t + O(t^2)\right\} v_3 .$$

For a point $\mathfrak{w} \in P^2$, denote by $\pi_i \mathfrak{w}$ the projection of the point $\mathfrak{w}$ from $\mathfrak{v}_i(p_0)$ into the line $\{\mathfrak{v}_j(p_0), \mathfrak{v}_k(p_0)\}$, $j \neq i \neq k \neq j$. Then it is easy to see that

(2.50) $$\varrho_{12}(t) := (\mathfrak{v}_1(p_0), \mathfrak{v}_2(p_0); \pi_3 \mathfrak{v}_1(u_{(1)}(t), v_{(0)}(t)), \pi_3 \mathfrak{v}_2(u_{(2)}(t), v_{(2)}(t))) =$$

$$= a_1 a_4 f \frac{du_{(1)}(0)}{dt} g \frac{dv_{(1)}(0)}{dt} t^2 + O(t^3) ,$$

$$\varrho_{13}(t) := (\mathfrak{v}_1(p_0), \mathfrak{v}_3(p_0); \pi_2 \mathfrak{v}_1(u_{(1)}(t), v_{(1)}(t)), \pi_2 \mathfrak{v}_3(u_{(3)}(t), v_{(3)}(t))) =$$

$$= g \frac{dv_{(1)}(0)}{dt} \left(b_3 f \frac{du_{(3)}(0)}{dt} + b_4 g \frac{dv_{(3)}(0)}{dt}\right) t^2 + O(t^3),$$

$$\varrho_{23}(t) := (\mathfrak{v}_2(p_0), \mathfrak{v}_3(p_0); \pi_1 \mathfrak{v}_2(u_{(2)}(t), v_{(2)}(t)), \pi_1 \mathfrak{v}_3(u_{(3)}(t), v_{(3)}(t))) =$$

$$= f \frac{du_{(2)}(0)}{dt} \left(b_2 f \frac{du_{(3)}(0)}{dt} + b_3 g \frac{dv_{(3)}(0)}{dt}\right) t^2 + O(t^3) .$$

Let us write

(2.51) $$t_{(i)} := d\gamma_i \left(\left.\frac{d}{dt}\right|_{t=0}\right) = \frac{du_{(i)}(0)}{dt} \left.\frac{\partial}{\partial u}\right|_{p_0} + \frac{dv_{(i)}(0)}{dt} \left.\frac{\partial}{\partial v}\right|_{p_0} \in T_{p_0}(D) ;$$

we get

(2.52) $$\frac{d^2 \varrho_{12}(0)}{dt^2} = 2 a_1 a_4 \omega^1(t_{(1)})\, \omega^2(t_{(2)}) ,$$

$$\frac{d^2\varrho_{13}(0)}{dt^2} = 2\omega^2(t_{(1)})\{b_3\,\omega^1(t_{(3)}) + b_4\,\omega^2(t_{(3)})\}\,,$$

$$\frac{d^2\varrho_{23}(0)}{dt^2} = 2\,\omega^1(t_{(2)})\{b_2\,\omega^1(t_{(3)}) + b_3\,\omega^2(t_{(3)})\}\,.$$

Let us study the osculating quadrics of our surface. Let $\gamma: (-h, h) \to D$ be a curve, $\gamma(0) = p_0$; let γ be given by $u = u(t)$, $v = v(t)$; $p_0 = (u(0), v(0))$. Because of

$$\frac{\partial m}{\partial u} = f v_1\,, \quad \frac{\partial m}{\partial v} = g v_2 \tag{2.53}$$

we get

$$\begin{aligned} m(u(t), v(t)) = m(p_0) + \left\{f\frac{du(0)}{dt}\,v_1(p_0) + g\,\frac{dv(0)}{dt}\,v_2(p_0)\right\} t + \\ + \frac{1}{2}\left\{\left(\frac{\partial f}{\partial u} + f r_1\right)\left(\frac{du(0)}{dt}\right)^2 v_1(p_0) + a_1 f^2\left(\frac{du(0)}{dt}\right)^2 v_2(p_0) + \right. \\ + 2fg\,\frac{du(0)}{dt}\,\frac{dv(0)}{dt}\,v_3(p_0) + \\ + a_4 g^2\left(\frac{dv(0)}{dt}\right)^2 v_1(p_0) + \left(\frac{\partial g}{\partial v} + g s_2\right)\left(\frac{dv(0)}{dt}\right)^2 v_2(p_0) + \\ \left. + f\,\frac{d^2u(0)}{dt^2}\,v_1(p_0) + g\,\frac{d^2v(0)}{dt^2}\,v_2(p_0)\right\} t^2 + O(t^3)\,; \end{aligned} \tag{2.54}$$

we have used the identity $\partial f/\partial v + f r_2 = 0$ following from $d\omega^1 = \omega^1 \wedge \omega_1^1$. For a point $X \in A^3$ let us introduce the local coordinates by

$$X = m(p_0) + X^1\,v_1(p_0) + X^2\,v_2(p_0) + X^3\,v_3(p_0)\,; \tag{2.55}$$

for the curve (2.54) we have

$$\begin{aligned} X^1 &= f\,\frac{du(0)}{dt}\,t + \frac{1}{2}\left\{\left(\frac{\partial f}{\partial u} + f r_1\right)\left(\frac{du(0)}{dt}\right)^2 + a_4 g^2\left(\frac{dv(0)}{dt}\right)^2 + \right. \\ &\quad \left. + f\,\frac{d^2u(0)}{dt^2}\right\} t^2 + O(t^3)\,, \\ X^2 &= g\,\frac{dv(0)}{dt}\,t + \frac{1}{2}\left\{a_1 f^2\left(\frac{du(0)}{dt}\right)^2 + \left(\frac{\partial g}{\partial v} + g s_2\right)\left(\frac{dv(0)}{dt}\right)^2 + \right. \\ &\quad \left. + g\,\frac{d^2v(0)}{dt^2}\right\} t^2 + O(t^3)\,, \\ X^3 &= fg\,\frac{du(0)}{dt}\,\frac{dv(0)}{dt}\,t^2 + O(t^3)\,. \end{aligned} \tag{2.56}$$

The general quadric being given by

$$(2.57)\qquad A_{ij}X^iX^j + A_iX^i + A = 0\,,$$

it is easy to see that the most general osculating quadric, i.e., the quadric having the contact of order 2 with (2.54) for each curve γ, is given by

$$(2.58)\qquad A_{33}(X^3)^2 + 2A_{12}(X^1X^2 - X^3) + 2A_{13}X^1X^3 + 2A_{23}X^2X^3 = 0\,.$$

Introducing, in the projective extension $A^3 \cup P^2$ of A^3, the homogeneous coordinates by

$$(2.59)\qquad X^1 = x^1/x^0\,,\quad X^2 = x^2/x^0\,,\quad X^3 = x^3/x^0\,,$$

we get it in the form

$$(2.60)\qquad A_{33}(x^3)^2 + 2A_{12}(x^1x^2 - x^0x^3) + 2A_{13}x^1x^3 + 2A_{23}x^2x^3 = 0\,.$$

Its intersection with the improper plane P^2 is given by (2.60) and $x^0 = 0$, i.e.,

$$(2.61)\qquad A_{33}(x^3)^2 + 2A_{12}x^1x^2 + 2A_{13}x^1x^3 + 2A_{23}x^2x^3 = 0\,.$$

By definition, the *Lie quadric* is the osculating quadric intersecting P^2 in the Lie conic (2.47); its equation is

$$(2.62)\qquad b_3(X^3)^2 - 2X^1X^2 + 2X^3 = 0\,.$$

We easily see that it is a paraboloid for $b_3 = 0$; for $b_3 \neq 0$, its center is $m - b_3^{-1}v_3$. Using the spacialization (1.28), we get the geometrical description of the *normal vector* v_3.

Let us suppose (1.28), and let us recall (1.38) and (1.40). One of the Laplace transforms of $\mathfrak{R}_1$ (or $\mathfrak{R}_2$) defined in (2.9) is the point $\mathfrak{v}_1$ (or $\mathfrak{v}_2$), the other one is situated on the line $\{\mathfrak{R}_1, \mathfrak{U}_1\}$ (or $\{\mathfrak{R}_2, \mathfrak{U}_2\}$, respectively) with

$$(2.63)\qquad \mathfrak{U}_1 = (c_1v_1 + a_1v_2)\,,\quad \mathfrak{U}_2 = (a_4v_1 + c_2v_2)\,;$$

indeed,

$$(2.64)\qquad \begin{aligned} t_2(b_2v_1 - a_1v_3) &= 3\,\omega_1^1(t_2)\,(b_2v_1 - a_1v_3) - \varepsilon'(c_1v_1 + a_1v_2)\,,\\ t_1(b_4v_2 - a_4v_3) &= 3\,\omega_2^2(t_1)\,(b_4v_2 - a_4v_3) - \varepsilon'(a_4v_1 + c_2v_2)\,. \end{aligned}$$

Thus we get the geometric characterization of the invariants

$$(2.65)\qquad \begin{aligned} I_6 &:= (\mathfrak{v}_1, \mathfrak{v}_2; \mathfrak{P}_1, \mathfrak{U}_1) = b_2c_1(\varepsilon'a_1)^{-1}\,,\\ I_7 &:= (\mathfrak{v}_2, \mathfrak{v}_1; \mathfrak{P}_2, \mathfrak{U}_2) = b_4c_2(\varepsilon'a_4)^{-1}\,. \end{aligned}$$

It is easy to calculate

$$(2.66)\qquad \begin{aligned} &t_2(c_1v_1 + a_1v_2) =\\ &= 2\,\omega_1^1(t_2)\,(c_1v_1 + a_1v_2) + (c_{12} + a_1a_4)\,v_1 + b_2v_2 + c_1v_3\,; \end{aligned}$$

let

$$(2.67)\qquad \mathfrak{B} := t_2\mathfrak{U}_1 \cap \{\mathfrak{v}_1, \mathfrak{v}_3\} = ((c_2c_{12} + a_1a_4c_2 - a_4b_2)\,v_1 + c_1c_2v_3)\,.$$

Now,

(2.68) $$I_8 := (\mathfrak{v}_1, \mathfrak{v}_3; \mathfrak{R}_1, \mathfrak{B}) = -I_6^{-1}(\varepsilon'^{-1}c_{12} + I_2 - I_1 I_7^{-1}),$$

which gives the geometrical description of the invariant c_{12}.

