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CONDITIONS FOR FACTORABLE RELATIONS

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Let A, B be algebras of the same type. A binary relation R on the product $A \times B$ is called *factorable* whenever $R = R_A \times R_B$ for some binary relations R_A on A and R_B on B. A variety V has *factorable* congruences (tolerances) whenever every congruence (tolerance, respectively) on $A \times B$, $A, B \in V$, has this property.

From [5] we know that a variety V has factorable congruences iff the congruence condition

$$\langle\langle x, x\rangle, \langle y, y\rangle\rangle \in \Theta \Rightarrow \langle\langle x, z\rangle, \langle y, z\rangle\rangle \in \Theta$$

holds for any congruence Θ on the product $A \times B$, $x, y \in A \in V$, $x, y, z \in B \in V$. In the recent paper [4] we have proved that a variety V has factorable congruences whenever the square $A \times A$, $x, y \in A \in V$, has the same property. However, two congruence conditions, namely

$$\langle\langle x, x\rangle, \langle y, y\rangle\rangle \in \Theta \Rightarrow \langle\langle x, y\rangle, \langle y, y\rangle\rangle \in \Theta \quad \text{see [2], and}$$

$$\langle\langle x, x\rangle, \langle y, x\rangle\rangle \in \Theta \Rightarrow \langle\langle x, y\rangle, \langle y, y\rangle\rangle \in \Theta\,, \quad \text{see [6]}\,,$$

are needed in [4].

The aim of the present paper is to show that a single congruence (tolerance) condition formulated on the product $A \times A \times A$, $x, y \in A \in V$, is enough for factorability of congruences (tolerances, respectively) on the whole variety V.

Let us recall that $\Theta(\langle\langle a_1, b_1, c_1\rangle, \langle a'_1, b'_1, c'_1\rangle\rangle, \ldots, \langle\langle a_m, b_m, c_m\rangle, \langle a'_m, b'_m, c'_m\rangle\rangle)$ $(T(\langle\langle a_1, b_1, c_1\rangle, \langle a'_1, b'_1, c'_1\rangle\rangle, \ldots, \langle\langle a_m, b_m, c_m\rangle, \langle a'_m, b'_m, c'_m\rangle\rangle))$ denotes the congruence (tolerance, respectively) on the product $A \times B \times C$ of similar algebras A, B, C generated by $\langle\langle a_1, b_1, c_1\rangle, \langle a'_1, b'_1, c'_1\rangle\rangle, \ldots, \langle\langle a_m, b_m, c_m\rangle, \langle a'_m, b'_m, c'_m\rangle\rangle \in$ $\in A \times B \times C \times A \times B \times C$.

The symbol $\mathbf{w}$ stands for a finite sequence $w_1, \ldots, w_n$.

Theorem 1. *For a variety V, the following conditions are equivalent:*

(1) *V has factorable congruences;*

(2) *the congruence condition $\langle\langle x, x, x\rangle, \langle y, y, x\rangle\rangle \in \Theta \Rightarrow \langle\langle x, x, y\rangle, \langle y, x, y\rangle\rangle \in$ $\in \Theta$ holds for any congruence Θ on the product $A \times A \times A$, $x, y \in A \in V$.*

Proof. (1) $\Rightarrow$ (2): Let Θ be an arbitrary congruence on the product $A \times A \times A$, $x, y \in A$. By hypothesis $\Theta = \Theta_1 \times \Theta_2 \times \Theta_3$ for some congruences Θ_1, Θ_2 and Θ_3

on A. Then $\langle\langle x, x, x\rangle, \langle y, y, x\rangle\rangle \in \Theta$ yields $\langle x, y\rangle \in \Theta_1$, $\langle x, y\rangle \in \Theta_2$ and $\langle x, x\rangle \in \Theta_3$. Since $\langle x, x\rangle \in \Theta_2$ and $\langle y, y\rangle \in \Theta_3$ by reflexivity, we have also $\langle\langle x, x, y\rangle, \langle y, x, y\rangle\rangle \in \Theta_1 \times \Theta_2 \times \Theta_3 = \Theta$, as required.

(2) $\Rightarrow$ (1): Take $A = F_V(x, y)$, the V-free algebra with free generators x and y. Further take $\Theta = \Theta(\langle\langle x, x, x\rangle, \langle y, y, x\rangle\rangle)$ on the product $A \times A \times A$. Then the assumption of (2) is fulfilled and thus $\langle\langle x, x, y\rangle, \langle y, x, y\rangle\rangle \in \Theta(\langle\langle x, x, x\rangle, \langle y, y, x\rangle\rangle)$ Applying the binary scheme from (1) to this relation we get the identities

(α) $x = d_1(x, y, \boldsymbol{a}(x, y))$,
(β) $x = d_1(x, y, \boldsymbol{b}(x, y))$,
(γ) $y = d_1(x, x, \boldsymbol{c}(x, y))$,

(α) $d_i(y, x, \boldsymbol{a}(x, y)) = d_{i+1}(x, y, \boldsymbol{a},(x, y))$,
(β) $d_i(y, x, \boldsymbol{b}(x, y)) = d_{i+1}(x, y, \boldsymbol{b}(x, y))$,
(γ) $d_i(x, x, \boldsymbol{c}(x, y)) = d_{i+1}(x, x, \boldsymbol{c}(x, y))$, $1 \leqq i < m$,

(α) $y = d_m(y, x, \boldsymbol{a}(x, y))$,
(β) $x = d_m(y, x, \boldsymbol{b}(x, y))$,
(γ) $y = d_m(x, x, \boldsymbol{c}(x, y))$

for some binary terms $a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n$ and $(2 + n)$-ary terms $d_1, \ldots, d_m$. It is known, see [4], that the above identities $(\alpha), (\beta), (\gamma)$ ensure the factorability of congruences. Notice that the identities $(\alpha), (\beta)$ $((\alpha), (\gamma))$ were already used in the former papers [2] ([6], respectively).

