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CONGRUENCE PROPERTIES OF DISTRIBUTIVE DOUBLE p-ALGEBRAS

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1. INTRODUCTION

Congruence relations and their properties on distributive p-algebras (alias distributive lattices with pseudocomplementation) are, generally speaking, well understood. However, this is not the case for distributive double p-algebras even though there are numerous papers on them; see, for example, [1]–[4], [8]–[12], and [15]–[17]. An exception to this rule are the congruence regular double p-algebras for which various characterizations have been given by J. C. Varlet [15] and T. Katriňák [9]. Amongst these, two in particular are pertinent to this work. One identifies the congruence regular double p-algebras with those that are distributive and whose determination congruence is the equality relation whereas the other identifies them with those that are distributive and whose poset of prime ideals contains no 3-element chain. Congruence regular double p-algebras play an important role in the general theory of double p-algebras. In fact, they form a variety and a satisfactory theory concerning their congruences already exists. In particular, it is known that congruence regular double p-algebras are congruence permutable and have the principal join property (meaning that all compact congruences are principal).

In § 3, we show that a distributive double p-algebra is congruence permutable iff every class of its determination congruence, regarded as a distributive lattice, is congruence permutable; in other words, relatively complemented. It is then shown that, should a double p-algebra have a non-empty core, every class of the determination congruence is congruence permutable iff the core is. The more general result is then applied to show that every distributive double p-algebra can be embedded into one that is congruence permutable. From another stance, we show that congruence permutable distributive double p-algebras are precisely those whose poset of prime ideals contains no 4-element chain. This result leads us quite naturally to pose the question: Which bounded distributive lattices have the property that their poset of prime ideals contains no n-element chain? Not only does our answer provide us with an intrinsic characterization of congruence permutable distributive double p-algebras but also turns out to be particularly relevant in § 4.

Two characterizations of distributive double p-algebras having the principal join property are obtained in § 4. With this in view, § 4 begins with two characterizations of distributive lattices having the principal join property. The characterizations of distributive double p-algebras having the principal join property then follow. The first identifies such algebras with those whose determination congruence classes, regarded as lattices, have the principal join property. Again, should an algebra have a non-empty core, it is shown that every class of the determination congruence has the P.J.P. iff the core does. The second identifies them as those algebras whose poset of prime ideals contains no 5-element chain. The latter result is then applied to show that a variety of distributive double p-algebras has the principal join property (meaning that every algebra in the variety has the property) iff it is congruence regular. Consequently, a variety of distributive double p-algebras is congruence permutable iff it too is congruence regular.

It is a pleasure to acknowledge conversations with J. Sichler regarding this paper.

2. PRELIMINARIES

An algebra $(L; \vee, \wedge, {}^*, {}^+, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ is called a (*distributive*) *double p-algebra* if $(L; \vee, \wedge, 0, 1)$ is a bounded (distributive) lattice in which, for any $a \in L$, a^* is characterized by $a \wedge x = 0$ iff $x \leqq a^*$ and a^+ is characterized dually.

For the standard rules of computation in double p-algebras the reader is referred to [1]. The principal ones necessary for an understanding of this paper are: $a \leqq a^{**} \leqq a^{*+}$, $a = a^{**} \wedge (a \vee a^*)$, $(a \vee b)^* = a^* \wedge b^*$, $(a \wedge b)^{**} = a^{**} \wedge b^{**}$, and their duals.

If, in any double p-algebra L, we write $D^*(L) = \{x \in L: x^* = 0\}$, then $D^*(L)$ is a filter in L and $D^*(L) = \{x \vee x^*: x \in L\}$. The set $D^+(L)$ is defined dually, is an ideal, and $D^+(L) = \{x \wedge x^+: x \in L\}$. The *core of* L is the set $K = D^*(L) \cap D^+(L)$.

A filter F in a double p-algebra is said to be *normal* if $f \in F$ implies $f^{+*} \in F$. If F is a filter in a lattice L or double p-algebra L, then $\Theta(F)$ will denote the smallest lattice or double p-algebra congruence on L collapsing F. In the event that L is a double p-algebra, the corresponding lattice congruence on the lattice reduct of L will be denoted by $\Theta_{\mathrm{lat}}(F)$. If L is a lattice or double p-algebra, then $\theta(a, b)$ will denote the principal lattice or double p-algebra congruence on L collapsing the pair $a, b \in L$. In the event that L is a double p-algebra, S is any sublattice of the lattice reduct, and $a, b \in S$, then $\theta_{\mathrm{lat}S}(a, b)$ will denote the principal lattice congruence on S collapsing the pair a, b.

The relation Φ defined on a double p-algebra L by $a \equiv b(\Phi)$ iff $a^* = b^*$ and $a^+ = b^+$ is a congruence, called *the determination congruence of* L, the congruence classes of which will be referred to as the *determination classes of* L. For example, if the core of L is non-empty, then it is a determination class. For any algebra A

(be it a lattice or double p-algebra), subalgebra S of A, and congruences θ, ψ of A, $\theta \upharpoonright S$ will denote the restriction of θ to S, $[a]\,\theta$ will denote the class of θ containing $a \in L$ and $\theta \circ \psi$ will denote the relational product of θ and ψ. An algebra A is *congruence permutable* if $\theta \circ \psi = \psi \circ \theta$, for all congruences θ, ψ on A, and *congruence regular* if $\theta = \psi$ whenever θ and ψ have a class in common. For all other unexplained notation or terminology the reader is referred to [5] or [7].

3. CONGRUENCE PERMUTABILITY

We begin with a useful result which holds for any (universal) algebra.

Lemma 3.1. *An algebra A is congruence permutable iff every pair of principal congruences on A permute.*

Proof. Suppose that every pair of principal congruences on A permute. Let θ and ψ be arbitrary congruences on A, $x, y \in A$, and $x \equiv y(\theta \circ \psi)$. Then $x \equiv t(\theta)$ and $t \equiv y(\psi)$, for some $t \in A$. Clearly, $x \equiv y(\theta(x, t) \circ \theta(t, y))$ and, therefore, $x \equiv$ $\equiv y(\theta(t, y) \circ \theta(x, t))$. Since $\theta(x, t) \leqq \theta$ and $\theta(t, y) \leqq \psi$, $x \equiv y(\psi \circ \theta)$. □

Lemma 3.2. *If a distributive double p-algebra L has permuting congruences then every determination class of L is relatively complemented.*

Proof. It is enough to show that every determination class of L has permuting principal (lattice) congruences. Let C be a determination class of L containing elements a, b with $a \leqq b$. Since distributive lattices have equationally definable principal congruences, $\theta_{\mathrm{lat}C}(a, b) = \theta_{\mathrm{lat}L}(a, b) \upharpoonright C$. Moreover, $\theta_{\mathrm{lat}L}(a, b)$ preserves * and $^+$, since $\theta_{\mathrm{lat}L}(a, b) \leqq \Phi$, and therefore $\theta_{\mathrm{lat}L}(a, b) = \theta(a, b)$. Thus, $\theta_{\mathrm{lat}C}(a, b) =$ $= \theta(a, b) \upharpoonright C$. Next, observe that, for $a, b, c, d \in C$ satisfying $a \leqq b$ and $c \leqq d$,

$$\theta(a, b) \upharpoonright C \circ \theta(c, d) \upharpoonright C = (\theta(a, b) \circ \theta(c, d)) \upharpoonright C\,.$$