3. In this section we present one example of a hyperbolic surface.

Definition. Let m: $D \to A^3$ be a surface. It is called *transitive* if, for each $p, q \in D$, there is an affine mapping $\mathscr{A}_{p,q}$: $A^3 \to A^3$ and neighborhoods $U_p, U_q \subset D$ of the points p and q, respectively, such that $\mathscr{A}_{p,q} \circ m|_{U_p} = m|_{U_q}$.

Proposition 2. *Let m: $D \to A^3$ be a transitive hyperbolic non-maximal surface with $I_1 \neq 1$, $I_2 \neq 0$ and $\mathfrak{T}_1 = \mathfrak{v}_2, \mathfrak{T}_2 = \mathfrak{v}_1$. Then there is, in A^3, a fixed frame $\{O; v_1, v_2, v_3\}$ generating the coordinates of a point $P \in A^3$ by $P = O + xv_1 + yv_2 + zv_3$ and such that $m(D)$ is a part of the surface*

(3.1) $$(x^2 + y^2)\, z = 1\,.$$

Proof. Because of the suppositions and (2.8), (2.16), (2.41) and (2.42) we have

(3.2) $$b_3 \neq 0\,, \quad a_1 a_4 \neq 0\,, \quad b_2 b_4 \neq b_3^2\,;$$

(3.3) $$b_1 = 0\,, \quad b_5 = 0\,.$$

It follows from (1.18) that we may choose the frames of our surface in such a way that

(3.4) $$a_1 = a_4 = 1\,;$$

then $\alpha = \beta = 1$, and the functions b_i are invariants. Our surface being transitive, all its invariants must be constant.

First of all, let us determine all transitive hyperbolic surfaces (thus we do not take into account the equations (3.3)). From (1.23), (3.4) and $b_i \in R$ we get

(3.5) $$\begin{aligned} \omega_1^1 &= -\tfrac{1}{3}(2b_1 + b_4)\,\omega^1 - \tfrac{1}{3}(2b_2 + b_5)\,\omega^2\,,\\ \omega_2^2 &= -\tfrac{1}{3}(b_1 + 2b_4)\,\omega^1 - \tfrac{1}{3}(b_2 + 2b_5)\,\omega^2\,,\\ \omega_3^2 &= b_2\omega^1 + b_3\omega^2\,, \quad \omega_3^1 = b_3\omega^1 + b_4\omega^2\,. \end{aligned}$$

Inserting these relations into (1.25), we obtain

(3.6) $$\begin{aligned} &2b_2b_4 - b_1(b_2 + b_5) = 9(b_3 - 1)\,,\\ &3b_3(b_1 + b_4) - 2b_2(2b_2 + b_5) = -3b_4\,,\\ &2b_4(b_1 + 2b_4) - 3b_3(b_2 + b_5) = 3b_2\,,\\ &b_5(b_1 + b_4) - 2b_2b_4 = 9(1 - b_3)\,. \end{aligned}$$

Thus each transitive hyperbolic surface is given by the completely integrable system

(3.7) $$\begin{aligned} &\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,; \quad \mathrm{d}v_1 = \omega_1^1 v_1 + \omega^1 v_2 + \omega^2 v_3\,,\\ &\mathrm{d}v_2 = \omega^2 v_1 + \omega_2^2 v_2 + \omega^1 v_3\,, \quad \mathrm{d}v_3 = \omega_3^1 v_1 + \omega_3^2 v_2 + (\omega_1^1 + \omega_2^2)\, v_3\,, \end{aligned}$$

where $\omega_1^1, \omega_2^2, \omega_3^2, \omega_3^1$ are given by (3.5) and the invariants $b_i \in R$ satisfy (3.6).

Now, suppose (3.3). The equations (3.6) reduce to

$$(3.8)\qquad 2b_4b_4 = 9(b_3 - 1), \quad 3b_3b_4 - 4b_2^2 = -3b_4, \quad 4b_4^2 - 3b_2b_3 = 3b_2\,.$$

From (3.8_1) we obtain

$$(3.9)\qquad b_3 = \tfrac{2}{9}b_2b_4 + 1\,,$$

and inserting this into $(3.8_{2,3})$, we have

$$(3.10)\qquad b_2b_4^2 - 6b_2^2 = -9b_4\,, \quad 6b_4^2 - b_2^2b_4 = 9b_2\,.$$

Multiplying the first equation by b_2, the second by b_4 and adding, we get $b_2 = b_4$; (3.10_1) reduces then to $b_2^3 - 6b_2^2 + 9b_2 = 0$ with the solutions $b_2 = 0$ and $b_2 = 3$. In the case $b_2 = 3$, we have $b_4 = 3$ and $b_3 = 3$, see (3.9), i.e., $I_1 = 1$, a contradiction. Thus

$$(3.11)\qquad b_2 = b_4 = 0\,, \quad b_3 = 1\,,$$

and the system (3.7) reduces to

$$(3.12)\qquad \mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,; \quad \mathrm{d}v_1 = \omega^1 v_2 + \omega^2 v_3\,, \quad \mathrm{d}v_2 = \omega^2 v_1 + \omega^1 v_3\,,$$
$$\mathrm{d}v_3 = \omega^1 v_1 + \omega^2 v_2\,.$$

The system (3.12) being completely integrable, it is sufficient to find one of its solutions. It is easy to see that we may take

$$(3.13)\qquad m = O + u\cos v\mathbf{v}_1 + u\sin v\mathbf{v}_2 + u^{-2}\mathbf{v}_3\,; \quad v_3 = m - O\,,$$
$$v_1 = \tfrac{1}{2}u(\cos v - \sqrt{(3)}\sin v)\,\mathbf{v}_1 - \tfrac{1}{2}u(\sin v + \sqrt{(3)}\cos v)\,\mathbf{v}_2 + u^{-2}\mathbf{v}_3\,,$$
$$v_2 = -\tfrac{1}{2}u(\cos v + \sqrt{(3)}\sin v)\,\mathbf{v}_1 - \tfrac{1}{2}u(\sin v - \sqrt{(3)}\cos v)\,\mathbf{v}_2 + u^{-2}\mathbf{v}_3\,,$$

which satisfies (3.12) with

$$(3.14)\qquad \omega^1 = -(u^{-1}\,\mathrm{d}u + \tfrac{1}{3}\sqrt{(3)}\,\mathrm{d}v)\,, \quad \omega^2 = -(u^{-1}\,\mathrm{d}u - \tfrac{1}{3}\sqrt{(3)}\,\mathrm{d}v)\,;$$

here, $u > 0$, $v \in \mathbb{R}$. Of course, the point m is then situated on the surface (3.1). QED.

Problem. Determine *all* transitive hyperbolic surfaces.

ELLIPTIC CASE

4. Similarly, let us study elliptic surfaces $m\colon D \to A^3$ having no real asymptotic vectors. With each point $m(p)$, $p \in D$, let us associated a frame $\{m; e_1, e_2, e_3\}$; we have

$$(4.1)\qquad \mathrm{d}m = \tau^i e_i\,, \quad \mathrm{d}e_i = \tau_i^j e_j\,; \quad i, j, \ldots = 1, 2, 3\,,$$

with the integrability conditions

$$(4.2)\qquad \mathrm{d}\tau^i = \tau^j \wedge \tau_j^i\,, \quad \mathrm{d}\tau_i^j = \tau_i^k \wedge \tau_k^j\,.$$

Similarly to the case of the hyperbolic surfaces, it is possible to show that the frames may be chosen in such a way that

(4.3) $$\tau^3 = 0\,; \quad \tau_1^3 = \tau^1\,, \quad \tau_2^3 = \tau^2$$

with the differential consequences

(4.4) $$(2\tau_1^1 - \tau_3^3) \wedge \tau^1 + (\tau_1^2 + \tau_2^1) \wedge \tau^2 = 0\,,$$
$$(\tau_1^2 + \tau_2^1) \wedge \tau^1 + (2\tau_2^2 - \tau_3^3) \wedge \tau^2 = 0\,.$$

Once again, it may be proved that the frames may be chosen such that there are functions $A_2, A_3\colon D \to \mathbb{R}$ and the equations (4.4) are satisfied if we take

