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COMPLETIONS AND CLOSURES OF CYCLICALLY ORDERED GROUPS

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1. INTRODUCTION

The notion of a cyclically ordered group is due to Rieger [17]. A representation theorem for cyclically ordered groups has been proved by S. Swierczkowski [18]. Further results on cyclically ordered groups were established in [2], [3], [4], [7], [8], [9], [11], [19], [20] and [21]. Cf. also [5], Chap. IV, § 6. In [22], [23] and [24] more general notion (which can be termed a partially cyclically ordered group) was dealt with.

The completion $d_1(G)$ of a cyclically ordered group G has been defined in [2]. The elments of $d_1(G)$ are certain cuts of G as introduced (in a more general setting) in [16]. The cyclically ordered group G is called *complete* if it coincides with $d_1(G)$ (under the natural injection of G into $d_1(G)$).

If G_1 is a subgroup of a cyclically ordered group G, then G_1 is always considered to be cyclically ordered by means of the inherited cyclic order.

Linearly ordered groups can be viewed as a particular case of cyclically ordered groups.

We can introduce the notion of a closed subgroup of a cyclically ordered group in such a way that for the particular case of linearly ordered groups this notion has the usual meaning. (Cf. Section 3 below.)

Assume that G is a subgroup of a cyclically ordered group H. If for each closed subgroup G' of H with $G \subseteq G'$ the relation $G' = H$ is valid, then H is said to be *c-generated* by G.

For each cyclically ordered group G we denote by $\mathscr{C}(G)$ the class of all complete cyclically ordered groups H such that

(a) G is a subgroup of H;

(b) H is c-generated by G.

The elements of $\mathscr{C}(G)$ will be called *c-closures* of G.

In the present paper it will be proved that for each cyclically ordered group G the class $\mathscr{C}(G)$ contains, up to isomorphisms, only one element, namely $d_1(G)$.

This generalizes a result of [10] concerning lineatly ordered groups (the case of archimedean linearly ordered groups was dealt with earlier in [13]).

Let us remark that if $\mathscr{C}(G)$ is the class defined in an analogous way for the case when G is a lattice ordered group, then there can exist, in general, a subclass $\mathscr{C}_1$ of $\mathscr{C}(G)$ such that $\mathscr{C}_1$ is a proper class and the lattice ordered groups belonging to $\mathscr{C}_1$ are mutually nonisomorphic. (Cf. [13], [14].) A similar situation occurs in the theory of Boolean algebras [15] and of vector lattices [6].

2. COMPLETIONS

This section can be viewed as a continuation of [2]. Some new results on completions of cyclically ordered groups will be established; they will be then applied in Section 3 to investigate the relations between completions and closures.

For the basic notions and notation concerning cyclically ordered sets and cyclically ordered groups cf. [2], Section 1.

A cyclically ordered group will be written as $(G; +, [\,])$; if not missunderstanding can occur, then we write G instead of $(G; +, [\,])$.

Let $C(G)$ be the completion of the cyclically ordered set $(G; [\,])$ (cf. [16] or [2], Section 2). We recall the following definition (cf. [2], p. 161).

Let G_1 be a subset of $C(G)$ with $G \subseteq G_1$. Suppose that a binary operation $+_1$ is defined on G_1 such that the following conditions are fulfilled:

(i) $(G_1; +_1)$ is a cyclically ordered group (under the cyclic order inherited from $C(G)$).

(ii) $(G; +)$ is a subgroup of $(G_1; +)$.

Then $(G_1; +_1)$ is said to be an extension of G in $C(G)$ (we shall write G_1 instead of $(G_1; +_1)$).

Let $C_0(G)$ be the set of all extensions of G in $C(G)$. For $G_1, G_2 \in C_0(G)$ we put $G_1 \leqq G_2$ if G_1 is a subgroup of G_2. Then $C_0(G)$ turns out to be a partially ordered set. If $C_0(G)$ possesses a greatest element $d_1(G)$, then $d_1(G)$ is said to be a *completion* of the cyclically ordered group G.

In [2] it has been proved that the completion $d_1(G)$ does exist for each cyclically ordered group G.

Let K be as in [2], Section 1. The largest linearly ordered subgroup of a cyclically ordered group G will be denoted by G_0 (cf., e.g., [2], Lemma 13). It is obvious that G_0 is a normal subgroup of G.