Indeed, if $x, y \in C$, $x \equiv t(\theta(a, b))$ and $t \equiv y(\theta(c, d))$, for some $t \in L$, then, since $\theta(a, b) \leqq \Phi$, $t \in [x]\,\Phi = C$. Therefore, $x \equiv y(\theta(a, b) \upharpoonright C \circ \theta(c, d) \upharpoonright C)$. Thus,

$$(\theta(a, b) \circ \theta(c, d)) \upharpoonright C \subseteq \theta(a, b) \upharpoonright C \circ \theta(c, d) \upharpoonright C\,.$$

The reverse inclusion is obvious. We conclude that

$$\theta_{\mathrm{lat}C}(a, b) \circ \theta_{\mathrm{lat}C}(c, d) = (\theta(a, b) \circ \theta(c, d)) \upharpoonright C$$

and so, by Lemma 3.1, the congruence permutability of C follows from that of L. □

Lemma 3.6. *If every determination class of a distributive double p-algebra is relatively complemented, then any pair of congruences of L below Φ permute.*

Proof. Suppose that θ and ψ are congruences on L and $\theta, \psi \leqq \Phi$. If $x, t, y \in L$, $x \equiv t(\theta)$ and $t \equiv y(\psi)$, then x, t, y all belong to the same determination class C of L and so $x \equiv y(\theta \upharpoonright C \circ \psi \upharpoonright C)$. It follows that $x \equiv y(\psi \upharpoonright C \circ \theta \upharpoonright C)$, since C is congruence permutable, and therefore $x \equiv y(\psi \circ \theta)$. Thus, θ and ψ permute. □

Our objective is to show that under the hypothesis of Lemma 3.3 all congruences of L permute.

Lemma 3.4. *Congruences θ and ψ of a lattice L permute iff, for all $x, y \in L$ with $x \leqq y$, the following statements are equivalent:*

(i) *there exists $t \in [x, y]$ with $x \equiv t(\theta)$ and $t \equiv y(\psi)$,*

(ii) *there exists $s \in [x, y]$ with $x \equiv s(\psi)$ and $s \equiv y(\theta)$.*

Proof. If θ and ψ permute then the equivalence is easily verified. Suppose now that the statements are equivalent, $a \equiv b(\theta)$ and $b \equiv c(\psi)$. Then $a \equiv a \vee b(\theta)$, $a \vee b \equiv a \vee b \vee c(\psi)$, $c \equiv b \vee c(\psi)$, and $b \vee c \equiv a \vee b \vee c(\theta)$, so that there exist $e, f \in L$ with $a \leqq e \leqq a \vee b \vee c$, $c \leqq f \leqq a \vee b \vee c$, $a \equiv e(\psi)$, $e \equiv a \vee b \vee \vee c(\theta)$, $c \equiv f(\theta)$, and $f \equiv a \vee b \vee c(\psi)$. Clearly, $a \equiv e(\psi)$, $e = e \wedge (a \vee b \vee \vee c) \equiv e \wedge f(\psi)$, $c \equiv f(\theta)$, and $f = f \wedge (a \vee b \vee c) \equiv f \wedge e(\theta)$. Thus, $a \equiv d(\psi)$ and $d \equiv c(\theta)$ when $d = e \wedge f$. □

It is known (see [4]) that if L is a distributive double p-algebra and F is a normal filter of L then $\Theta(F) = \Theta_{\mathrm{lat}L}(F)$. Furthermore, if $a \in L$ and elements $a^{n(+*)}$ are defined in L inductively by $a^{0(+*)} = a$ and $a^{(k+1)(+*)} = a^{k(+*)+*}$, for $k \geqq 0$, then $N(a)$, the principal normal filter of L generated by a, is given by

$$N(a) = \{x \in L\colon x \geqq a^{n(+*)}, \text{ for some } n < \omega\}.$$

Consequently, $\theta(a, 1) = \Theta(N(a))$ and, therefore, $x \equiv y(\theta(a, 1))$ iff $x \wedge a^{n(+*)} = y \wedge a^{n(+*)}$, for some $n < \omega$.

Lemma 3.5. *If L is a distributive double p-algebra then, for any $a, b, c \in L$, $\theta(a, 1)$ permutes with $\theta_{\mathrm{lat}L}(b, 1)$ and $\theta_{\mathrm{lat}L}(0, c)$.*

Proof. Suppose that $x, y, t \in L$, $x \equiv t(\theta(a, 1))$ and $t \equiv y(\theta_{\mathrm{lat}L}(b, 1))$. Then $x \wedge a^{n(+*)} = t \wedge a^{n(+*)}$, for some $n < \omega$, and so $x \equiv t(\theta_{\mathrm{lat}L}(a^{n(+*)}, 1))$. Now $\theta_{\mathrm{lat}L}(a^{n(+*)}, 1)$ and $\theta_{\mathrm{lat}L}(b, 1)$ are both congruences of the form $\Theta_{\mathrm{lat}L}(F)$, for some filter F of L, and it is well known that such congruences permute on any distributive lattice L. Therefore, there exists $s \in L$ such that $x \equiv s(\theta_{\mathrm{lat}L}(b, 1))$ and $s \equiv y(\theta_{\mathrm{lat}L}(a^{n(+*)}, 1))$. Clearly, $\theta_{\mathrm{lat}L}(a^{n(+*)}, 1) \subseteq \theta(a, 1)$ and so $x \equiv y(\theta_{\mathrm{lat}L}(b, 1) \circ \theta(a, 1))$. That $\theta(a, 1)$ permutes with $\theta_{\mathrm{lat}L}(0, c)$ can be proved by a dual argument using the observation that, for any $a \in L$, $\theta(a, 1) = \theta(0, a^+)$. □

With the aid of Lemmas 3.4 and 3.5 we prove:

Lemma 3.6. *If L is a distributive double p-algebra then, for any $a, b, c \in L$ with $b \equiv c(\Phi)$, $\theta(a, 1)$ permutes with $\theta(b, c)$.*

Proof. Without loss of generality, we may assume that $b \leqq c$. Suppose first that $t \in [x, y]$, $x \equiv t(\theta(a, 1))$ and $t \equiv y(\theta(b, c))$. Then $x \wedge a^{n(+*)} = t \wedge a^{n(+*)}$, for some $n < \omega$, and so $x \wedge a^{n(+*)} \equiv y \wedge a^{n(+*)}(\theta(b, c))$. It follows, on defining $s = (y \wedge a^{n(+*)}) \vee x$ that $s \in [x, y]$, $x \equiv s(\theta(b, c))$ and $s \equiv y(\theta(a, 1))$, since $a^{n(+*)} \equiv 1(\theta(a, 1))$. Thus, one half of the equivalence in Lemma 3.4 is established. For the other half, suppose that $s \in [x, y]$, $x \equiv s(\theta(b, c))$ and $s \equiv y(\theta(a, 1))$. Recall that $\theta(b, c) = \theta_{\mathrm{lat}L}(b, c)$, since $b \equiv c(\Phi)$, and $\theta_{\mathrm{lat}L}(b, c) = \theta_{\mathrm{lat}L}(b, 1) \cap \theta_{\mathrm{lat}L}(0, c)$. Now, by Lemmas 3.4 and 3.5, there exist $p, q \in [x, y]$ such that $x \equiv p(\theta(a, 1))$, $p \equiv$ $\equiv y(\theta_{\mathrm{lat}L}(b, 1))$, $x \equiv q(\theta(a, 1))$, and $q \equiv y(\theta_{\mathrm{lat}L}(0, c))$. Let $t = p \vee q$. Then $t \in$ $\in [x, y]$, $x \equiv t(\theta(a, 1))$, and $t \equiv y(\theta(b, c))$. □