(4.5) $$2\tau_1^1 - \tau_3^3 = -A_3\tau^1 + A_2\tau^2\,, \quad \tau_1^2 + \tau_2^1 = A_2\tau^1 + A_3\tau^2\,,$$
$$2\tau_2^2 - \tau_3^3 = A_3\tau^1 - A_2\tau^2\,.$$

Introducing the auxiliary form

(4.6) $$\varphi := \tfrac{1}{2}(\tau_1^2 - \tau_2^1)\,,$$

we may write

(4.7) $$\tau_1^1 = \tfrac{1}{2}(\tau_3^3 - A_3\tau^1 + A_2\tau^2)\,, \quad \tau_2^2 = \tfrac{1}{2}(\tau_3^3 + A_3\tau^1 - A_2\tau^2)\,,$$
$$\tau_1^2 = \tfrac{1}{2}(A_2\tau^1 + A_3\tau^2) + \varphi\,, \quad \tau_2^1 = \tfrac{1}{2}(A_2\tau^1 + A_3\tau^2) - \varphi\,.$$

The differentiation of (4.5) yields the equalities

(4.8) $$-(\mathrm{d}A_3 - \tfrac{1}{2}A_3\tau_3^3 + 3A_2\varphi - 3\tau_3^1) \wedge \tau^1 +$$
$$+ (\mathrm{d}A_2 - \tfrac{1}{2}A_2\tau_3^3 - 3A_3\varphi + \tau_3^2) \wedge \tau^2 = 0\,,$$
$$(\mathrm{d}A_2 - \tfrac{1}{2}A_2\tau_3^3 - 3A_2\varphi + \tau_3^2) \wedge \tau^1 +$$
$$+ (\mathrm{d}A_3 - \tfrac{1}{2}A_3\tau_3^3 + 3A_2\varphi + \tau_3^1) \wedge \tau^2 = 0\,,$$
$$(\mathrm{d}A_3 - \tfrac{1}{2}A_3\tau_3^3 + 3A_2\varphi + \tau_3^1) \wedge \tau^1 -$$
$$- (\mathrm{d}A_2 - \tfrac{1}{2}A_2\tau_3^3 - 3A_3\varphi - 3\tau_3^2) \wedge \tau^2 = 0\,,$$

and the existence of functions $B_1, \ldots, B_5\colon D \to \mathbb{R}$ such that

(4.9) $$-\mathrm{d}A_3 + \tfrac{1}{2}A_3\tau_3^3 - 3A_2\varphi + 3\tau_3^1 = B_1\tau^1 + B_2\tau^2\,,$$
$$\mathrm{d}A_2 - \tfrac{1}{2}A_2\tau_3^3 - 3A_3\varphi + \tau_3^2 = B_2\tau^1 + B_3\tau^2\,,$$
$$\mathrm{d}A_3 - \tfrac{1}{2}A_3\tau_3^3 + 3A_2\varphi + \tau_3^1 = B_3\tau^1 + B_4\tau^2\,,$$
$$-\mathrm{d}A_2 + \tfrac{1}{2}A_2\tau_3^3 + 3A_3\varphi + 3\tau_3^2 = B_4\tau^1 + B_5\tau^2\,,$$

i.e.,

(4.10) $$\tau_3^1 = \tfrac{1}{4}(B_1 + B_3)\,\tau^1 + \tfrac{1}{4}(B_2 + B_4)\,\tau^2\,,$$
$$\tau_3^2 = \tfrac{1}{4}(B_2 + B_4)\,\tau^1 + \tfrac{1}{4}(B_3 + B_5)\,\tau^2\,,$$
$$\mathrm{d}A_2 - \tfrac{1}{2}A_2\tau_3^3 - 3A_3\varphi = \tfrac{1}{4}(3B_2 - B_4)\,\tau^1 + \tfrac{1}{4}(3B_3 - B_5)\,\tau^2\,,$$
$$\mathrm{d}A_3 - \tfrac{1}{2}A_3\tau_3^3 + 3A_2\varphi = \tfrac{1}{4}(3B_3 - B_1)\,\tau^1 + \tfrac{1}{4}(3B_4 - B_2)\,\tau^2\,.$$

If we write

$$(4.11)\qquad \mathrm{D}B_1 := \mathrm{d}B_1 - B_1\tau_3^3 - 4B_2\varphi\,, \quad \mathrm{D}B_2 := \mathrm{d}B_2 - B_2\tau_3^3 + (B_1 - 3B_3)\,\varphi\,,$$
$$\mathrm{D}B_3 := \mathrm{d}B_3 - B_3\tau_3^3 + 2(B_2 - B_4)\,\varphi\,,$$
$$\mathrm{D}B_4 := \mathrm{d}B_4 - B_4\tau_3^3 + (3B_3 - B_5)\,\varphi\,, \quad \mathrm{D}B_5 := \mathrm{d}B_5 - B_5\tau_3^3 + 4B_4\varphi\,,$$

the prolongation of (4.10) is

$$(4.12)\qquad (\mathrm{D}B_1 + \mathrm{D}B_3) \wedge \tau^1 + (\mathrm{D}B_2 + \mathrm{D}B_4) \wedge \tau^2 =$$
$$= \{A_3(B_2 + B_4) + \tfrac{1}{2}A_2(B_1 - B_5)\}\,\tau^1 \wedge \tau^2\,,$$
$$(\mathrm{D}B_2 + \mathrm{D}B_4) \wedge \tau^1 + (\mathrm{D}B_3 + \mathrm{D}B_5) \wedge \tau^2 =$$
$$= \{\tfrac{1}{2}A_3(B_1 - B_5) - A_2(B_2 + B_4)\}\,\tau^1 \wedge \tau^2\,,$$
$$(3\mathrm{D}B_2 - \mathrm{D}B_4) \wedge \tau^1 + (3\mathrm{D}B_3 - \mathrm{D}B_5) \wedge \tau^2 =$$
$$= -\tfrac{3}{2}A_3(B_1 + 2B_3 + B_5 - 4A_2^2 - 4A_3^2)\,\tau^1 \wedge \tau^2\,,$$
$$(3\mathrm{D}B_3 - \mathrm{D}B_1) \wedge \tau^1 + (3\mathrm{D}B_4 - \mathrm{D}B_2) \wedge \tau^2 =$$
$$= \tfrac{3}{2}A_2(B_1 + 2B_3 + B_5 - 4A_2^2 - 4A_3^2)\,\tau^1 \wedge \tau^2\,.$$

Let us complexify our affine space A^3 and its improper plane P^2. With each point $m(p)$ of our surface let us associate the frame

$$(4.13)\qquad v_1 := e_1 - \mathrm{i}e_2\,, \quad v_2 := e_1 + \mathrm{i}e_2\,, \quad v_3 := 2e_3\,;$$

for these frames we get the equations

$$(4.14)\qquad \mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,; \qquad \mathrm{d}v_1 = \omega_1^1 v_1 + \omega_1^2 v_2 + \omega^2 v_3\,,$$
$$\mathrm{d}v_2 = \omega_2^1 v_1 + \omega_2^2 v_2 + \omega^1 v_3\,, \quad \mathrm{d}v_3 = \omega_3^1 v_1 + \omega_3^2 v_2 + \omega_3^3 v_3$$

with

$$(4.15)\qquad \omega_1^1 = \tfrac{1}{2}\tau_3^3 + \mathrm{i}\varphi\,, \quad \omega_2^2 = \bar{\omega}_1^1\,, \quad \omega_3^3 = \tau_3^3\,, \quad \omega_3^1 = \tau_3^1 + \mathrm{i}\tau_3^2\,, \quad \omega_3^2 = \bar{\omega}_3^1\,,$$
$$(4.16)\qquad \omega_1^2 = -(A_3 + \mathrm{i}A_2)\,\omega^1\,, \quad \omega_2^1 = \bar{\omega}_1^2\,.$$

Comparing them with (1.20) and (1.23), we have

$$(4.17)\qquad a_1 = -(A_2 + \mathrm{i}A_2)\,, \quad a_4 = \bar{a}_1\,;$$
$$(4.18)\qquad b_1 = \tfrac{1}{4}(B_1 - 6B_3 + B_5) + \mathrm{i}(B_4 - B_2)\,,$$
$$b_2 = \tfrac{1}{4}(B_1 - B_5) - \tfrac{1}{2}\mathrm{i}(B_2 + B_4)\,,$$
$$b_3 = \tfrac{1}{4}(B_1 + 2B_3 + B_5)\,, \quad b_4 = \bar{b}_2\,, \quad b_5 = \bar{b}_1\,.$$

The invariants (2.8), (2.16), (2.19) and (2.43) are then

$$(4.19)\qquad I_1 = b_2\bar{b}_2 b_3^{-2}\,, \quad I_2 = a_1\bar{a}_1 b_3^{-1}\,, \quad I_3 = \bar{a}_1 b_2^2(a_1 b_3 \bar{b}_2)^{-1}\,,$$
$$I_3' = \bar{I}_3\,, \quad I_4 = -\tfrac{1}{2}b_1 b_2(a_1^2 b_3)^{-1}\,, \quad I_5 = \bar{I}_4\,,$$

and we get the *real-valued invariants*

$$(4.20)\qquad J_1 := I_1\,, \quad J_2 := I_2\,, \quad J_3 := I_3 + I_3'\,, \quad J_4 := I_4 + I_5\,, \quad J_5 := \mathrm{i}(I_4 - I_5)\,.$$