First, let $G_0 = \{0\}$. In [2] it was shown that $d_1(G) = G$ if G is finite, and $d_1(G)$ is isomorphic to K if G is infinite.

Next, let $G_0 \neq \{0\}$. In [2], Section 6, a cyclically ordered group G_1 was constructively described and it was proved ([2], Theorem 6.2) that G_1 is a completion of G.

In the rest of this section we assume that $G_0 \neq \{0\}$.

Let $g \in G$. For $g(1)$ and $g(2)$ in $g + G_0$ we put $g(1) \leqq g(2)$ if the relation

(1) $$g(1) - g \leqq g(2) - g$$

is valid in G_0.

If g' is another element of $g + G_0$, then in view of $g - g' \in G_0$, (1) is equivalent to

$$g(1) - g + (g - g') \leqq g(2) - g + (g - g');$$

hence the relation $\leqq$ on the set $g + G_0$ is independent of the particular choice of the element g of the set $g + G_0$.

The following assertion is obvious (it was expressed already in [2]).

2.1. Lemma. *Let $g \in G$. Then the relation $\leqq$ on $g + G_0$ is a linear order. For each $g' \in g + G_0$, the mapping $t \to t + g'$ (where t runs over G_0) is an isomorphism of the linearly ordered set G_0 onto $g + G_0$.*

2.2. Lemma. *Let $g \in G$, $\{x_i\}_{i \in I} \subseteq g + G_0$, $x \in g + G_0$, $t \in G$. Assume that $x = \bigvee_{i \in I} x_i$ holds in $g + G_0$. Then*

$$x + t = \bigvee_{i \in I} (x_i + t), \quad t + x = \bigvee_{i \in I} (t + x_i)$$

are valid in $g + t + G_0$.

Proof. We have $\{x_i + t\}_{i \in I} \subseteq g + t + G_0$ and $x + t \in g + t + G_0$. Next, 2.1 yields $x - g = \bigvee_{i \in I} (x_i - g)$ in G_0. By applying 2.1 again we obtain that

$$(x - g) + (g + t) = \bigvee_{i \in I} ((x_i - g) + (g + t))$$

holds in $g + t + G_0$, hence the first of the desired assertions is valid. The scond assertion can be proved similarly.

Analogously we can verify the assertion dual to 2.2.

Let $D(G_0)$ and $m(G_0)$ be as in [2] (Sections 4 and 5). In view of [2], p. 165 we can assume that $D(G_0) \subseteq C(G)$; because of $m(G_0) \subseteq D(G_0)$ we obtain $m(G_0) \subseteq C(G)$.

Let H' be a completion of G. According to [2], Lemma 5.6, $m(G_0)$ is asubgroup of H'. Since $m(G_0)$ is a linearly ordered group we get $m(G_0) \subseteq H'_0$. Thus $m(G_0)$ is a subgroup of H'_0.

Let $h' \in H'_0$. Then [2], Lemma 6.3 yields that there is $g \in G$ such that $g + h' \in m(G_0)$. If $g \notin H'_0$, then $g + h' \in g + h' + H'_0 = g + H'_0 \neq H'_0$, which is a contradiction. Therefore $g \in H'_0$. Hence the subgroup of H generated by g is linearly ordered. Thus g belongs to $G_0 \subseteq m(G_0)$. Then we have $h' \in m(G_0)$. Summarizing, we conclude that $H'_0 = m(G_0)$. Thus from 6.1, 6.2 and 6.3 in [2] we obtain:

2.3. Lemma. *Let H' be a completion of G. Then*

(i) $H'_0 = m(G_0)$;

(ii) *for each $h' \in H'$ there is $g \in G$ such that $h' \in g + H'_0$.*

In what follows, H' has the same meaning as in 2.3.

2.4. Lemma. *Let $g \in G$. Next, let $\{g_i\}_{i \in I}$ and $\{g_j\}_{j \in J}$ be subsets of $g + G_0$ such that* *the relations*

$$\bigwedge_{i\in I, j\in J} (g_j - g_i) = 0\,, \quad \bigwedge_{i\in I, j\in J} (-g_i + g_j) = 0$$

are valid in G_0. *Then there is a uniquely determined element* $h' \in H'$ *such that* $h' = \bigvee_{i\in I} g_i = \bigwedge_{j\in J} g_j$ *is valid in* $g + H'_0$.