Corollary 3.7. *If F is a normal filter of a distributive double p-algebra L, then $\Theta(F)$ permutes with every congruence of L below Φ.*

Proof. Let $a, b, c \in L$, $a \equiv b(\Theta(F))$ and $b \equiv c(\theta)$, where θ is a congruence of L below Φ. Then $a \wedge f = b \wedge f$, for some $f \in F$, so that $a \equiv b(\Theta(f, 1))$, $b \equiv c(\theta(b, c))$, and $b \equiv c(\Phi)$. By Lemma 3.6, $a \equiv d(\theta(b, c))$ and $d \equiv c(\theta(f, 1))$, for some $d \in L$. Therefore, $a \equiv d(\theta)$ and $d \equiv c(\Theta(F))$, since $\theta(b, c) \leqq \theta$ and $\theta(f, 1) \leqq \Theta(F)$. □

In [1] it was shown that any congruence θ on a double p-algebra L can be represented as the join of two simpler congruences; one of which is below the determination congruence Φ of L while the other is of the form $\Theta(F)$, for some normal filter F of L. Specifically, we have $\theta = \Theta(\mathrm{Cok}\,\theta) \vee (\theta \wedge \Phi)$, where $\mathrm{Cok}\,\theta = [1]\,\theta$ which is a normal filter of L.

Combining this with Corollary 3.7, we obtain:

Lemma 3.8. *Any congruence θ on a distributive double p-algebra L can be represented in the form*

$$\theta = \Theta_{\mathrm{lat}L}(\mathrm{Cok}\,\theta) \circ (\theta \wedge \Phi)\,. \quad \square$$

A direct consequence of Lemmas 3.2, 3.3, 3.8, and Corollary 3.7 is:

Theorem 3.9. *A distributive double p-algebra is congruence permutable iff its determination classes are all relatively complemented.* □

Lemma 3.10. *Let L be a distributive double p-algebra with non-empty core K If K is relatively complemented, then any pair of congruences below Φ permute.*

Proof. Let $\theta, \psi \leqq \Phi$. Suppose that $x, t, y \in L$, $x \leqq t \leqq y$, $x \equiv t(\theta)$, and $t \equiv y(\psi)$. We can always choose $k \in K$ satisfying $k \geqq y \wedge y^+$; indeed, if $k' \in K$ is arbitrary, then $k = (y \wedge y^+) \vee k'$ will suffice. Now, let $l = (x \vee x^*) \wedge k$, $m = (t \vee x^*) \wedge k$, and $n = (y \vee y^*) \wedge k$. Then $l \leqq m \leqq n$, $\{l, m, n\} \subseteq K$, $l \equiv m(\theta)$, and $m \equiv n(\psi)$, since $x^* = y^*$. Because $\theta \restriction K$ permutes with $\psi \restriction K$, there exists $m' \in K$ such that $l \leqq m' \leqq n$, $l \equiv m'(\psi)$, and $m' \equiv n(\theta)$, by Lemma 3.4. It follows, on using the identity $a = a^{**} \wedge (a \vee a^*)$, that

$$x = x \vee (x \wedge k) = x \vee (x^{**} \wedge l) \equiv x \vee (x^{**} \wedge m')(\psi)$$

and $x \vee (x^{**} \wedge m') \equiv x \vee (y^{**} \wedge n)(\theta)$, since $x^{**} = y^{**}$, $= x \vee (y \wedge k) =$ $= y \wedge (x \vee k)$, since $x \leqq y$. On writing $s = x \vee (x^{**} \wedge m')$, we have shown, thus far, that $x \leqq s \leqq y \wedge (x \vee k)$, $x \equiv s(\psi)$, and $s \equiv y \wedge (x \vee k)(\theta)$. Consequently, $x \leqq s \leqq y$, and $s = x^{++} \vee (s \wedge x^+) \equiv x^{++} \vee (y \wedge (x \vee k) \wedge x^+)(\theta) = y^{++} \vee$

$\vee ((y \wedge y^+) \wedge (x \vee k))$, since $x^+ = y^+$, $= y^{++} \vee (y \wedge y^+)$, since $y \wedge y^+ \leqq k$, $= y$, so $s \equiv y(\theta)$. On interchanging the roles of θ and ψ in the above and applying Lemma 3.4 we see that θ and ψ permute. □

The conjunction of Lemmas 3.2, 3.8, 3.10, and Corollary 3.7 yields:

Corollary 3.11. *Let L be a distributive double p-algebra with non-empty core K. Then L is congruence permutable iff K (as a distributive lattice) is congruence permutable.* □

A consequence of Theorem 3.9 is the following.

Corollary 3.12. *Every distributive double p-algebra can be embedded into a congruence permutable distributive double p-algebra.*

Proof. It is known that any determination class of any subdirectly irreducible distributive double p-algebra contains at most two elements (see [12]) and so any subdirectly irreducible distributive double p-algebra is congruence permutable. Therefore, it is enough, by the subdirect product theorem, to show that the class of all congruence permutable distributive double p-algebras is closed under arbitrary products. This, though, is clear. Indeed, if $(L_i: i \in I)$ is any family of congruence permutable distributive double p-algebras, Φ_i is the determination congruence on L_i for every $i \in I$, Φ is the determination congruence on $L = \Pi(L_i: i \in I)$, and $a \in L$, then it is easy to see that $[a]\,\Phi = \Pi([a(i)]\,\Phi_i: i \in I)$. Therefore, $[a]\,\Phi$ is relatively complemented, since each $[a(i)]\,\Phi_i$ is relatively complemented, by Theorem 3.9, and the class of relatively complemented distributive lattices is closed under arbitrary products. □

The imposition of congruence permutability, like congruence regularity, on a distributive double p-algebra severely restricts the height of its poset of prime ideals. Our next objective is to characterize congruence permutable distributive double p-algebras in terms of the height of their poset of prime ideals. With this in mind, recall that any prime ideal I of a sublattice S of a distributive lattice L can be extended to L in the sense that there is a prime ideal I' of L satisfying $I' \cap S = I$.

Lemma 3.13. *Let C be a convex sublattice of a distributive lattice L and let I and J be prime ideals of C. If I', J' are extensions of I, J to L, respectively, and $I \subset J$, then $I' \subset J'$.*

Proof. Let $a \in I$ and $b \in C \setminus J$. If $x \in I' \setminus J'$, then $(a \vee x) \wedge b \in I \setminus J$, which is absurd. Therefore $I' \subseteq J'$ and, obviously, $I' \neq J'$. □

Our next lemma is crucial and will be required again in § 4.

Lemma 3.14. *Let L be a distributive double p-algebra and let n be an integer $\geqq 1$. Then the poset of prime ideals of L contains an $(n + 2)$-element chain iff the poset of prime ideals of some determination class of L contains an n-element chain.*

Proof. Suppose that $I_0 \subset \ldots \subset I_{n+1}$ is an $(n + 2)$-element chain of prime ideals of L. Let $a \in I_1 \setminus I_0$, $b \in I_{n+1} \setminus I_n$, $d = a \vee a^*$, and $e = b \wedge b^+$. Observe that $a^* \in I_0$ and so $d \in I_1 \setminus I_0$. Moreover, $b^+ \notin I_{n+1}$ and so $e \in I_{n+1} \setminus I_n$. Now consider the interval $I = [d, d \vee e]$ of L. Clearly, $I_1 \cap I \subset \ldots \subset I_n \cap I$ is an n-element chain of prime ideals of I. In addition, $d \equiv d \vee e(\Phi)$, since $d, d \vee e \in D^*(L)$ and $e \in D^+(L)$. Thus, I is a convex sublattice of some determination class of L and so there is an n-element chain in the poset of prime ideals of this class by Lemma 3.13.