The invariant forms $(2.48_{1,2})$ are

$$(4.21) \qquad \varphi = \tfrac{1}{4}(A_2^2 + A_3^2)\{(\tau^1)^2 + (\tau^2)^2\}\,,$$
$$\psi = \tfrac{1}{4}(B_1 + 2B_3 + B_5)\{(\tau^1)^2 + (\tau^2)^2\}\,;$$

from $(2.48_{3,4})$ we get the real-valued invariant forms

$$(4.22) \qquad \Psi := \psi_2 + \psi_4 = \tfrac{1}{8}(B_1 - B_5)\{(\tau^1)^2 - (\tau^2)^2\} + \tfrac{1}{2}(B_2 + B_4)\,\tau^1\tau^2\,,$$
$$\Psi' := \mathrm{i}(\psi_2 - \psi_4) = \tfrac{1}{4}(B_2 + B_4)\{(\tau^1)^2 - (\tau^2)^2\} - \tfrac{1}{4}(B_1 - B_5)\,\tau^1\tau^2\,.$$

In the improper plane P^2 we introduce the local coordinates by

$$(4.23) \qquad x \in \xi^1 e_1 + \xi^2 e_2 + \xi^3 e_3\,.$$

Comparing them with (2.45) and (4.13), we have

$$(4.24) \qquad x^1 = \tfrac{1}{2}(\xi^1 + \mathrm{i}\xi^2)\,, \quad x^2 = \tfrac{1}{2}(\xi^1 - \mathrm{i}\xi^2)\,, \quad x^3 = \tfrac{1}{2}\xi^3\,,$$

and the equation of the *Lie conic* (2.47) is

$$(4.25) \qquad (\xi^1)^2 + (\xi^2)^2 - \tfrac{1}{8}(B_1 + 2B_3 + B_5)(\xi^3)^2 = 0\,.$$

Let us *suppose that the Lie conic is regular* at each point (such a surface will be called *non-maximal*); we are then in the position to choose the frames in such a way that

$$(4.26) \qquad B_1 + 2B_3 + B_5 = -8\varepsilon\,, \quad \varepsilon = \pm 1\,;$$

$\varepsilon = 1$ in the case of an imaginary Lie conic, $\varepsilon = -1$ in the real case. Define $B_0\colon D \to \to \mathbb{R}$ by

$$(4.27) \qquad B_1 = B_0 - B_3 - 4\varepsilon\,, \quad B_5 = -(B_0 + B_3 + 4\varepsilon)\,.$$

Then the equations $(4.12_{1,2})$ reduce to

$$(4.28) \qquad \{\mathrm{d}B_0 - B_0\tau_3^3 - 2(B_2 + B_4)\,\varphi + 4\varepsilon\tau_3^3\} \wedge \tau^1 +$$
$$+ \{\mathrm{d}(B_2 + B_4) - (B_2 + B_4)\,\tau_3^3 + 2B_0\varphi\} \wedge \tau^2 =$$
$$= \{A_3(B_2 + B_4) + A_2B_0\}\,\tau^1 \wedge \tau^2\,,$$
$$\{\mathrm{d}(B_2 + B_4) - (B_2 + B_4)\,\tau_3^3 + 2B_0\varphi\} \wedge \tau^1 -$$
$$- \{\mathrm{d}B_0 - B_0\tau_3^3 - 2(B_2 + B_4)\,\varphi - 4\varepsilon\tau_3^3\} \wedge \tau^2 =$$
$$= \{A_3B_0 - A_2(B_2 + B_4)\}\,\tau^1 \wedge \tau^2\,;$$

this yield the existence of functions $C_1, C_2\colon D \to \mathbb{R}$ such that

$$(4.29) \qquad \tau_3^3 = C_1\tau^1 + C_2\tau^2\,.$$

Define

$$(4.30) \qquad \tau := \varphi + \tfrac{1}{2}(C_2\tau^1 - C_1\tau^2)\,;$$

then

$$(4.31) \qquad \mathrm{d}\tau^1 = -\tau^2 \wedge \tau\,, \quad \mathrm{d}\tau^2 = \tau^1 \wedge \tau\,.$$

From (4.29) we have

$$(4.32) \qquad (\mathrm{d}C_1 - C_2\tau) \wedge \tau^1 + (\mathrm{d}C_2 + C_1\tau) \wedge \tau^2 = 0\,,$$

and consequently,

$$(4.33) \qquad \mathrm{d}C_1 - C_2\tau = C_{11}\tau^1 + C_{12}\tau^2\,, \quad \mathrm{d}C_2 + C_1\tau = C_{12}\tau^1 + C_{22}\tau^2\,.$$

On our surface $m\colon D \to A^3$ let us consider the *metric*, see (4.21),

$$(4.34) \qquad \mathrm{d}s^2 := \tfrac{1}{2}|\psi| = (\tau^1)^2 + (\tau^2)^2\,.$$

The Gauss curvature $\varkappa$ of $\mathrm{d}s^2$ is given, because of (4.31), by

$$(4.35) \qquad \mathrm{d}\tau = -\varkappa\tau^1 \wedge \tau^2\,,$$

and a direct calculation gives the following

Proposition 3. (Theorema egregium) *We have*

$$(4.36) \qquad 2\varkappa = 2\varepsilon + A_2^2 + A_3^2 + C_{11}C_{22}\,.$$

With the use of (4.27) and (4.30), the equations (4.10) take the form

$$(4.37) \qquad \tau_3^1 = \tfrac{1}{4}(B_0 - 4\varepsilon)\,\tau^1 + \tfrac{1}{4}(B_2 + B_4)\,\tau^2\,,$$
$$\tau_3^2 = \tfrac{1}{4}(B_2 + B_4)\,\tau^1 - \tfrac{1}{4}(B_0 + 4\varepsilon)\,\tau^2\,,$$

$$(4.38) \qquad \mathrm{d}A_2 - 3A_3\tau = \tfrac{1}{4}(3B_2 - B_4 + 2A_2C_1 - 6A_3C_2)\,\tau^1 +$$
$$+ \tfrac{1}{4}(4B_3 + B_0 + 4\varepsilon + 2A_2C_2 + 6A_3C_1)\,\tau^2\,,$$
$$\mathrm{d}A_3 + 3A_2\tau = \tfrac{1}{4}(4B_3 - B_0 + 4\varepsilon + 2A_3C_1 + 6A_2C_2)\,\tau^1 +$$
$$+ \tfrac{1}{4}(3B_4 - B_2 + 2A_3C_2 - 6A_2C_1)\,\tau^2\,.$$

Inserting into (4.12), we get

$$(4.39) \qquad \{\mathrm{d}B_0 - 2(B_2 + B_4)\,\tau\} \wedge \tau^1 + \{\mathrm{d}(B_2 + B_4) + 2B_0\tau\} \wedge \tau^2 =$$
$$= \{A_3(B_2 + B_4) + A_2B_0 + 4\varepsilon C_2\}\,\tau^1 \wedge \tau^2\,,$$
$$\{\mathrm{d}(B_2 + B_4) + 2B_0\tau\} \wedge \tau^1 - \{\mathrm{d}B_0 - 2(B_2 + B_4)\,\tau\} \wedge \tau^2 =$$
$$= \{A_3B_0 - A_2(B_2 + B_4) - 4\varepsilon C_1\}\,\tau^1 \wedge \tau^2\,;$$

$$(4.40) \qquad \{3\mathrm{d}B_2 - \mathrm{d}B_4 + 2(B_0 - 8B_3 - 8\varepsilon)\,\tau\} \wedge \tau^1 +$$
$$+ \{4\mathrm{d}B_3 + \mathrm{d}B_0 + 2(3B_2 - 5B_4)\,\tau\} \wedge \tau^2 =$$
$$= \{6A_3(2\varepsilon + A_2^2 + A_3^2) + 2(B_0 - 2B_3 - 2\varepsilon)\,C_1 - 4B_4C_2\}\,\tau^1 \wedge \tau^2\,,$$
$$\{4\mathrm{d}B_3 - \mathrm{d}B_0 + 2(5B_2 - 3B_4)\,\tau\} \wedge \tau^1 +$$
$$+ \{3\mathrm{d}B_4 - \mathrm{d}B_2 + 2(B_0 + 8B_3 + 8\varepsilon)\,\tau\} \wedge \tau^2 =$$
$$= \{-6A_2(2\varepsilon + A_2^2 + A_3^2) + 4B_2C_1 + 2(B_0 + 2B_3 + 2\varepsilon)\,C_2\}\,\tau^1 \wedge \tau^2\,.$$

From (4.33),

$$(4.41) \qquad (\mathrm{d}C_{11} - 2C_{12}) \wedge \tau^1 + \{\mathrm{d}C_{12} + (C_{11} - C_{22})\,\tau\} \wedge \tau^2 = \varkappa C_2\tau^1 \wedge \tau^2\,,$$
$$\{\mathrm{d}C_{12} + (C_{11} - C_{22})\,\tau\} \wedge \tau^1 + (\mathrm{d}C_{22} + 2C_{12}\tau) \wedge \tau^2 = -\varkappa C_1\tau^1 \wedge \tau^2$$

and

(4.42) $$dC_{11} - 2C_{12}\tau = C_{111}\tau^1 + C_{112}\tau^2 ,$$
$$dC_{12} + (C_{11} - C_{22})\tau = C_{121}\tau^1 + C_{122}\tau^2 ,$$
$$dC_{22} + 2C_{12}\tau = C_{221}\tau^1 + C_{222}\tau^2$$

with

(4.43) $$C_{121} - C_{112} = \varkappa C_2 , \quad C_{122} - C_{221} = \varkappa C_1 .$$