Proof. Put $g'_i = g_i - g$ and $g'_j = g_j - g$ for each $i \in I$ and each $j \in J$. Then $g'_j - g'_i = g_j - g_i$, whence $\bigwedge_{i\in I, j\in J} (g'_j - g'_i) = 0$ holds in G_0. Similarly, $\bigwedge_{i\in I, j\in J} (-g'_i + g'_j) = 0$ is valid in G_0. Thus according to [1] there is $h'_1 \in m(G_0)$ such that

(1) $\qquad \bigvee_{i\in I} g'_i = \bigwedge_{j\in J} g'_j = h'_1$

holds in $m(G_0)$. Then in view of 2.3 (i), the relation (1) is valid in H'_0. Hence according to 2.2, the relation

$$\bigvee_{i\in I} g_i = \bigwedge_{j\in J} g_j = h_1 + g$$

holds in $g + H'_0$. Now it suffices to put $h' = h'_1 + g$. The uniqueness of the element h' is obvious.

Now let H'' be any cyclically ordered group such that

(a) G is a subgroup of H'';

(b) $H''_0 = m(G_0)$;

(c) for each $h'' \in H''$ there is $g \in G$ with $h'' \in g + H''_0$.

If $h'' \in H''$, then we consider the linear order $\leqq$ on the set $h'' + H''_0$ which is defined in an analogous way as in 2.1. Let g be as in (c). If $t \in H''_0 \cap G$, then $t + g \in (h'' + H''_0) \cap G$ (since H''_0 is a normal subgroup of H''). In view of (b), each element of H''_0 is a join of some elements of G_0; thus according to 2.2 we obtain:

2.5. Lemma. *Let* $h'' \in H''$. *Then there is a set* $\{g_i\}_{i\in I} \subseteq (h'' + H''_0) \cap G$ *such that* $h'' = \bigvee_{i\in I} g_i$.

In a dual way we obtain:

2.6. Lemma. *Let* $h'' \in H''$. *Then there is a set* $\{g_j\}_{j\in J} \subseteq (h'' + H''_0) \cap G$ *such that* $h'' = \bigwedge_{j\in J} g_j$.

2.7. Lemma. *Let* $h'' \in H''$. *Next, let* $\{g_i\}_{i\in I}$ *and* $\{g_j\}_{j\in J}$ *be as in 2.5 or 2.6, respectively. Then* $\bigwedge_{i\in I, j\in J} (g_j - g_i) = 0$ *and* $\bigwedge_{i\in I, j\in J} (-g_i + g_j) = 0$ *are valid in* G_0.

Proof. By way of contradiction, assume that the first assertion of the lemma fails to hold. Then there is $0 < c \in G_0$ such that $g_j - g_i \geqq c$ in G_0 for each $i \in I$ and each $j \in J$. If $i \in I$, then in the linearly ordered set $h'' + H'_0 = g + H'_0$ (where g is as in (c) above) we would have $g_j \geqq x_i + c$, from which we obtain $h'' \geqq x_i + c$, hence $h'' - c \geqq x_i$ and thus $h'' - c \geqq h''$, which is a contradiction. The second relation can be proved analogously.

Let us remark that in the proof of 2.4 we have used only the fact that $G \subseteq H'$ and the conditions (i), (ii). Thus in view of (a), (b) nad (c) above, 2.4 remains valid if H' is replaced by H''; let the corresponding assertion be denoted as Lemma 2.4′.

2.8. Lemma. *Let* $g \in G$, $\{g_i\}_{i\in I} \subseteq g + G_0$, $h' \in g + H'_0$. *Assume that* $h' = \bigvee_{i\in I} g_i$ *is valid in the linearly ordered set* $g + H'_0$. *Then there is* $h'' \in g + H''_0$ *such that* $h'' = \bigvee_{i\in I} g_i$ *holds in* $g + H''_0$.

Proof. Let us apply Lemma 2.6 for H' instead of H'' and for the element h'. Thus there exists a set $\{g_j\}_{j\in J}$ in $g + G_0$ such that $\bigwedge_{j\in J} g_j = 0$. In view of 2.7 (applied for h' and H') the relations

$$\bigwedge_{i\in I, j\in J} (g_j - g_i) = \bigwedge_{i\in I, j\in J} (-g_i + g_j) = 0$$

are valid in G_0. Hence according to 2.4′ we obtain that there is $h'' \in g + H''_0$ such that $h'' = \bigvee_{i\in I} g_i$ is valid in $g + H''_0$.