For the converse, suppose that C is a determination class of L and let $I_1 \subset \ldots \subset I_n$ be an n-element chain in its poset of prime ideals. Lemma 3.13 guarantees the existence of an n-element chain $I'_1 \subset \ldots \subset I'_n$ in the poset of prime ideals of L. With the intention of expanding this to an $(n + 2)$-element chain, let $a \in I_1$ and $b \notin I_n$. Observe that $I'_n \vee (b] \neq L$, since otherwise we must have $b \vee i = 1$, for some $i \in I'_n$, so that $i \geqq b^+ = a^+$ and, therefore, $1 = a \vee a^+ \in I'_n$. Consequently, there is a prime ideal I'_{n+1} of L properly containing I'_n. A complimentary argument assures the existence of a prime ideal I'_0 of L properly contained in I'_1. Thus, there is an $(n + 2)$-element chain in the poset of prime ideals of L. □

On recalling that a distributive lattice is relatively complemented iff its poset of prime ideals is an antichain, we infer from Theorem 3.9 and Lemma 3.14 that

Theorem 3.15. *A distributive double p-algebra is congruence permutable iff there is no 4-element chain in its poset of prime ideals.* □

Theorem 3.15 tells us that the congruence permutability of a distributive double p-algebra depends solely on the structure of the poset of prime ideals of its lattice reduct. Naturally, further insight would be gained if we had an intrinsic characterization of those bounded distributive lattices whose poset of prime ideals contains no 4-element chain. We intend being a little more ambitious by answering the question: Which bounded distributive lattices have no n-element chain in their poset of prime ideals? We begin with

Lemma 3.16. *Let I be an ideal of a distributive lattice L and let $a \in L \setminus I$. Let I_a denote the ideal $\{x \in L: x \wedge a \in I\}$ and $I_a^+ = (a] \vee I_a$. If P is a prime ideal such that $I_a^+ \subseteq P$ then there exists a prime ideal Q such that $I \subseteq Q \subseteq P$ and $a \notin Q$ (so, in particular, $Q \subset P$).*

Proof. Consider the filter $L \setminus P$ and let $F = [a) \vee (L \setminus P)$. We claim that $F \cap I_a = \emptyset$. Indeed, if not, then there is an $x \in L \setminus P$ such that $a \wedge x \in I_a$; in other words, $a \wedge x \in I$ and, therefore, $x \in I_a \subseteq P$ which is absurd. Hence there exists a prime ideal $Q \supseteq I_a \supseteq I$ satisfying $Q \cap F = \emptyset$. Clearly, $Q \subseteq P$ and $a \notin Q$. □

In the present context, we are interested in the case where $n = 3$ in the following

Theorem 3.17. *Let L be a bounded distributive lattice. There is no $(n + 1)$-element chain in the poset of prime ideals of L iff, for any $a_0, \ldots, a_{n-1} \in L$ with $a_0 \leqq \ldots \leqq a_{n-1}$, there exist $a'_0, \ldots, a'_{n-1} \in L$ such that*

$$a_0 \wedge a'_0 = 0\,, \quad a_i \vee a'_i = a_{i+1} \wedge a'_{i+1}\,, \quad \textit{for} \quad 0 \leqq i < n-1\,, \quad \textit{and}$$
$$a_{n-1} \vee a'_{n-1} = 1\,.$$

Proof. Suppose that the condition is satisfied but, contrary to the statement, there exist prime ideals $P_0 \subset \ldots \subset P_n$. Choose $a_0, \ldots, a_{n-1} \in L$ such that $a_0 < \ldots \ldots < a_{n-1}$ and $a_i \in P_{i+1} \setminus P_i$, for $i \in \{0, \ldots, n-1\}$. Then $a'_0 \in P_0$, since $a_0 \wedge a'_0 = = 0 \in P_0$, and so $a_0 \vee a'_0 \in P_1$. Thus, $a_1 \wedge a'_1 \in P_1$ so $a'_1 \in P_1$ and $a_1 \vee a'_1 \in P_2$. Repetition of this process yields the contradiction $1 = a_{n-1} \vee a'_{n-1} \in P_n$.

Suppose, now, that there is no $(n+1)$-element chain in the poset of prime ideals of L and let $0 \leqq a_0 \leqq \ldots \leqq a_{n-1} \leqq 1$. In the event that $0 = a_0$ or $a_i = a_{i+1}$, for some i with $0 \leqq i < n-1$, or $a_{n-1} = 1$, it is possible to find suitable elements a'_i, for $0 \leqq i \leqq n-1$, in the set $\{a_i\colon 0 \leqq i \leqq n-1\} \cup \{0, 1\}$. Thus, without loss of generality, we may assume that $0 < a_0 < \ldots < a_{n-1} < 1$.

For the sake of convenience we will write $a_{-1} = 0$, $a_n = 1$, and proceed by defining some ideals in L. Let $I_{a_{-1}} = \{0\}$, $I_{a_0} = \{x \in L\colon a_0 \wedge x \in I_{a_{-1}}\}$ and $I^+_{a_0} = (a_0] \vee I_{a_0}$. Now, for $i \in \{0, \ldots, n\}$, define $I_{a_{i+1}} = \{x \in L\colon a_{i+1} \wedge x \in I^+_{a_i}\}$ and $I^+_{a_{i+1}} = (a_{i+1}] \vee \vee I_{a_{i+1}}$. Observe that $I_{a_i} \subseteq I^+_{a_i} \subseteq I_{a_{i+1}}$, for $i \in \{0, \ldots, n-1\}$.

We claim that $a_{i+1} \in I^+_{a_i}$, for some $i \in \{0, \ldots, n-1\}$. Suppose, to the contrary, that $a_{i+1} \notin I^+_{a_i}$, for any $i \in \{0, \ldots, n-1\}$. Then, in particular, $1 = a_n \notin I^+_{a_{n-1}}$ and so there is a prime ideal P_{n-1} such that $P_{n-1} \supseteq I^+_{a_{n-1}}$. By Lemma 3.16, there exists a prime ideal P_{n-2} such that $P_{n-1} \supset P_{n-2} \supseteq I^+_{a_{n-2}}$. Continuing in this manner, we produce an $(n+1)$-element chain $P_{n-1} \supset P_{n-2} \supset \ldots \supset P_0 \supset P_{-1} \supseteq I^+_{a_{-1}} = \{0\}$ in the poset of prime ideals of L, contrary to assumption. Thus, $a_{i+1} \in I^+_{a_i}$, for some $i \in \{0, \ldots, n-1\}$.