5. Let us study several objects associated with an elliptic surface $m: D \to A^3$. Let the frames be chosen in such a way that (4.3), (4.7) and (4.10) are valid. Given an arbitrary point $p_0 \in D$, there is its coordinate neighborhood $U \subset D$ such that we may write, on U,

(5.1) $$\tau^1 = f\,du , \quad \tau^2 = g\,dv , \quad \varphi = g_1 du + \varrho_2\,dv , \quad \tau_3^3 = \sigma_1\,du + \sigma_2\,dv ;$$
$$f = f(u, v), \ldots, \sigma_2 = \sigma_2(u, v) ; \quad fg \neq 0 .$$

Then

(5.2) $$\frac{\partial m}{\partial u} = fe_1 , \quad \frac{\partial m}{\partial v} = ge_2 ,$$
$$\frac{\partial e_1}{\partial u} = \tfrac{1}{2}(\sigma_1 - A_3 f)\,e_1 + \tfrac{1}{2}(A_2 f + 2\varrho_1)\,e_2 + fe_3 ,$$
$$\frac{\partial e_1}{\partial v} = \tfrac{1}{2}(\sigma_2 + A_2 g)\,e_1 + \tfrac{1}{2}(A_3 g + 2\varrho_2)\,e_2 ,$$
$$\frac{\partial e_2}{\partial u} = \tfrac{1}{2}(A_2 f - 2\varrho_1)\,e_1 + \tfrac{1}{2}(\sigma_1 + A_3 f)\,e_2 ,$$
$$\frac{\partial e_2}{\partial v} = \tfrac{1}{2}(A_3 g - 2\varrho_2)\,e_1 + \tfrac{1}{2}(\sigma_2 - A_2 g)\,e_2 + ge_3 .$$

Inserting into

(5.3) $$d\tau^1 = \tau^1 \wedge \tau_1^1 + \tau^2 \wedge \tau_2^1 , \quad d\tau^2 = \tau^1 \wedge \tau_1^2 + \tau^2 \wedge \tau_2^2 ,$$

we get

(5.4) $$\frac{\partial f}{\partial v} = -\tfrac{1}{2}\sigma_2 f - \varrho_1 g , \quad \frac{\partial g}{\partial u} = -\tfrac{1}{2}\sigma_1 g + \varrho_2 f .$$

Let $\gamma: (-h, h) \to D$ be a curve with $\gamma(0) = p_0$; let γ be given by $u = u(t)$, $v = v(t)$. We have

(5.5) $$m(u(t), v(t)) = m + (fu'e_1 + gv'e_2)\,t +$$
$$+ \frac{1}{2}\left[\left\{\frac{1}{2}\left(2\frac{\partial f}{\partial u} + \sigma_1 f + A_3 f^2\right)e_1 + \tfrac{1}{2}(A_2 f^2 + 2\varrho_1 f)\,e_2 + f^2 e_3\right\}u'^2 +\right.$$
$$+ \{(A_2 fg - 2\varrho_1 g)\,e_1 + (A_3 fg + 2\varrho_2 f)\,e_2\}\,u'v' +$$

$$+ \left\{\tfrac{1}{2}(A_3\varrho^2 - 2g\varrho_2)\, e_1 + \frac{1}{2}\left(2\frac{\partial g}{\partial v} + \sigma_2 g - A_2\varrho^2\right)e_2 + g^2 e_3\right\} v'^2 +$$

$$+ fu''e_1 + \varrho v''e_2\Big]\, t^2 + O(t^3)\,,$$

where all expressions on the right-hand side are to be calculated at $t = 0$ and $u' = \mathrm{d}u(0)/\mathrm{d}t, \ldots, v'' = \mathrm{d}^2v(0)/\mathrm{d}t^2$. Introducing the local coordinates X^i with respect to the frame $\{m; e_1, e_2, e_3\}$ at the point $m(p_0)$ by

$$X = m + X^1e_1 + X^2e_2 + X^3e_3\,, \tag{5.6}$$

we may write $m(u(t), v(t))$ as

$$X^1 = fu't + \frac{1}{4}\left\{\left(2\frac{\partial f}{\partial u} + \sigma_1 f + A_3 f^2\right)u'^2 + 2(A_2fg - 2\varrho_1 g)\, u'v' + \right. \tag{5.7}$$

$$\left. + (A_3g^2 - 2\varrho_2 g)\, v'^2 + 2fu''\right\} t^2 + O(t^3)\,,$$

$$X^2 = gv't + \frac{1}{4}\left\{(A_2f^2 + 2\varrho_1 f)\, u'^2 + 2(A_3fg + 2\varrho_2 f)\, u'v' + \right.$$

$$\left. + \left(2\frac{\partial g}{\partial v} + \sigma_2 g - A_2\varrho^2\right)v'^2 + 2gv''\right\} t^2 + O(t^3)\,,$$

$$X^3 = \tfrac{1}{2}(f^2u'^2 + g^2v'^2)\, t^2 + O(t^3)\,.$$

Consider a quadric given by

$$A_{ij}X^iX^j + A_iX^i + A = 0\,. \tag{5.8}$$

We get an osculating quadric of our surface $m\colon D \to A^3$ as follows: we insert (5.7) into (5.8); then the terms at t^0, t^1, t^2 must be equal to zero for each choice of $u', \ldots, v''$. We easily find that *the most general osculating quadric is given by*

$$(X^1)^2 + (X^2)^2 + A_{33}(X^3)^2 + 2(A_{13}X^1 + A_{23}X^2 - 1)\, X^3 = 0\,. \tag{5.9}$$

In the projective extension $A^3 \cup P^2$ of A^3, consider the homogeneous coordinates $X^i = \xi^i/\xi^0$; then the intersection of (5.9) with P^2 is

$$(\xi^1)^2 + (\xi^2)^2 + (A_{33}\xi^3 + 2A_{13}\xi^1 + 2A_{23}\xi^2)\, \xi^3 = 0\,. \tag{5.10}$$

Let the *Lie quadric* be defined as the osculating quadric intersecting P^2 in the Lie conic (4.25); the equation of the Lie quadric is

$$(X^1)^2 + (X^2)^2 - \tfrac{1}{8}(B_1 + 2B_3 + B_5)\,(X^3)^2 - 2X^3 = 0\,. \tag{5.11}$$

It is easy to see that the Lie quadric is a paraboloid if and only if $B_1 + 2B_3 + B_5 = 0$.

Let us suppose, for a moment, that our surface is non-maximal. Then we may choose the frames in such a way that we have (4.26). It is easy to see that the center S

of the Lie quadric (5.11) is

(5.12) $$S = m + \varepsilon e_3\,;$$

this gives the geometrical construction of the *normal vector* e_3.

Let us now return to a general elliptic surface. Denote by N the normal bundle of m (formed by the affine normal straight lines) and by T its tangent planes bundle. For a vector $v \in V^3$ and a point $m(p)$ we may write

(5.13) $$v = v_p^T + v_p^N\,; \quad v_p^T \in T_p\,, \quad v_p^N \in N_p\,;$$

let us introduce the projections

(5.14) $$\mathrm{pr}_p^N\colon V^3 \to N_p\,, \quad \mathrm{pr}_p^T\colon V^3 \to T_p \quad \text{by} \quad \mathrm{pr}_p^N(v) = v_p^N\,, \quad \mathrm{pr}_p^T(v) = v_p^T\,.$$

Proposition 4. *Let* $m\colon D \to A^3$ *be an elliptic surface with the associated frames satisfying* (4.3), (4.7) *and* (4.10). *Then the N-valued quadratic form*

(5.15) $$\mathrm{d}\sigma^2 := \{(\tau^1)^2 + (\tau^2)^2\}\, e_3$$

on D *is invariant. Let* $p \in D$, $t_p \in T_p(D)$ *and let* $\gamma\colon (-h, h) \to D$ *be a curve such that* $\gamma(0) = p$, $\mathrm{d}\gamma(\mathrm{d}/\mathrm{d}t|_0) = t_p$; *we have*

(5.16) $$\mathrm{pr}_p^N\left(\left.\frac{\mathrm{d}^2 m(\gamma(t))}{\mathrm{d}t^2}\right|_{t=0}\right) = \{(\tau^1(t_p))^2 + (\tau^2(t_p))^2\}\, e_3(p)\,.$$

Proof. Because of (5.5),

(5.17) $$\frac{\mathrm{d}^2 m}{\mathrm{d}t^2} = \left\{\frac{1}{2}\left(2\frac{\partial f}{\partial u} + \sigma_1 f - A_3 f^2\right) u'^2 + (A_2 fg - 2\varrho_1 g)\, u'v' + \right.$$
$$\left. + \tfrac{1}{2}(A_3 g^2 - 2\varrho_2 g)\, v'^2 + f u''\right\} e_1 +$$
$$+ \left\{\tfrac{1}{2}(A_2 f^2 + 2\varrho_1 f)\, u'^2 + (A_3 fg + 2\varrho_2 f)\, u'v' + \right.$$
$$\left. + \frac{1}{2}\left(2\frac{\partial g}{\partial v}\,\sigma_2 g - A_2 g^2\right) v'^2 + g v''\right\} e_2 + (f^2 u'^2 + g^2 v'^2)\, e_3$$

with

(5.18) $$t_p = u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v} \in T_p(D)\,, \quad u' = \frac{\mathrm{d}u(0)}{\mathrm{d}t}\,, \quad v' = \frac{\mathrm{d}v(0)}{\mathrm{d}t}\,,$$

the curve γ being given by $u = u(t)$, $v = v(t)$. From $(5_{1,2})$ we obtain

(5.19) $$\omega^1(t_p) = u'f\,, \quad \omega^2(t_p) = v'f\,.$$

QED.