Theorem 2. *For a variety V, the following conditions are equivalent:*

(1) *V has factorable tolerances;*

(2) *the tolerance condition*

$$\langle\langle x, x, x\rangle, \langle y, y, x\rangle\rangle, \langle\langle y, y, y\rangle, \langle y, y, x\rangle\rangle \in T \Rightarrow$$
$$\Rightarrow \langle\langle x, y, y\rangle, \langle y, y, x\rangle\rangle \in T$$

holds for any tolerance T on the product $A \times A \times A$, $x, y \in A \in V$.

Proof. (1) $\Rightarrow$ (2): Let T be a tolerance on $A \times A \times A$, $x, y \in A \in V$. Since T is of the form $T = T_1 \times T_2 \times T_3$ for some tolerances T_1, T_2 and T_3 on A, we have $\langle x, y\rangle, \langle y, y\rangle \in T_1$, $\langle x, y\rangle, \langle y, y\rangle \in T_2$ and $\langle x, x\rangle, \langle y, x\rangle \in T_3$. In particular, $\langle x, y\rangle \in T_1$, $\langle y, y\rangle \in T_2$, $\langle y, x\rangle \in T_3$ and thus $\langle\langle x, y, y\rangle, \langle y, y, x\rangle\rangle \in T_1 \times T_2 \times T_3 = T$.

(2) $\Rightarrow$ (1): The tolerance $T(\langle\langle x, x, x\rangle, \langle y, y, x\rangle\rangle, \langle\langle y, y, y\rangle, \langle y, y, x\rangle\rangle)$ on the product $F_V(x, y) \times F_V(x, y) \times F_V(x, y)$ evidently satisfies the assumptions from (2). Hence $\langle\langle x, y, y\rangle, \langle y, y, x\rangle\rangle \in T(\langle\langle x, x, x\rangle, \langle y, y, x\rangle\rangle, \langle\langle y, y, y\rangle, \langle y, y, x\rangle\rangle)$. By a standard argument, see [1] again, we get binary terms $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, $c_1, \ldots, c_n$ and a $(4 + n)$-ary term t such that

(α) $x = t(x, y, y, y, \boldsymbol{a}(x, y))$,

$(\beta)\ y = t(x, y, y, y, \boldsymbol{b}(x, y))$,
$(\gamma)\ y = t(x, x, y, x, \boldsymbol{c}(x, y))$,
$(\alpha)\ y = t(y, x, y, y, \boldsymbol{a}(x, y))$,
$(\beta)\ y = t(y, x, y, y, \boldsymbol{b}(x, y))$,
$(\gamma)\ x = t(x, x, x, y, \boldsymbol{c}(x, y))$

are identities in $\boldsymbol{V}$.

First, consider the identities $(\alpha), (\beta)$. Interchanging the variables x and y in (β) we obtain

$(\alpha)\ x = t(x, y, y, y, \boldsymbol{a}(x, y))$,
$(\beta)\ x = t(y, x, x, x, \boldsymbol{b}(y, x))$,
$(\alpha)\ y = t(y, x, y, y, \boldsymbol{a}(x, y))$,
$(\beta)\ x = t(x, y, x, x, \boldsymbol{b}(y, x))$.

Defining

$t_1(u, v, \boldsymbol{w}) = t(u, v, w_{n+1}, w_{n+2}, w_1, \ldots, w_n)$,
$\boldsymbol{f}(x, y) = a_1(x, y), \ldots, a_n(x, y), y, y,$ and
$\boldsymbol{g}(x, y) = b_1(y, x), \ldots, b_n(y, x), x, x$

we find the identities

$$(\Sigma_1)\quad \begin{aligned} x &= t_1(x, y, \boldsymbol{f}(x, y)),\\ x &= t_1(y, x, \boldsymbol{g}(x, y)),\\ y &= t_1(y, x, \boldsymbol{f}(x, y)),\\ x &= t_1(x, y, \boldsymbol{g}(x, y)). \end{aligned}$$

Further, take the identities $(\alpha), (\gamma)$:

$(\alpha)\ x = t(x, y, y, y, \boldsymbol{a}(x, y))$,
$(\gamma)\ y = t(x, x, y, x, \boldsymbol{c}(x, y))$,
$(\alpha)\ y = t(y, x, y, y, \boldsymbol{a}(x, y))$,
$(\gamma)\ x = t(x, x, x, y, \boldsymbol{c}(x, y))$.

By setting $t_2 = t$, $\boldsymbol{h} = \boldsymbol{a}$, and $\boldsymbol{k} = \boldsymbol{c}$ we get exactly the identities (Σ_2) from [3; Thm. 2 (4)]. As stated in this theorem the identities (Σ_1) and (Σ_2) together guarantee the factorability of tolerances on a variety $\boldsymbol{V}$.

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