It follows from the above that there is an $i \in \{0, \ldots, n-1\}$ and an $x_i \in I_{a_i}$ such that $a_{i+1} \leqq a_i \vee x_i$. Moreover, $a_i \wedge x_i \leqq a_{i-1} \vee x_{i-1}$, for some $x_{i-1} \in I_{a_{i-1}}$, since $a_i \wedge x_i \in I^+_{a_{i-1}}$. Proceeding in this fashion, for $j \in \{0, \ldots, i\}$, there exist $x_j, x_{j-1} \in \in L$ such that $a_j \wedge x_j \leqq a_{j-1} \vee x_{j-1}$; in particular, $a_0 \wedge x_0 = 0$. Now, for $j \in \in \{0, \ldots, i\}$, we define $x'_j \in [a_{j-1}, a_{j+1}]$ by $x'_j = (x_j \vee a_{j-1}) \wedge a_{j+1}$ and show that these elements enjoy the same basic properties as the elements x_j. First observe that $a_i \vee x'_i = (x_i \vee a_i) \wedge a_{i+1} = a_{i+1}$, since $a_{i+1} \leqq a_i \vee x_i$, and $a_0 \wedge x'_0 = = a_0 \wedge x_0 = 0$. Next, we verify that $a_{j+1} \wedge x'_{j+1} \leqq a_j \vee x'_j$, for $j \in \{0, \ldots, i-1\}$. Indeed, $a_{j+1} \wedge x'_{j+1} = (x_{j+1} \vee a_j) \wedge a_{j+1} = (a_{j+1} \wedge x_{j+1}) \vee a_j \leqq (a_j \vee x_j) \wedge \wedge a_{j+1} = a_j \vee x'_j$. Now we are in a position to define the elements we seek.

Let

$$a'_j = x'_0\,, \quad \text{for} \quad j = 0\,, \quad x'_j \vee x'_{j-1}\,, \quad \text{for} \quad 0 < j \leqq i\,,$$
$$a_{j+1}\,, \quad \text{for} \quad i+1 \leqq j \leqq n-1\,.$$

First, observe that $a_i \vee a'_i = a_i \vee x'_i \vee x'_{i-1} = a_{i+1} \vee x'_{i-1} = a_{i+1}$ and $a_0 \wedge \wedge a'_0 = a_0 \wedge x'_0 = 0$. It remains only to show that $a_{j+1} \wedge a'_{j+1} = a_j \vee a'_j$. Calculating, we have $a_{j+1} \wedge a'_{j+1} = a_{j+1} \wedge (x'_{j+1} \vee x'_j) = (a_{j+1} \wedge x'_{j+1}) \vee x'_j$ and $a_j \vee a'_j = a_j \vee (x'_j \vee x'_{j-1}) = (a_j \vee x'_j) \vee x'_j \vee x'_{j-1}$. It follows, since $a_{j+1} \wedge$

$\wedge\ x'_{j+1} \leqq a_j \vee x'_j$, that $a_{j+1} \wedge a'_{j+1} \leqq a_j \vee a'_j$ and, since $a_j \leqq a_{j+1}$ and $x'_j \vee \vee\ x'_{j-1} \leqq x'_{j+1} \vee x'_j$, that $a_j \vee a'_j \leqq a_{j+1} \wedge a'_{j+1}$. Thus, the proof is complete. □

Besides giving an intrinsic characterization of congruence permutable distributive double p-algebras, Theorem 3.17 (in the case where $n = 2$) yields a new characterization of the congruence regular double p-algebras and will prove to be illuminating in the next section.

4. THE PRINCIPAL JOIN PROPERTY

We will say that an algebra A (be it a lattice or a double p-algebra) has the *principal join property* (henceforth abbreviated P.J.P.) if the join of any pair of principal congruences on A is again principal; equivalently, if every compact congruence on A is principal. I. Chajda [6] calls such an algebra *congruence principal* and R. W. Quackenbush [14] shows that every congruence principal variety (i.e., variety in which every algebra is congruence principal) can be characterized by a Mal'cev condition.

In this section we give two characterizations of distributive double p-algebras having the P.J.P. The first solution we offer is modulo the solution of the same problem for distributive lattices and so our initial objective is to characterize distributive lattices satisfying the P.J.P. We start with a pair of lemmas whose conjunction reduces the problem to the bounded case.

In the next three lemmas and proofs, L will denote a distributive lattice, $a, b, c, d \in \in L$, $a \leqq b$, and $c \leqq d$.

Lemma 4.1. *$\theta(a, b) \restriction [c, d]$ is a principal congruence on $[c, d]$.*

Proof. We will show that $\theta(a, b) \restriction [c, d] = \theta_{[c,d]}(a', b')$, where $a' = (a \vee c) \wedge d$ and $b' = (b \vee c) \wedge d$. Clearly, $\theta_{[c,d]}(a', b') \subseteq \theta(a, b) \restriction [c, d]$, since $a', b' \in [c, d]$ and $a' \equiv b'(\theta(a, b))$. For the reverse inclusion, suppose that $x, y \in [c, d]$ and $x \equiv y(\theta(a, b))$. Then $x \wedge a = y \wedge a$ and $x \vee b = y \vee b$. Therefore, $x \wedge a' = = (x \wedge a) \vee c = (y \wedge a) \vee c = y \wedge a'$ and $x \vee b' = (x \vee b \vee c) \wedge d = = (y \vee b \vee c) \wedge d = y \vee b'$, from which it follows that $x \equiv y(\theta_{[c,d]}(a', b'))$. □

Lemma 4.2. *Let $\theta = \theta(a, b) \vee \theta(c, d)$ and let I be the interval $[a \wedge c, b \vee d]$ of L. Then θ is a principal congruence on L iff $\theta \restriction I$ is a principal congruence on I.*

Proof. If θ is principal then so is $\theta \restriction I$, by Lemma 4.1. Suppose then that $\theta \restriction I = = \theta_I(e, f)$, for some $e, f \in I$ with $e \leqq f$. Clearly, $\theta(e, f) \leqq \theta$. Moreover, $\theta(a, b) \leqq \leqq \theta(e, f)$, since $a \equiv b(\theta_I(e, f))$ and $\theta_I(e, f) = \theta(e, f) \restriction I$. Similarly, $\theta(c, d) \leqq \theta(e, f)$ and so $\theta \leqq \theta(e, f)$. Thus, $\theta = \theta(e, f)$. □

Lemma 4.3. *L has the P.J.P. iff every interval of L has the P.J.P.*

Proof. If L has the P.J.P. then so does every interval of L, since the P.J.P. is preserved under the formation of homomorphic images.

Suppose that every interval of L has the P.J.P. Let $\theta = \theta(a, b) \vee \theta(c, d)$ and $I = [a \wedge c, b \vee d]$. Then, since I is a convex sublattice of L, $\theta \restriction I = \theta(a, b) \restriction I \vee \vee \theta(c, d) \restriction I = \theta_I(a, b) \vee \theta_I(c, d) = \theta_I(e, f)$, for some $e, f \in L$, and so θ is principal by Lemma 4.2. $\square$

The problem now is to characterize those bounded distributive lattices with the P.J.P. In this connection we prove.

Lemma 4.4. *A bounded distributive lattice L has the P.J.P. iff, for all $p, q \in L$, $\theta(0, p) \vee \theta(q, 1)$ is principal.*

Proof. Suppose that $\theta(0, p) \vee \theta(q, 1)$ is principal, for all $p, q \in L$. Let $\theta = = \theta(a, b) \vee \theta(c, d)$, where $a, b, c, d \in L$ with $a \leqq b$ and $c \leqq d$. Then

$$\begin{aligned} \theta &= [\theta(0, b) \wedge \theta(a, 1)] \vee [\theta(0, d) \wedge \theta(c, 1)] = \\ &= \theta(0, b \vee d) \wedge [\theta(a, 1) \vee \theta(0, d)] \wedge [\theta(0, b) \vee \theta(c, 1)] \wedge \\ &\wedge \theta(a \wedge c, 1), \end{aligned}$$

since the congruence lattice of L is distributive and $\theta(0, x) \vee \theta(0, y) = \theta(0, x \vee y)$ and $\theta(x, 1) \vee \theta(y, 1) = \theta(x \wedge y, 1)$, for any $x, y \in L$. Thus, θ is a meet of finitely many principal congruences and so must be principal, since distributive lattices have the principal intersection property; in other words, the intersection of any two principal congruences is principal. $\square$