Proposition 5. *Let* $m\colon D \to A^3$ *be an elliptic surface. Then the N-valued cubic form*

(5.20) $$\chi(\omega^1, \omega^2) := -\tfrac{1}{2}\{A_3(\omega^1)^3 - 3A_2(\omega^1)^2\,\omega^2 - 3A_3\omega^1(\omega^2)^2 + A_2(\omega^2)^3\}\, e_3$$

on D is invariant, and we have

$$(5.21)\qquad \mathrm{pr}_p^N\left(\frac{\mathrm{d}^3 m(\gamma(t))}{\mathrm{d}t^3}\bigg|_{t=0}\right) - \tfrac{3}{2}\mathrm{pr}_p^N\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{pr}_{\gamma(t)}^N \frac{\mathrm{d}^2 m(\gamma(t))}{\mathrm{d}t^2}\bigg|_{t=0}\right)\right\} = \chi(\omega^1(t_p), \omega^2(t_p));$$

for the notation, see Proposition 4.

Proof. We complete (5.2) by

$$(5.22)\qquad \frac{\partial e_3}{\partial u} = \tfrac{1}{4}(B_1 + B_3) f e_1 + \tfrac{1}{4}(B_2 + B_4) f e_2 + \sigma_1 e_3\,,$$

$$\frac{\partial e_3}{\partial v} = \tfrac{1}{4}(B_2 + B_4)\, g e_1 + \tfrac{1}{4}(B_3 + B_5)\, g e_2 + \sigma_2 e_3\,.$$

From (5.17), we get (in the obvious, nevertheless simplified notation)

$$(5.23)\qquad \mathrm{pr}^N \frac{\mathrm{d}^3 m}{\mathrm{d}t^3} = \frac{1}{2}\left\{\left(6f\frac{\partial f}{\partial u} + 3\sigma_1 f^2 - A_3 f^3\right)u'^3 + 3(A_2 f^2 g - 2\varrho_1 f g)\, u'^2 v' + \right.$$

$$+ 3(A_3 f g^2 + 2\varrho_2 f g)\, u' v'^2 + \left(6g\frac{\partial g}{\partial v} + 3\sigma_2 g^2 - A_2 g^3\right)v'^3 +$$

$$\left. + 6f^2 u' u'' + 6g^2 v' v''\right\} e_3\,,$$

$$(5.24)\qquad \mathrm{pr}^N\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{pr}^N \frac{\mathrm{d}^2 m}{\mathrm{d}t^2}\right)\right\} = \mathrm{pr}^N\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[(f^2 u'^2 + g^2 v'^2)\, e_3\right]\right\} =$$

$$= \left\{\left(2f\frac{\partial f}{\partial u} + \sigma_1 f^2\right)u'^3 - 2\varrho_1 f g u'^2 v' + 2\varrho_2 f g u' v'^2 + \right.$$

$$\left. + \left(2g\frac{\partial g}{\partial v} + \sigma_2 g^2\right)v'^3 + 2f^2 u' u'' + 2g^2 v' v''\right\} e_3\,,$$

and the proof is (almost) complete. QED.

Now, let $m\colon D \to A^3$ be a non-maximal surface, i.e., we suppose that the associated frames satisfy in addition (4.26), (4.27) and (4.29); e_3 is the normal vector. Then

$$(5.29)\qquad \frac{\partial e_3}{\partial u} = \tfrac{1}{4}(B_0 - 4\varepsilon) f e_1 + \tfrac{1}{4}(B_2 + B_4) f e_2 + C_1 f e_3\,,$$

$$\frac{\partial e_3}{\partial v} = \tfrac{1}{4}(B_2 + B_4)\, g e_1 - \tfrac{1}{4}(B_0 + 4\varepsilon)\, g e_2 + C_2 g e_3\,.$$

Proposition 6. *Let $m\colon D \to A^3$ be a non-maximal elliptic surface. Then the 1-form*

$$(5.30)\qquad \mathscr{C}(\tau^1, \tau^2) := C_1 \tau^1 + C_2 \tau$$

on D is invariant, and we have (in the notation of Proposition 4),

$$\text{(5.31)} \qquad \mathrm{pr}_p^N \left(\frac{\mathrm{d}e_3(\gamma(t))}{\mathrm{d}t} \bigg|_{t=0} \right) = \mathscr{C}(\tau^1(t_p), \tau^2(t_p))\, e_3(p)\,.$$

Proof follows immediately from (5.29). QED.

Proposition 7. *Let $m\colon D \to A^3$ be a non-maximal elliptic surface. Then the quadratic form*

$$\text{(5.32)} \qquad \mathscr{B}(\tau^1, \tau^2) := B_0(\tau^1)^2 + 2(B_2 + B_4)\,\tau^1\tau^2 - B_0(\tau^2)^2$$

on D is invariant. Let $\mathrm{d}s^2$ be the metric (4.34) *induced on D, and let $\langle\, ,\rangle$ be the associated scalar product. Then*

$$\text{(5.33)} \qquad \left\langle \frac{\mathrm{d}m(\gamma(t))}{\mathrm{d}t}\bigg|_{t=0},\; \mathrm{pr}_p^T \left(\frac{\mathrm{d}e_3(\gamma(t))}{\mathrm{d}t}\bigg|_{t=0} \right) + \varepsilon \frac{\mathrm{d}m(\gamma(t))}{\mathrm{d}t}\bigg|_{t=0} \right\rangle = \tfrac{1}{4}\mathscr{B}(\tau^1(t_p), \tau^2(t_p))\,.$$

Proof. We have

$$\text{(5.34)} \qquad \mathrm{pr}^T \frac{\mathrm{d}e_3}{\mathrm{d}t} = \tfrac{1}{4}\{(B_0 - 4\varepsilon)fu' + (B_2 + B_4)\,gv'\}\, e_1 + $$
$$+ \tfrac{1}{4}\{(B_2 + B_4)fu' - (B_0 + 4\varepsilon)\,gv'\}\, e_2\,, \quad \frac{\mathrm{d}m}{\mathrm{d}t} = fu'e_1 + gv'e_2\,,$$

and the proof now consists in direct verification of (5.33). QED.

Let us investigate the following *problem*: on a domain $D \subset \mathbb{R}^2$, let forms

$$\text{(5.35)} \qquad \mathrm{d}s^2 = (\tau^1)^2 + (\tau^2)^2\,,$$
$$\Phi = A_3(\tau^1)^3 - 3A_2(\tau^1)^2\tau^2 - 3A_3\tau^1(\tau^2)^2 + A_2(\tau^2)^3\,; \quad A_2^2 + A_3^2 \neq 0\,,$$

and further, $\varepsilon = \pm 1$ be given. Is there a non-maximal elliptic surface $m\colon D \to A^3$ (at least locally, i.e., a surface $m\colon D' \to A^3$ with $D' \subset D$) such that (7.1_1) is its affine metric and $\chi = -\frac{1}{2}\Phi e_3$? For the definition of χ, see (5.20).