2.9. Lemma. *Let* $g \in G$, $\{g_i\}_{i\in I} \subseteq g + G_0$, $\{g_k\}_{k\in K} \subseteq g + G_0$, $h' \in g + H'_0$. *Assume that* $h' = \bigvee_{i\in I} g_i = \bigvee_{k\in K} g_k$ *is valid in the linearly ordered set* $g + H'_0$. *Let* h'' *be as in* 2.8. *Then* $h'' = \bigvee_{k\in K} g_k$ *holds in* $g + H''_0$.

Proof. According to 2.8 there is $h''(1) \in g + H''_0$ such that $h''(1) = \bigvee_{k\in K} g_k$ is valid in $g + H''_0$. We have to verify that $h'' = h''(1)$. By way of contradiction, suppose that $h'' \neq h''(1)$. Then without loss of generality we can assume that $h'' < h''(1)$ in $g + H''_0$. Hence $h''(1) - h'' > 0$ is valid in $H''_0 = m(G_0)$. Thus there is $0 < c \in G_0$ such that $c \leqq h''(1) - h''$ is valid in H''_0. Let $\{g_j\}_{j\in J}$ be as in the proof of 2.8. In view of 2.7 (applied for h', H', $\{g_k\}_{k\in K}$ and $\{g_j\}_{j\in J}$) we obtain that $\bigwedge_{i\in I, k\in K} (g_k - g_i) = 0$ is valid in G_0. But

$$g_k - g_i \geqq h''(1) - h'' \geqq c \quad (i \in I,\ k \in K)$$

is valid in H''_0, whence $g_k - g_i \geqq c$ is valid in G_0 for each $i \in I$ and each $k \in K$, which is a contradiction.

2.10. Lemma. *Let* g, $\{g_i\}_{i\in I}$ *and* h' *be as in* 2.8. *Next, let* $g(1)$, $\{g_{i(1)}\}_{i(1)\in I(1)}$ *and* $h'(1)$ *have analogous meanings. Then* $\bigvee_{i\in I, i(1)\in I(1)} (g_i + g_{i(1)}) = h' + h'(1)$ *is valid in* $g + g(1) + H'_0$.

Proof. According to 2.2 we have

$$h' + h'(1) = \bigvee_{i(1)\in I(1)} (h' + g_{i(1)}) = \bigvee_{i\in I} \bigvee_{i(1)\in I(i)} (g_i + g_{i(1)}) =$$
$$= \bigvee_{i\in I, i(1)\in I(1)} (g_i + g_{i(1)})$$

in $g + g(1) + H'_0$.

2.11. Lemma. *Let* $h'(1), h'(2), h'(3) \in H'$. *Then* $[h'(1), h'(2), h'(3)]$ *if and only if some of the following conditions is fulfilled*:

(i) *there are distinct elements* $g_1, g_2, g_3 \in G$ *such that* $h'(i) \in g_i + H'_0$ $(i = 1, 2, 3)$ *and* $[g_1, g_2, g_3]$;

(ii) *there are* $g_1, g_3 \in G$ *such that* $h(1) \in g_1 + H'_0$, $h(2) \in g_1 + H'_0$, $h(3) \in g_3 + H'_0$, $g_3 \notin g_1 + H'_0$ *and* $h(1) < h(2)$ *in* $g_1 + H'_0$;

(iii) *there are* $g_1, g_2 \in G$ *such that* $h(2), h(3) \in g_2 + H'_0$, $g_1 \notin g_2 + H'_0$ *and* $h(2) < h(3)$ *in* $g_2 + H'_0$;

(iv) *there are* $g_2, g_3 \in G$ *such that* $h(1), h(3) \in g_3 + H'_0$, $g_2 \notin g_3 + H'_0$ *and* $h(3) < h(1)$ *in* $g_3 + H'_0$;

(v) *there is* $g \in G$ *such that* $h(1), h(2), h(3) \in g + H'_0$ *and some of the conditions*

$$h(1) < h(2) < h(3) \quad or \quad h(2) < h(3) < h(1) \quad or \quad h(3) < h(1) < h(2)$$

is valid in $g + H'_0$.