Theorem 4.5. *For a bounded distributive lattice L the following are equivalent:*

(i) *L has the P.J.P.,*

(ii) *there is no 3-element chain in the poset of prime ideals of L,*

(iii) *for all $a, b \in L$ with $a \leqq b$, there exist $a', b' \in L$ such that $a \wedge a' = 0$, $a \vee a' = b \wedge b'$, and $b \vee b' = 1$.*

Proof. Suppose that $P \subset Q \subset R$ is a chain of prime ideals of L. Choose $a \in Q \setminus P$ and $b \in R \setminus Q$ with $0 < a < b < 1$. We claim that $\theta = \theta(0, a) \vee \theta(b, 1)$ is not principal. Suppose, to the contrary, that $\theta = \theta(c, d)$, for some $c, d \in L$ with $c < d$. Then $a \equiv 0(\theta(c, d))$ so that $a \wedge c = 0$ and $a \vee d = d$ from which we infer that $c \in P$ and $a \leqq d$. In addition, $b \equiv 1(\theta(c, d))$ so that $b \wedge c = c$ and $b \vee d = 1$ from which we infer that $d \notin R$ and $c \leqq b$. Consequently, $a \vee c \leqq b \wedge d$, $a \vee c \in Q \setminus P$ and $b \wedge d \in R \setminus Q$. Therefore, $a \vee c < b \wedge d$. Now, $a \vee c \equiv b \wedge d(\theta \wedge \theta(a, b))$, since θ collapses $[c, d]$ and $\theta(a, b)$ collapses $[a, b]$. But $\theta \wedge \theta(a, b) = \omega$ (in fact, θ is the complement of $\theta(a, b)$) and we have a contradiction. Thus, (i) implies (ii).

The equivalence of (ii) and (iii) is the case $n = 2$ of Theorem 3.17.

Finally, suppose that (iii) holds. In order to show that (i) holds, it is enough, by Lemma 4.4, to show that $\theta(0, b) \vee \theta(c, 1)$ is principal, for any $b, c \in L$. On applying condition (iii) to the pair $a = c \wedge b$, $c \in L$, we obtain elements $a', c' \in L$ such that $a \wedge a' = 0$, $a \vee a' = c \wedge c'$, and $c \vee c' = 1$. We claim that $\theta(0, b) \vee \theta(c, 1) = = \theta(a', b \vee c')$. With this in mind, note that $b \wedge a' = b \wedge c \wedge a' = a \wedge a' = 0$, since $c \leqq a'$, so that $b \wedge a' = 0 \wedge a'$, and $b \vee (b \vee c') = 0 \vee (b \vee c')$. Therefore,

$b \equiv 0(\theta(a', b \vee c'))$ and so $\theta(0, b) \leqq \theta(a', b \vee c')$. In addition, $\theta(c, 1) \leqq \theta(a', b \vee c')$ since $\theta(a', b \vee c')$ collapses $[c \wedge c', c']$ which is perspective to $[c, 1]$. Thus, $\theta(0, b) \vee \vee \theta(c, 1) \leqq \theta(a', b \vee c')$. For the reverse inclusion, first note that $a' \equiv c \wedge \wedge c'(\theta(0, b))$, since $\theta(0, b)$ collapses $[0, a]$ which is perspective to $[a', c \wedge c']$, $c \wedge c' \equiv c'(\theta(c, 1))$ and $c' \equiv b \vee c'(\theta(0, b))$. It follows from this sequence of congruences that $a' \equiv b \vee c'(\theta(0, b) \vee \theta(c, 1))$ and so our claim is substantiated. □

The following lemma, in conjunction with Lemma 4.3 and Theorem 4.5 facilitates the characterization of distributive (not necessarily bounded) lattices having the P.J.P.

Lemma 4.6. *A distributive lattice L has no n-element chain in its poset of prime ideals iff no interval of L has an n-element chain in its poset of prime ideals.*

Proof. It follows as a consequence of Lemma 3.13 that if L has no n-element chain in its poset of prime ideals then neither does any interval of L.

Suppose now that L has an n-element chain $P_1 \subset \ldots \subset P_n$ in its poset of prime ideals. Choose $a_1 \in P_1$, $a_{n+1} \in L \setminus P_n$, and $a_{i+1} \in P_{i+1} \setminus P_i$ whenever $1 \leqq i \leqq n - 1$. Define $a'_1, \ldots, a'_{n+1} \in L$ by $a'_1 = a_1$ and $a'_{i+1} = a_{i+1} \vee a'_i$ whenever $1 \leqq i \leqq n$. Clearly, $a'_i \in P_1$, $a'_{n+1} \in L \setminus P_n$, $a'_{i+1} \in P_{i+1} \setminus P_i$ whenever $1 \leqq i \leqq n - 1$, and $a'_1 < a'_2 < \ldots < a'_{n+1}$. Let $I = [a'_1, a'_{n+1}]$. Then $P_i \cap I$ is a prime ideal of I, for each i with $1 \leqq i \leqq n$, and $P_1 \cap I \subset \ldots \subset P_n \cap I$. Thus, if no interval of L has an n-element chain in its poset of prime ideals then neither does L. □

Corollary 4.7. *If L is a distributive lattice then the following are equivalent:*

(i) *L has the P.J.P.,*

(ii) *there is no 3-element chain in the poset of prime ideals of L,*

(iii) *for all $a, b, x, y \in L$ with $a \leqq x \leqq y \leqq b$, there exist $x', y' \in L$ such that*

$$x \wedge x' = a\,, \quad x \vee x' = y \wedge y'\,, \quad \text{and} \quad y \vee y' = b\,. \quad \square$$

With this in hand, we begin our investigation of distributive double p-algebras satisfying the P.J.P.

Lemma 4.8. *If a distributive double p-algebra has the P.J.P., then so does every determination class of L.*

Proof. Suppose that L has the P.J.P. Let C be a determination class of L and $a, b, c, d \in C$ with $a \leqq b$ and $c \leqq d$. Then $\theta(a, b) \vee \theta(c, d)$ must be a principal congruence on L which, since $\theta(a, b) \leqq \Phi$ and $\theta(c, d) \leqq \Phi$, must be generated by a pair of elements in some determination class of L. It follows that $\theta = \theta_{\mathrm{lat}L}(a, b) \vee \vee \theta_{\mathrm{lat}L}(c, d)$ is a principal lattice congruence on L, since $\theta(x, y) = \theta_{\mathrm{lat}L}(x, y)$ whenever x, y belong to some determination class of L. By the proof of Lemma 4.2, $\theta = \theta_{\mathrm{lat}L}(e; f)$ for some e, f satisfying $e \leqq f$ and belonging to the interval $I = = [a \wedge c,\ b \vee d]$ of L. Therefore, $\theta_{\mathrm{lat}C}(a, b) \vee \theta_{\mathrm{lat}C}(c, d) = \theta_{\mathrm{lat}L}(a, b) \restriction C \vee \vee \theta_{\mathrm{lat}L}(c, d) \restriction C = (\theta_{\mathrm{lat}L}(a, b) \vee \theta_{\mathrm{lat}L}(c, d)) \restriction C$, since C is a convex sublattice of L, $= (\theta_{\mathrm{lat}L}(e, f)) \restriction C = \theta_{\mathrm{lat}C}(e, f)$, since $e, f \in I \subseteq C$. Thus, C has the P.J.P. □

The following sequence of lemmas will establish the converse.