Thus the 1-forms τ^1, τ^2 as well as the functions $A_2, A_3\colon D \to \mathbb{R}$ are given. From (4.31), we may determine the 1-form τ and the Gauss curvature $\varkappa$. The left-hand sides of (4.38) are known; we may rewrite (4.38) as

$$\text{(5.36)} \qquad \mathrm{d}A_2 - 3A_3\tau = p_1\tau^1 + p_2\tau^2\,, \quad \mathrm{d}A_3 + 3A_2\tau = p_3\tau^1 + p_4\tau^2$$

with $p_i\colon D \to \mathbb{R}$ known. Comparing (4.38) and (5.36), we get

$$\text{(5.37)} \qquad B_2 = \tfrac{3}{2}p_1 + \tfrac{1}{2}p_4 + 2A_3C_2\,, \quad B_4 = \tfrac{1}{2}p_1 + \tfrac{3}{2}p_4 + 2A_2C_1\,,$$
$$B_3 = \tfrac{1}{2}p_2 + \tfrac{1}{2}p_3 - \varepsilon - A_2C_2 - A_3C_1\,,$$
$$B_0 = 2p_2 - 2p_3 + 2A_2C_2 - 2A_3C_1\,.$$

The exterior differentiation of (5.36) yields

$$\text{(5.38)} \qquad \{\mathrm{d}p_1 - (p_2 + 3p_3)\,\tau\} \wedge \tau^1 + \{\mathrm{d}p_2 + (p_1 - 3p_4)\,\tau\} \wedge \tau^2 =$$
$$= 3A_3\varkappa\tau^1 \wedge \tau^2\,,$$

$$\{dp_3 + (3p_1 - p_4)\tau\} \wedge \tau^1 + \{dp_4 + (p_3 + 3p_2)\tau\} \wedge \tau^2 =$$
$$= -3A_2\varkappa\tau^1 \wedge \tau^2 ;$$

Cartan's lemma implies the existence of (known) functions $p_{ij}\colon D \to \mathbb{R}$ such that

(5.39) $$dp_1 - (p_2 + 3p_3)\tau = p_{11}\tau^1 + p_{12}\tau^2 ,$$
$$dp_2 + (p_1 - 3p_4)\tau = p_{21}\tau^1 + p_{22}\tau^2 ,$$
$$dp_3 + (3p_1 - p_4)\tau = p_{31}\tau^1 + p_{32}\tau^2 ,$$
$$dp_4 + (p_3 + 3p_2)\tau = p_{41}\tau^1 + p_{42}\tau^2 ;$$

(5.40) $$p_{21} - p_{12} = 3A_3\varkappa , \quad p_{32} - p_{41} = 3A_2\varkappa .$$

From (5.37) we find, using (4.33),

(5.41) $$dB_0 - 2(B_2 + B_4)\tau =$$
$$= 2(p_{21} - p_{31} - p_3C_1 + p_1C_2 - A_3C_{11} + A_2C_{12})\tau^1 +$$
$$+ 2(p_{22} - p_{32} - p_4C_1 + p_2C_2 + A_2C_{22} - A_3C_{12})\tau^2 ,$$
$$d(B_2 + B_4) + 2B_0\tau =$$
$$= 2(p_{11} + p_{41} + p_1C_1 + p_3C_2 + A_2C_{11} + A_3C_{12})\tau^1 +$$
$$+ 2(p_{12} + p_{42} + p_2C_1 + p_4C_2 + A_2C_{12} + A_3C_{22})\tau^2 ;$$
$$A_3(B_2 + B_4) + A_2B_0 + 4\varepsilon C_2 =$$
$$= 2\{(p_2 - p_3)A_2 + (p_1 + p_4)A_3 + (A_2^2 + A_3^2 + 2\varepsilon)C_2\} ,$$
$$A_3B_0 - A_2(B_2 + B_4) - 4\varepsilon C_1 =$$
$$= 2\{(p_2 - p_3)A_3 - (p_1 + p_4)A_2 - (A_2^2 + A_3^2 + 2\varepsilon)C_1\} .$$

Inserting these into (4.39) we get

(5.42) $$A_2(C_{11} - C_{22}) + 2A_3C_{12} + (p_1 + p_4)C_1 +$$
$$+ (p_3 - p_2 - A_2^2 - A_3^2 - 2\varepsilon)C_2 =$$
$$= q_1 := p_{22} - p_{11} - p_{41} - p_{32} + (p_2 - p_3)A_2 + (p_1 + p_4)A_3 ,$$
$$A_3(C_{11} - C_{22}) - 2A_2C_{12} +$$
$$+ (p_3 - p_2 + A_2^2 + A_3^2 + 2\varepsilon)C_1 - (p_1 + p_4)C_2 =$$
$$= q_2 := p_{21} - p_{31} + p_{12} + p_{42} - (p_1 + p_4)A_2 + (p_2 - p_3)A_3 ,$$

q_1, q_2 being again known functions. Further,

(5.43) $$3dB_2 - dB_4 + 2(B_0 - 8B_3 - 8\varepsilon)\tau =$$
$$= 2(2p_{11} - p_1C_1 + 3p_3C_2 - A_2C_{11} + 3A_3C_{12})\tau^1 +$$
$$+ 2(2p_{12} - p_2C_1 + 3p_4C_2 - A_2C_{12} + 3A_3C_{22})\tau^2 ,$$

$$4\mathrm{d}B_3 + \mathrm{d}B_0 + 2(3B_2 - 5B_4)\,\tau =$$
$$= 2(2p_{21} - 3p_3C_1 - p_1C_2 - A_2C_{12} - 3A_3C_{11})\,\tau^1 +$$
$$+ 2(2p_{22} - 3p_4C_1 - p_2C_2 - A_2C_{22} - 3A_3C_{12})\,\tau^2\,,$$
$$4\mathrm{d}B_3 - \mathrm{d}B_0 + 2(5B_2 - 3B_4)\,\tau =$$
$$= 2(2p_{31} - p_3C_1 - 3p_1C_2 - A_3C_{11} - 3A_2C_{12})\,\tau^1 +$$
$$+ 2(2p_{32} - p_4C_1 - 3p_2C_2 - 3A_2C_{22} - A_3C_{12})\,\tau^2\,,$$
$$3\mathrm{d}B_4 - \mathrm{d}B_2 + 2(B_0 + 8B_3 + 8\varepsilon)\,\tau =$$
$$= 2(2p_{41} + 3p_1C_1 - p_3C_2 + 3A_2C_{11} - A_3C_{12})\,\tau^1 +$$
$$+ 2(2p_{42} + 3p_2C_1 - p_4C_2 + 3A_2C_{12} - A_3C_{22})\,\tau^2\,;$$

inserting from these equalities into (4.40), we get identities. Using (5.42) and (4.36), we are in the position to evaluate

(5.44) $$C_{ij} = r_{ij}C_1 + r'_{ij}C_2 + r''_{ij}\,;\quad i, j = 1, 2\,;$$

with known functions $r_{ij}, r'_{ij}, r''_{ij}: D \to \mathbb{R}$. Inserting into (4.41) we get two linear equations for C_1, C_2; thus C_1, C_2 are known. As integrability conditions we have (4.32).

Hence we see that we can (in principle) determine a non-maximal elliptic surface by its metric and cubic forms; nevertheless, this method is not very effective.

GLOBAL THEOREMS

6. Before presenting global theorems, let us introduce the following

Definition. Let $m: D \to A^3$ be a non-maximal elliptic surface. Then $m \in \mathrm{Cl}(\mathscr{C})$ if, on D,

(6.1) $$C_1 = C_2 = 0\,;$$

$m \in \mathrm{Cl}(\mathscr{B})$ if, on D,

(6.2) $$B_0 = B_2 + B_4 = 0\,.$$

The geometrical meaning of these conditions is given in Propositions 6 and 7 respectively. The following proposition is trivial because of (4.39):

Proposition 8. *Let* $m \in \mathrm{Cl}(\mathscr{B})$; *then* $m \in \mathrm{Cl}(\mathscr{C})$.

In what follows, m is supposed to be non-maximal and elliptic.

Theorem 1. *Let D be compact and orientable, $m: D \to A^3$ a surface with $\varepsilon = 1$. Let $\mathscr{S}(m)$ denote its surface area with respect to its metric* (4.34). *Then*

(6.3) $$\mathscr{S}(m) = \int_D \varkappa\,\mathrm{d}v - \tfrac{1}{2}\int_D (A_2^2 + A_3^2)\,\mathrm{d}v\,,\quad \mathrm{d}v := \tau^1 \wedge \tau^2\,.$$

Proof. Consider the 1-form $\Omega := *\tau_3^3 = -C_2\tau^1 + C_1\tau^2$, $*$ being the Hodge operator with respect to ds^2. Then, using (4.36),

(6.4) $$d\Omega = (C_{11} + C_{22})\,dv = (2\varkappa - 2 - A_2^2 - A_3^2)\,dv\,,$$

and (6.3) follows from the Stokes theorem. QED.

Theorem 2. *Let D be compact and orientable, $m\colon D \to A^3$ a surface with $\varepsilon = 1$. Let $\varkappa \leqq 1$ on D. Then $m(D)$ is an ellipsoid.*

Proof. From $\int_D d\Omega = 0$ we have

(6.5) $$2 \int_D (\varkappa - 1)\,dv = \int_D (A_2^2 + A_3^2)\,dv\,.$$

The supposition $\varkappa \leqq 1$ on D implies

(6.6) $$\varkappa = 1$$

and

(6.7) $$A_2 = A_3 = 0\,.$$

This and (4.38) imply

(6.8) $$3B_2 - B_4 = 0\,,\quad 4B_3 + B_0 + 4 = 0\,,\quad 3B_4 - B_2 = 0\,,$$
$$4B_3 - B_0 + 4 = 0\,,$$

i.e., by virtue of (4.27),

(6.9) $$B_0 = 0\,,\quad B_1 = -3\,,\quad B_2 = 0\,,\quad B_3 = -1\,,\quad B_4 = 0\,,\quad B_5 = -3\,.$$

Thus $m \in \mathrm{Cl}(\mathscr{B})$ and, according to Proposition 8, $m \in \mathrm{Cl}(\mathscr{C})$, and we have (6.1). Because of (4.30) and (4.7), (4.10), our surface is given by the system

(6.10) $$dm = \tau^1 e_1 + \tau^2 e_2\,;\quad de_1 = \tau e_2 + \tau^1 e_3\,,\quad de_2 = -\tau e_1 + \tau^2 e_3\,,$$
$$de_3 = -\tau^1 e_1 - \tau^2 e_2\,.$$

Because of (4.31), this system is completely integrable. It is therefore sufficient to find one solution of (6.10), any other one being an affine transformation of the former. Let $\{O; E_1, E_2, E_3\}$ be a fixed frame in A^3. Then, with $\alpha \in \langle 0, 2\pi \rangle$, $\beta \in \langle 0, 2\pi \rangle$,

(6.11) $$m = O + \sin\alpha \sin\beta E_1 + \cos\alpha \sin\beta E_2 + \cos\beta E_3\,,$$
$$e_1 = \cos\alpha E_1 - \sin\alpha E_2\,,$$
$$e_2 = \sin\alpha\cos\beta E_1 + \cos\alpha\cos\beta E_2 - \sin\beta E_3\,,\quad v_3 = O - m$$

satisfy (6.10) with

(6.12) $$\tau^1 = \sin\beta\,d\alpha\,,\quad \tau^2 = d\beta\,,\quad \tau = -\cos\beta\,d\alpha\,.$$

Introducing the affine coordinates by $m = O + XE_1 + YE_2 + ZE_3$, we see that the point m is situated on the „sphere" $X^2 + Y^2 + Z^2 = 1$. QED.