Proof. This is an immediate consequence of 2.3 and 2.1.

We define a mapping φ of H' into H'' as follows.

Let $h' \in H'$. According to 2.5 (applied for h' and H') there is a subset $\{g_i\}_{i \in I}$ of $(h' + H'_0) \cap G$ such that $h' = \bigvee_{i \in I} g_i$ holds in $h' + H'_0$. Next, there is $g \in \in (h' + H'_0) \cap G$. Let h'' be as in 2.8. Then we put $\varphi(h') = h''$.

It is easy to verify that h'' does not depend on the particular choice of the element g of the set $(h' + H'_0) \cap G$. This fact and Lemma 2.9 imply that the mapping φ is correctly defined.

Next, 2.8 yields that φ is an injection. From 2.5, 2.6, 2.7 and 2.4 we obtain that φ is surjective. Further, from 2.10 and from the assertion analogous to 2.10 concerning H'' we infer that φ is a homomorphism with respect to the operation $+$. Summarizing, we have

2.12. Lemma. *φ is an isomorphism of the group H' onto H''.*

2.13. Lemma. *Let $h(1), h(2) \in H'$ such that $h(1) - h(2) \in H_0$ and $h(1) \leqq h(2)$ in the linearly ordered set $h(1) + H'_0$. Then $\varphi(h(1)) \leqq \varphi(h(2))$, and conversely.*

Proof. This is an immediate consequence of the definition of φ.

2.14. Lemma. *φ is an isomorphism of the cyclically ordered set H' onto the cyclically ordered set H''.*

Proof. This follows from 2.13 and 2.11.

It is obvious that $\varphi(g) = g$ for each $g \in G$.

The following Theorem is a consequence of 2.3, 2.12 and 2.14.

2.15. Theorem. *Let G be a cyclically ordered group with $G_0 \neq \{0\}$. Let H' be the completion of G and let H'' be a cyclically ordered group such that G is a subgroup of H''. Then the following conditions are equivalent:*

(α) *There is an isomorphism φ of H' onto H'' such that $\varphi(g) = g$ for each $g \in G$.*

(β) *H'' satisfies the conditions* (b) *and* (c).

3. CLOSURES

If $(L; \leqq)$ is a linearly ordered set, then the cyclic order on L which is generated by the linear order $\leqq$ will be denoted by $[\ ]_{\leqq}$. More thoroughly: for distinct elements a, b and c of L we put $[a, b, c]_{\leqq}$ if and only if

(1) $\qquad a < b < c \quad$ or $\quad b < c < a \quad$ or $\quad c < a < b$

is valid.

Let H be a cyclically ordered group. A linear order on the set H will be called *admissible* if the cyclic order $[\,]_{\leqq}$ coincides with the original order $[\,]$ defined on H.

Let G^0 be a subset of a cyclically ordered group H. Consider the following condition:

(c_1) If $\leqq$ is an admissible linear order on H, $\emptyset \neq X \subseteq G^0$, and if h is an element of H such that the relation $h = \sup X$ holds in $(H; \leqq)$, then $h \in G^0$.

If the condition (c_1) and also the condition (c_2) dual to (c_1) are valid, then H will be said to be a *closed subset* of the cyclically ordered group H.

Let us consider the particular case when $(H; \leqq)$ is a linearly ordered group. Then H is a cyclically ordered group under the cyclic order $[\,]_{\leqq}$.

Let $A \neq \emptyset$ and $B \neq \emptyset$ be subsets of H such that $A \cap B = \emptyset$, $A \cup B = H$ and $a < b$ for each $a \in A$ and each $b \in B$. Let $x, y \in H$. We put $x \leqq (A, B)\, y$ it some of the following conditions holds:

(i) $x \leqq y$, and either $x, y \in A$ or $x, y \in B$;

(ii) $x \in B$ and $y \in A$.

The following assertion is a consequence of [16], Corollary 3.9.