Lemma 4.9. *A congruence θ on a distributive double p-algebra L is principal iff it is of the form*

$$\theta = \theta(0, a) \vee \theta(b, c),$$

for some $a, b, c \in L$ with $b \leqq c$ and $b \equiv c(\Phi)$.

Proof. We start by showing that, for any $x, y \in L$ with $x \leqq y$,

$$\theta(x, y) = \theta(0, (y \wedge x^*) \vee (x \vee y^+)^+) \vee \theta(x, y \wedge (y \wedge x^*)^* \wedge \wedge (x \vee y^+)).$$

Let θ denote the congruence on the right hand side. Clearly,

$$(y \wedge x^*) \vee (x \vee y^+)^+ \equiv 0(\theta(x, y)) \quad \text{and}$$
$$y \wedge (y \wedge x^*)^* \wedge (x \vee y^+) \equiv x(\theta(x, y))$$

so that $\theta \leqq \theta(x, y)$. Now observe that

$$y \geqq y \wedge (y \wedge x^*)^* \wedge (x \vee y^+) \geqq y \wedge (y \wedge x^*)^* \wedge (x \vee y^+)^{+*} =$$
$$= y \wedge [(y \wedge x^*) \vee (x \vee y^+)^+]^* \equiv y(\theta(0, (y \wedge x^*) \vee (x \vee y^+)^+))$$

and so $y \wedge (y \wedge x^*)^* \wedge (x \vee y^+) \equiv y(\theta(0, (y \wedge x^*) \vee (x \vee y^+)^+))$. It follows that $x \equiv y(\theta)$ and, therefore, $\theta = \theta(x, y)$. Finally, let $z = y \wedge (y \wedge x^*)^* \wedge \wedge (x \vee y^+)$. Note that $x \leqq z$, since $x \leqq (y \wedge x^*)^*$, $x^* \wedge z = 0$, so that $x^* \leqq z^*$, and $x \vee z^+ = 1$, so that $x^+ \leqq z^+$. Consequently, $x \equiv z(\Phi)$.

With the intention of establishing the converse, suppose that $a, b, c \in L$ and $b \leqq c$. We claim that

$$\theta(0, a) \vee \theta(b, c) = \theta(a^* \wedge b, a \vee c).$$

Let $\theta = \theta(0, a) \vee \theta(b, c)$. Then $a^* \wedge b \equiv b(\theta)$, since $a^* \equiv 1(\theta)$, $b \equiv c(\theta)$ and $c \equiv a \vee c(\theta)$, since $a \equiv 0(\theta)$. Therefore, $a^* \wedge b \equiv a \vee c(\theta)$ and, so, $\theta(a^* \wedge b, a \vee c) \leqq \theta$. To prove the reverse inequality, it is enough to show that $\theta(0, a) \leqq \leqq \theta(a^* \wedge b, a \vee c)$. Let $d = (a \vee c) \wedge (a^* \wedge b)^*$. Then $d \equiv 0(\theta(a^* \wedge b, a \vee c))$ and $d \geqq a$, since $(a^* \wedge b)^* \geqq a^{**} \geqq a$, so that $a \equiv 0(\theta(a^* \wedge b, a \vee c))$ and our claim is substantiated. □

Remark. Our proof of the sufficiency part of Lemma 4.9 shows, in fact, that $\theta(0, a) \vee \theta(b, c)$ is principal, for any a, b, c in a (not necessarily distributive) double p-algebra.

Lemma 4.10. *Let L be a distributive double p-algebra and $b, c \in L$ with $b \leqq c$. Then*

(i) *if $b^+ = c^+$, then $\theta(b, c) = \theta(b \wedge c^+, c \wedge c^+)$,*

and

(ii) *if $b^* = c^*$, then $\theta(b, c) = \theta((b^* \wedge x) \vee b, (b^* \wedge x) \vee c)$, for any $x \in L$.*

Proof. Clearly,

$$\begin{aligned}
b &= b \wedge (c \vee c^+) = (b \wedge c) \vee (b \wedge c^+) \equiv \\
&\equiv (b \wedge c) \vee (c \wedge c^+)(\theta(b \wedge c^+, c \wedge c^+)) = \\
&= c \wedge (b \vee c^+) = c \wedge (b \vee b^+) = c
\end{aligned}$$

and so $\theta(b, c) \leqq \theta(b \wedge c^+, c \wedge c^+)$. Therefore (i) holds, since the reverse inclusion is obvious.

We also have

$$\begin{aligned}
c &= c \wedge [(b^* \wedge x) \vee c] \equiv \\
&\equiv c \wedge [(b^* \wedge x) \vee b] (\theta((b^* \wedge x) \vee b, (b^* \wedge x) \vee c)) = \\
&= c \wedge [(c^* \wedge x) \vee b] = c \wedge b = b
\end{aligned}$$

and so $\theta(b, c) \leqq \theta((b^* \wedge x) \vee b, (b^* \wedge x) \vee c)$. Therefore (ii) holds, since the reverse inclusion is again obvious. □

Lemma 4.11. *If every determination class of a distributive double p-algebra L has the P.J.P., then the join of any two principal congruences of L below Φ is principal.*

Proof. Suppose that $b, c, e, f \in L$, $b \leqq c$, $e \leqq f$, $\theta(b, c) \leqq \Phi$, and $\theta(e, f) \leqq \Phi$. We may assume, without loss of generality, that $b, c, e, f \in D^+(L)$, by Lemma 4.10 (i). Let $p = (b^* \wedge e) \vee b$, $q = (b^* \wedge e) \vee c$, $r = (e^* \wedge b) \vee e$, and $s = (e^* \wedge b) \vee f$. Observe that $p, q, r, s \in D^+(L)$ and

$$\theta(b, c) \vee \theta(e, f) = \theta(p, q) \vee \theta(r, s),$$

by Lemma 4.10 (ii). Furthermore, $p \equiv r(\Phi)$. Indeed, $p^+ = r^+ = 1$, $p^{**} = [(b \vee b^*) \wedge (e \vee b)]^{**} = (b \vee b^*)^{**} \wedge (e \vee b)^{**} = (e \vee b)^{**}$ and so $p^{**} = r^{**}$ by symmetry. Thus, $\{p, q, r, s\}$ is contained in some determination class C of L and so $\theta_{\mathrm{lat}C}(p, q) \vee \theta_{\mathrm{lat}C}(r, s) = \theta_{\mathrm{lat}C}(u, v)$, by hypothesis. It is now an easy matter to see that $\theta_{\mathrm{lat}L}(p, q) \vee \theta_{\mathrm{lat}L}(r, s) = \theta_{\mathrm{lat}L}(u, v)$ and, therefore, $\theta(b, c) \vee \theta(e, f)$ is a principal congruence on L. □

Lemmas 4.9 and 4.11 in conjunction with the fact that $\theta(0, x) \vee \theta(0, y) = \theta(0, x \vee y)$, for any x, y in a distributive double p-algebra L, show that if every determination class of L has the P.J.P., then so does L. In summary, we have

Theorem 4.12. *A distributive double p-algebra L has the P.J.P. iff every determination class of L has the P.J.P.* □

Lemma 4.13. *Let L be a distributive double p-algebra with non-empty core K. Then a congruence $\theta \leqq \Phi$ is principal iff it is of the form $\theta = \theta(k, l)$, for some $k, l \in K$ with $k \leqq l$.*