Theorem 3. *Let $D \subset \mathbb{R}^2$ be a bounded domain, ∂D its boundary, $m\colon D \to A^3$*

a surface with $\varepsilon = 1$, *and let* $\varkappa \leqq -1$ *on* D *Further, let* $\mathscr{C} \equiv 0$ *on* ∂D. *Then* $m(D)$ *is a part of a hyperboloid.*

Proof. We get, compare (6.4),

$$0 = \int_{\partial D} \Omega = \int_D \Omega = \int_D (2\varkappa + 2 - A_2^2 - A_3^2)\, dv\,, \tag{6.13}$$

which implies (6.7). From (4.38) we have

$$3B_2 - B_4 = 0\,, \quad 4B_3 + B_0 - 4 = 0\,, \quad 4B_3 - B_0 - 4 = 0\,, \quad 3B_4 - B_2 = 0\,, \tag{6.14}$$

i.e.,

$$B_0 = 0\,; \quad B_1 = 3\,, \quad B_2 = 0\,, \quad B_3 = 1\,, \quad B_4 = 0\,, \quad Bz = 3\,. \tag{6.15}$$

Once again, we have (6.1) using Proposition 8, and our surface is given by a completely integrable system

$$dm = \tau^1 e_1 + \tau^2 e_2\,; \quad de_1 = \tau e_2 + \tau^1 e_3\,, \quad de_2 = -\tau e_1 + \tau^2 e_3\,, \quad de_3 = \tau^1 e_1 + \tau^2 e_2\,. \tag{6.16}$$

Similarly to the proof of the preceding theorem, consider

$$\begin{aligned} m &= O + \sin\alpha \sinh\beta E_1 + \cos\alpha \sinh\beta E_2 + \cosh\beta E_3\,, \\ e_1 &= \cos\alpha E_1 - \sin\alpha E_2\,, \quad e_2 = \sin\alpha\cosh\beta E_1 + \cos\alpha\cosh\beta E_2 + \\ &+ \sinh\beta E_3\,, \quad e_3 = m - O \end{aligned} \tag{6.17}$$

with $\alpha \in \langle 0, 2\pi \rangle$, $\beta \in \mathbb{R}$. Then the equations (6.16) are satisfied if we take

$$\tau^1 = \sinh\beta\, d\alpha\,, \quad \tau^2 = d\beta\,, \quad \tau = -\cosh\beta\, d\alpha\,, \tag{6.18}$$

and our surface is the hyperboloid $Z^2 - X^2 - Y^2 = 1$. QED.

Theorem 4. *Let* D *be compact,* $m: D \to A^3$ *a surface. Let* $m \in \mathrm{Cl}(\mathscr{B})$ *and* $\varkappa > 0$ *on* D. *Then* $m(D)$ *is an ellipsoid.*

Proof. The equations (4.38) and (4.40) reduce to

$$\begin{aligned} dA_2 - 3A_3\tau &= B_2\tau^1 + (B_3 + \varepsilon)\tau^2\,, \\ dA_3 + 3A_2\tau &= (B_3 + \varepsilon)\tau^1 - B_2\tau^2\,; \end{aligned} \tag{6.19}$$

$$\begin{aligned} \{dB_2 - 4(B_3 + \varepsilon)\tau\} \wedge \tau^1 + (dB_3 + 4B_2\tau) \wedge \tau^2 &= 3\varkappa A_3 \tau^1 \wedge \tau^2\,, \\ (dB_3 + 4B_2\tau) \wedge \tau^1 - \{dB_2 - 4(B_3 + \varepsilon)\tau\} \wedge \tau^2 &= -3\varkappa A_2 \tau^1 \wedge \tau^2\,. \end{aligned} \tag{6.20}$$

From (6.20) we get the existence of functions $B_{ij}: D \to \mathbb{R}$ such that

$$\begin{aligned} &dB_2 - 4(B_3 + \varepsilon)\tau = B_{21}\tau^1 + B_{22}\tau^2\,, \quad dB_3 + 4B_2\tau = B_{31}\tau^1 + B_{32}\tau^2\,; \\ &B_{31} - B_{22} = 3\varkappa A_3\,, \quad B_{21} + B_{32} = 3\varkappa A_2\,. \end{aligned} \tag{6.21}$$

Further,

$$\tfrac{1}{2}d(A_2^2 + A_3^2) = \{A_2 B_2 + A_3(B_3 + \varepsilon)\}\tau^1 + \{A_2(B_3 + \varepsilon) - A_3 B_2\}\tau^2\,, \tag{6.22}$$

and the Laplacian Δf of a function $f: D \to \mathbb{R}$ being defined by $d * df = \Delta f . do$, we get

$$\tfrac{1}{2}\Delta(A_2^2 + A_3^2) = 2\{B_2^2 + (B_3 + \varepsilon)^2\} + 3\varkappa(A_2^2 + A_3^2)\,. \tag{6.23}$$

From the maximum principle we get (6.7) and $B_2 = 0$, $B_3 = -\varepsilon$. Proposition 3 yields $\varkappa = \varepsilon$, i.e., $\varepsilon = 1$ because of $\varkappa > 0$. Thus we may repeat the second part of the proof of Theorem 2. QED.

Theorem 5. *Let $m: D \to A^3$ be a surface with $m \in \mathrm{Cl}(\mathscr{C})$. Then either $m \in \mathrm{Cl}(\mathscr{B})$ or the points with vanishing form $\mathscr{B}$ from* (5.32) *are isolated.*

Proof. Let $m \in \mathrm{Cl}(\mathscr{C})$. Then (4.39) reduce to

$$(6.24) \qquad \{dB_0 - 2(B_2 + B_4)\tau\} \wedge \tau^1 + \{d(B_2 + B_4) + 2B_0\tau\} \wedge \tau^2 = = \{A_3(B_2 + B_4) + A_2B_0\}\,\tau^1 \wedge \tau^2\,,$$

$$\{d(B_2 + B_4) + 2B_0\tau\} \wedge \tau^1 - \{dB_0 - 2(B_2 + B_4)\tau\} \wedge \tau^2 = = \{A_3B_0 - A_2(B_2 + B_4)\}\,\tau^1 \wedge \tau^2\,.$$

Let $D' \subset D$ be a coordinate neighborhood in D with coordinates (u, v) chosen in such a way that

$$(6.25) \qquad \tau^1 = f\,du\,, \quad \tau^2 = g\,dv\,; \quad f = f(u, v) \neq 0\,, \quad g = g(u, v) \neq 0\,.$$

Because of (4.31),

$$(6.26) \qquad \tau = -g^{-1}\frac{\partial f}{\partial v}\,du + f^{-1}\frac{\partial g}{\partial u}\,dv\,,$$

and (6.24) read

$$(6.27) \qquad g\,\frac{\partial(B_2 + B_4)}{\partial u} - f\,\frac{\partial B_0}{\partial v} = \left(A_3fg - 2\frac{\partial g}{\partial u}\right)(B_2 + B_4) + \left(A_2fg + 2\frac{\partial f}{\partial v}\right)B_0,$$

$$f\,\frac{\partial(B_2 + B_4)}{\partial v} + g\,\frac{\partial B_0}{\partial u} = \left(A_2fg - 2\frac{\partial f}{\partial v}\right)(B_2 + B_4) - \left(A_3fg + 2\frac{\partial g}{\partial u}\right)B_0\,.$$

This system for the functions $B_2 + B_4$, B_0 being clearly elliptic, it either has the vanishing solution or the zeros of its solution are isolated. QED.

Theorem 6. *Let D be diffeomorphic to the sphere $S^2(1)$, let $m: D \to A^3$ be a surface with $m \in \mathrm{Cl}(\mathscr{C})$. Then $m \in \mathrm{Cl}(\mathscr{B})$.*

Proof. If $B_0^2 + (B_2 + B_4)^2 \neq 0$ on D, the equation $\mathscr{B}(\omega^1, \omega^2) = 0$ determines a net of curves on D. But this is impossible because of $D = S^2(1)$. Thus $B_0 = = B_2 + B_4 = 0$ at least at one point $p_0 \in D$. The rest of the proof follows from the unicity of the solution of the elliptic system (6.27) on the sphere. QED.

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