3.1. Lemma. *Let H be a linearly ordered group and let $\leqq (1)$ be a linear order on H. Then $\leqq(1)$ is an admissible linear order on H if and only if either $\leqq(1)$ coincides with $\leqq$, or there are $A, B \subseteq H$ satisfying the above conditions such that $\leqq(1)$ coincides with $\leqq(A, B)$.*

Lemma 3.1 yields:

3.2. Lemma. *Let H be a linearly ordered group and let $\emptyset \neq X \subseteq H$, $h \in H$. Let $\leqq(1)$ be an admissible linear order on H. Assume that $\sup X = h$ is valid in $(H; \leqq(1))$. Then there is a nonempty subset X_1 of X such that $\sup X_1 = h$ is valid in $(H; \leqq)$.*

3.3. Lemma. *Let H be a linearly ordered group and let G^0 be a subset of H. Then the following conditions are equivalent:*

(i) *G^0 is a closed subset of the cyclically ordered group H.*

(ii) *G^0 is a closed subset (in the usual sense) of the linearly ordered group H.*

Proof. This is a consequence of 3.2 and of the corresponding dual assertion.

Thus we have verified that in the case of linearly ordered groups the notion of closedness (as introduced above) coincides with the usual meaning of closedness as applied for lattices.

Now let G be a cyclically ordered group. The class $\mathscr{C}(G)$ was defined in the introduction. Let $H \in \mathscr{C}(G)$ and let H' be the completion of G. Our purpose is to show that H is isomorphic to H'.

Consider the factor cyclically ordered group G/G_0 (cf. [12]). From the Swierczkowski's Representation Theorem (cf. [18] or [2], Thm. 1.1) we obtain that G/G_0 is isomorphic to a subgroup of K. Since K is linearly ordered (let us remark that it fails to be a linearly ordered group), we can assume that G/G_0 is linearly ordered; this linear order is generated by the isomorphism under consideration.

Let g_1 and g_2 be elements of G. We put $g_1 \leqq^0 g_2$ if either $g_1 + G_0 < g_2 + G_0$ in the linear order of G/G_0, or $g_1 + G_0 = g_2 + G_0$ and $g_1 \leqq g_2$ in $g_1 + G_0$ (in the sense of Lemma 2.1).

Then we obviously have:

3.4. Lemma. *$\leqq^0$ is a linear order on G. Next, $\leqq^0$ is admissible with respect to the original cyclic order on G.*

3.5. Lemma. *Let $G_0 \neq \{0\}$. Let $g \in G$, $\emptyset \neq X \subseteq G$ and suppose that $\sup X = g$ holds in $(G; \leqq^0)$. Then $X_1 = X \cap (g + G_0) \neq \emptyset$ and $\sup X_1 = g$ holds in $g + G_0$.*

Proof. By way of contradiction, assume that $X_1 = \emptyset$. Since $G_0 \neq \{0\}$, it has no least element and hence $g + G_0$ has no least element. Thus there is $g' \in g + G_0$ with $g' < g$ in $g + G_0$. Then $x <^0 g'$ holds in $(G; \leqq^0)$ for each $x \in X$, therefore $\sup X \leqq^0 g'$, which is a contradiction. We have verified that $X_1 \neq \emptyset$. If $x_1 \in X_1$ and $x \in X \setminus X_1$, then $x < x_1$. Thus $\sup X = \sup X_1$ and hence $\sup X_1 = g$ holds in $(G; \leqq^0)$.

3.6. Lemma. *Let $G_0 \neq \{0\}$ and let $\emptyset \neq X \subseteq G$. Then the following conditions are equivalent:*

(i) *X is a closed subset of G.*

(ii) *If $g \in G$, $X_1 = X \cap (g + G_0) \neq \emptyset$, then X_1 is a closed subset of the linearly ordered set $g + G_0$.*

Proof. This is a consequence of 3.6.

3.7. Lemma. *$H_0 \neq \{0\}$ if and only if $G_0 \neq \{0\}$.*

Proof. Let $G_0 \neq \{0\}$. Since G_0 is a linearly ordered subgroup of H and H_0 is a largest linearly ordered subgroup of H we obtain that $H_0 \supseteq G_0$, thus $H_0 \neq \{0\}$.

Conversely, suppose that $H_0 \neq \{0\}$. By way of contradiction, assume that $G_0 = \{0\}$. Hence

$$H \neq G. \tag{1}$$

Thus according to the definition of H, the set G cannot be closed in H. Therefore in view of 3.6 (applied for H and G instead of G and X) there is $h \in H$, $\emptyset \neq X_1 \subseteq$ $\subseteq (h + H_0) \cap G$ such that $\sup X_1 = h$ or $\inf X_1 = h$ holds in $h + H_0$, and $h \notin G$. Let us consider the first case.