Proof. Suppose $\theta \leqq \Phi$ and $\theta = \theta(b, c)$, for some $b, c \in L$ with $b \leqq c$; in particular, $b \equiv c(\Phi)$. By Lemma 4.10 (i), we may assume that $b, c \in D^+(L)$. Let $k \in K$ be arbitrary. Then, by Lemma 4.10 (ii),

$$\theta(b, c) = \theta((b^* \wedge k) \vee b, (b^* \wedge k) \vee c) =$$
$$= \theta((b \vee b^*) \wedge (k \vee b), (c \vee c^*) \wedge (k \vee c)), \quad \text{since} \quad b^* = c^* .$$

Now, $k \vee b \in K$ so $l = (b \vee b^*) \wedge (k \vee b) \in K$. Similarly, $m = (c \vee c^*) \wedge \wedge (k \vee c) \in K$. Therefore, $\theta = \theta(l, m)$, $l, m \in K$, and $l \leqq m$. □

Lemma 4.14. *Let L be a distributive double p-algebra with non-empty core K. If K has the P.J.P., then the join of any two principal congruences of L below Φ is principal.*

Proof. Let $b, c, e, f \in L$, $b \leqq c$, $e \leqq f$, $\theta(b, c) \leqq \Phi$, and $\theta(e, f) \leqq \Phi$. By Lemma 4.13, we can assume that $b, c, e, f \in K$. Since K is a determination class of L, the claim follows by an argument similar to that in the conclusion of the proof of Lemma 4.11. □

By Lemmas 4.9 and 4.14, we infer:

Corollary 4.15. *Let L be a distributive double p-algebra with non-empty core K. Then L has the P.J.P. iff K has the P.J.P.* □

The conjunction of Lemma 3.14, Corollary 4.7, and Theorem 4.12 yields:

Corollary 4.16. *A distributive double p-algebra has the P.J.P. iff there is no 5-element chain in its poset of prime ideals.* □

Note, in passing, that Theorem 3.17 (with $n = 4$) and Corollary 4.16 provide yet another characterization of distributive double p-algebras that have the P.J.P.

Finally, we will say that a variety of distributive double has the P.J.P. if each of its members has the P.J.P. Recall that a variety of (distributive double p-) algebras is said to be congruence regular (congruence permutable) if each of its members is congruence regular (congruence permutable). Obviously, every congruence regular variety of distributive double p-algebras has the P.J.P. Our objective is to show that the converse is true. The key to success is provided by

Lemma 4.17. *Let L be a distributive double p-algebra that is not regular and let $V(L)$ be the (quasi-)variety of distributive double p-algebras generated by L. For any integer $n \geqq 1$, there is an extension of L in $V(L)$ whose poset of prime ideals has an n-element chain.*

Proof. Let $L_{\Phi}^{[n]} = \{(x_1, \ldots, x_n) \in L^n\colon x_1 \leqq \ldots \leqq x_n,\ x_1 \equiv x_n(\Phi)\}$. It is straightforward to veiify that $L_{\Phi}^{[n]}$ is a subalgebra of L^n and that $x \mapsto (x, \ldots, x)$ is an embedding of L into $L_{\Phi}^{[n]}$.

Now, L is not congruence regular and so we can assume that there is a determination class $[a]\,\Phi$ and an element $b \in [a]\,\Phi$ such that $a < b$. For $i \in \{0, \ldots, n\}$, define elements $s_i \in L_{\Phi}^{[n]}$ by $s_0 = (a, \ldots, a)$ and, for $i \in \{1, \ldots, n\}$, $s_i = (a, \ldots a, b, \ldots b)$ where the first $n - i$ and the last i entries are identical. Let $S = \{s_i\colon 0 \leqq i \leqq n\}$. Observe that $S \subseteq L_{\Phi}^{[n]}$ and $s_{i-1} \leqq s_i$, for $i \in \{1, \ldots, n\}$. Choose a prime ideal I_1 in $L_{\Phi}^{[n]}$ such that $s_0 \in I_1$ and $s_1 \notin I_1$. Now, $s_2 \notin I \vee (s_1]$, since otherwise $s_2 \leqq s_1 \vee i_1$, for some $i_1 = (x_1, \ldots, x_n) \in I_1$, and so, on considering penultimate components, $b \leqq a \vee \vee x_{n-1} \leqq a \vee x_n$ from which it follows that $s_1 \leqq s_0 \vee i_1 \in I_1$; contrary to $s_1 \notin I_1$. Choose a prime ideal I_2 in $L_{\Phi}^{[n]}$ satisfying $I_1 \vee (s_1] \subseteq I_2$ and $s_2 \notin I_2$. Obviously, $I_1 \subset I_2$ and repetition of the process yields an n-element chain $I_1 \subset \ldots \subset I_n$ in the poset of prime ideals of $L_{\Phi}^{[n]}$. □

As a simple consequence, we have

Corollary 4.18. *For a (quasi-)variety V of distributive double p-algebras, the following are equivalent:*

(i) *V has the P.J.P,*
(ii) *V is congruence permutable,*
(iii) *V is congruence regular.* □

The equivalence of (ii) and (iii) in Corollary 4.18 is rightfully attributable to V. Koubek and J. Sichler. In V. Koubek and J. Sichler [13], several characterizations of finitely generated congruence permutable varieties of distributive double p-algebras are given. For example, the main result of [13] states that a finitely generated variety of distributive double p-algebras is iso-universal iff it is not congruence permutable.

References

[1] *R. Beazer,* The determination congruence on double p-algebras , Algebra Universalis *6* (1976), 121—129.
[2] *R. Beazer,* Pseudocomplemented algebras with Boolean congruence lattices, J. Australian Math. Soc. *26* (1978), 163—168.
[3] *R. Beazer,* Congruence pairs of distributive double p-algebras with non-empty core, Houston J. Math. *6* (1980), 443—454.
[4] *R. Beazer,* Congruence uniform algebras with pseudocomplementation, Studia Sci. Math. Hungar. *20* (1985), 43—48
[5] *S. Burris* and *H. P. Sankappanavar,* A Course in Universal Algebra, Graduate Texts in Mathematics, Springer-Verlag, 1981.
[6] *I. Chajda,* A Mal'cev condition for congruence principal permutable varieties, Algebra Universalis *19* (1984), 337—340.
[7] *G. Grätzer,* General Lattice Theory, Birkhäuser Verlag, Basel and Stuttgart, 1978.
[8] *T. Hecht* and *T. Katriňák,* Principal congruences of p-algebras and double p-algebras, Proc. Amer. Math. Soc. *58* (1976), 25—31.
[9] *T. Katriňák,* The structure of distributive double p-algebras. Regularity and congruences, Algebra Universalis *3* (1973), 238—246.
[10] *T. Katriňák,* Congruence extension property for distributive double p-algebras, Algebra Universalis *4* (1974), 273—276.
[11] *T. Katriňák,* Congruence pairs on p-algebras with a modular frame, Algebra Universalis *8* (1978), 205—220.

[12] *T. Katriňák,* Subdirectly irreducible distributive double p-algebras, Algebra Universalis *10* (1980), 195—219.
[13] *V. Koubek* and *J. Sichler,* Universal varieties of distributive double p-algebras, Glasgow Math. J. *26* (1985), 121—133.
[14] *R. W. Quackenbush,* Varieties with n-principal compact congruences, Algebra Universalis *14* (1982), 292—296.
[15] *J. C. Varlet,* A regular variety of type $\langle 2, 2, 1, 1, 0, 0\rangle$, Algebra Universalis *2* (1972), 218—223.
[16] *J. C. Varlet,* Large congruences in p-algebras and double p-algebras, Algebra Universalis *9* (1979), 165—178.
[17] *J. C. Varlet,* Regularity in double p-algebras, Algebra Universalis *18* (1984), 95—105.

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