Assume that $\operatorname{card} X_1 > 1$. Hence we can choose distinct elements x and x' in X_1. Then $0 \neq x - x' \in H_0$ and, at the same time, $x - x' \in G$. This yields that $x - x' \in$ $\in G_0$, which is a contradiction. Therefore $\operatorname{card} X_1 = 1$ and hence $h \in G$. The case $h = \inf X$ in $h + H_0$ can be treated analogously. Thus G is closed in H and so we have arrived at a contradiction.

3.8. Lemma. Let $G_0 = \{0\}$. *Then there is an isomorphism φ of H' onto H such that $\varphi(g) = g$ for each $g \in G$.*

Proof. In view of 3.7 we have $H_0 = \{0\}$. According to [2] (Lemma 1.3 and Theorem 1.1) the cyclically ordered group H is isomorphic to a subgroup of K. Thus Theorem 7.5, [2] yields that some of the following conditions is valid:

(i) H is finite.

(ii) H is isomorphic to K.

First, suppose that (i) holds. Then G is finite. Now, since G c-generates H, we must have $G = H$. Hence G is complete and thus $G = H'$. Therefore $H = H'$.

Next, suppose that (ii) is valid. Because G c-generates H, the cyclically ordered group G must be infinite. Also, G is isomorphic to a subgroup of K. From [2], Theorem 7.3 we get that H' is isomorphic to K.

Hence there is an isomorphicm φ of H' onto H. We can obviously choose φ in such a way that all elements of G remain fixed.

Now suppose that $G_0 \neq \{0\}$; thus $H_0 \neq \{0\}$.

3.9. Lemma. *Let $h \in H$. There exists $g \in G$ such that $g \in h + H_0$.*

Proof. By way of contradiction, assume that there exists $h \in H$ such that $(h + H_0) \cap G = \emptyset$. Put

$$H_1 = \{h_1 \in H: (h_1 + H_0) \cap G \neq \emptyset\} .$$

Then $H_1 \neq H$. It is obvious that H_1 is a subgroup of H and $G \subseteq H_1$.

Next, 3.6 yields that H_1 is a closed subset of H. Thus G does not c-generate H, which is a contradiction.

Let T_1 be the system of all elements $h \in H$ having the property that $h = \bigvee_{i \in I} x_i$ in $h + H_0$ for some subset $\{x_i\}_{i \in I}$ of G. Next, let T_2 have the dual meaning and $T = T_1 \cup T_2$.

Since each $h + H_0$ is linearly ordered, by applying the same method as in [10], 2.1–2.7, and in view of 2.2 we obtain:

3.10. Lemma. *T is a closed subgroup of H and $G \subseteq T$; thus $T = H$. Moreover, $T_1 = T_2$.*

3.11. Lemma. *Let $h \in H_0$. There are subsets $\{g_i\}_{i \in I}$ and $\{g_j\}_{j \in J}$ of G_0 such that $\bigvee_{i \in I} g_i = h = \bigwedge_{j \in J} g_j$.*

Proof. Without loss of generality we may assume that $0 < h$ is valid in H_0. According to 3.10 there are subsets $\{g_i\}_{i \in I}$ and $\{g_j\}_{j \in J}$ of $H_0 \cap G$ such that $\bigvee_{i \in I} g_i = h = \bigwedge_{j \in J} g_j$. It is clear that all elements of $H_0 \cap G$ belong to G_0.

3.12. Lemma. $H_0 = m(G_0)$.

Proof. From the fact that H is a complete cyclically ordered group it follows that $m(H_0) = H_0$. Thus according to 3.11, $H_0 = m(G_0)$.

3.13. Lemma. *There is an isomorphism φ of H' onto H_0 such that $\varphi(g) = g$ for each $g \in G$.*

Proof. This is a consequence of 3.9, 3.12 and 2.15.

Now, 3.8 and 3.13 yield:

3.14. Theorem. *Let G be a cyclically ordered group. Next, let H' be a completion of G and $H \in \mathscr{C}(G)$. Then there is an isomorphism φ of H' onto H such that $\varphi(g) = g$ for each $g \in G$.*

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