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Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 2, 288–296

Persistent URL: http://dml.cz/dmlcz/102462

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

SMALLNESS OF SETS OF NONDIFFERENTIABILITY OF CONVEX FUNCTIONS IN NON-SEPARABLE BANACH SPACES

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(Received June 26, 1989)

INTRODUCTION

There exist many papers which investigate Asplund (or weak Asplund) spaces – Banach spaces in which each continuous convex function is Frechet (or Gateaux) differentiable at all points except those which belong to a first category set (cf.e.g. [6]).

It was shown e.g. in [1], [9], [8] that in some separable Banach spaces the above mentioned exceptional sets are not only of the first category but are small in some more restrictive senses.

The problem of finding the most strict sense of smallness was solved for Gateaux differentiability in separable Banach spaces in [9].

The case of Frechet differentiability in Banach spaces with a separable dual was considered in [7] and [8], where it was proved that in this case the exceptional set is angle small. There is a reason tc say (cf. [8]), that this result is close to the best possible one which is not known in the present time. Note that Problem 1 from [8] was solved in negative by S. V. Konjagin [4].

In the present article we obtain some results concerning the smallness of sets of nondifferentiability of continuous convex functions in some non-separable Banach spaces.

In the case of Frechet differentiability we obtain in a general Asplund space a result, which is only slightly weaker than the one obtained in [8] in separableAsplund spaces. The proof uses ideas contained in [3] and [8].

In the case of Gateaux differentiability we ate not able to obtain results in general weak Asplund spaces, but only in Asplund spaces and in spaces with a strictly convex dual. In these spaces we prove that the exceptional set is σ-cone supported. Also this result is not the best possible, but in the case of a separable space it is not too far from the best possible result of [9]. The proofs use the ideas of Kenderov's articles [2] and [3].

As usual, we have generalized our results to theorems concerning general multi-valued monotone operators.

DEFINITIONS

In the sequel we consider real Banach spaces. If X is a Banach space, $a \in X$ and $M \subset X$, then we put $a + M = \{a + x : x \in M\}$. If P is a metric space, then the open ball with the center $x \in P$ and the radius $r > 0$ is denoted by $B(x, r)$.

Definition 1. Let P be a metric space, $M \subset P$, $x \in P$, $R > 0$. Then we denote the supremum of the set of all $r > 0$ for which there exists $z \in P$ such that $B(z, r) \subset \subset B(x, R) - M$ by $\gamma(x, R, M)$. The number $\limsup_{R \to 0+} \gamma(x, R, M) R^{-1}$ is called the *porosity of M at x*. If the porosity of M at x is positive we say that *M is porous at x*. A set is said to be *porous* if it is porous at all its points. A set is termed *σ-porous* if it can be written as a union of countably may porous sets.

It is easy to see that any porous set is nowhere dense and therefore any σ-porous set is a first category set. On the other hand, in an arbitrary Banach space there exists a first category set which is not σ-porous (cf. [11]).

Definition 2. Let X be a Banach space. If $x^* \in X^*$, $x^* \neq o$, and $0 < \alpha < 1$, define

$$C(x^*, \alpha) = \{x \in X: \alpha \|x\| \, \|x^*\| < \langle x, x^* \rangle\} .$$

We say that a set $M \subset X$ is *α-cone porous at $x \in X$* (where $0 < \alpha < 1$) if there exists $R > 0$ such that for each $r > 0$ there exist $z \in B(x, r)$ and $o \neq x^* \in X^*$ such that

$$M \cap B(x, R) \cap (z + C(x^*, \alpha)) = \emptyset .$$

A subset of X is said to be *α-cone porous* if it is α-cone porous at all its points. A set is termed *$\sigma - \alpha$-cone porous* if it can be written as a union of countably many α-cone porous sets. A set is said to be *cone-small* if it is $\sigma - \alpha$-cone porous for each $0 < \alpha < 1$.

In the following definition, it will be convenient to use an another definition of a "cone".

Definition 3. Let X be a Banach space. If $v \in X$, $\|v\| = 1$, and $0 < c < 1$, define

$$A(v, c) = \{x: x = \lambda v + w, \ \lambda > 0, \ \|w\| < c\lambda\} = \bigcup_{\lambda > 0} \lambda B(v, c) .$$

A set $M \subset X$ is said to be *cone supported at $x \in M$* if there exist $R > 0$, $v \in X$, $\|v\| = 1$ and $0 < c < 1$ such that

$$M \cap B(x, R) \cap (x + A(v, c)) = \emptyset .$$

A subset of X is said to be *cone supported* if it is cone supported at all its points. A set is termed σ-cone supported if it can be written as a union of countably many cone supported sets.

Note 1. It is easy to prove that each "cone" $C(x^*, \alpha)$ contains a "cone" $A(v, c)$.

Note 2. In [8], the following terminology is used. A set $M \subset X$ is said to be *α-angle porous* if for every $x \in M$ and every $\varepsilon > 0$ one may find $z \in B(x, \varepsilon)$ and $x^* \in X^*$ such that $M \cap (z + C(x^*, \alpha)) = \emptyset$. A set is said to be *angle-small* if, for every α positive, it can be written as a countable union of α-angle porous sets. It is easy to see that each α-angle porous set is α-cone porous and each angle-small set is cone-small in an arbitrary Banach space. It is easy to prove that the notions of angle-small sets and cone-small sets coincide if X is a separable Banach space. It seems to be probable that these notions do not coincide in non-separable spaces.

Note 3. If X is a separable Banach space then $M \subset X$ is σ-cone supported iff M is a sparse set (it follows easily from Lemma 1 of [12]). A set is said to be a *sparse set* [10] if it can be covered by a countable union of Lipschitz hypersurfaces. Note that $M \subset X$ is a subset of a set of points of Gateaux non-differentiability of a continuous convex function f if and only if M is a d.c.-sparse set [9]. A set is said to be *d.c.-sparse* [10] if it can be covered by countably many of Lipschitz hypersurfaces such that each of them is determinated by a Lipschitz function which is a difference of two convex Lipschitz functions.

Note 4. Every σ-cone supported set is obviously σ-porous. Also every cone-small set is clearly σ-porous. In finite-dimensional spaces it is easy to prove that each cone-small set is σ-cone supported. In R^2 there exists a σ-cone supported set which is not cone-small (it follows easily from Remark 1 of [8], p. 221). It seems to be probable that in all infinite-dimensional spaces these two notions are uncomparable. Note that the set L (a subset of a separable Hilbert space) constructed in the proof of Theorem 2 from [4] is cone-small but is not σ-cone supported.

LEMMAS

The proofs of our results have the following ingredients.

(i) Ideas of Kenderov's proofs from [2] and [3].

(ii) Lemma 1 and Lemma 2 which deal with monotone operators. Lemma 1 is implicitely contained in the proof of Theorem 1 from [8]. Lemma 2 is possibly new.

(iii) An indirect method based on Corollary 1 and Corollary 2. In [3] (and in many other proofs) the fact that an "exceptional" set E is of the first category is proved indirectly. Supposing that E is of the second category it is proved that an appropriate subset of S is dense in a ball, and from the latter fact a contradiction is deduced. Our results on generalized porosity, which are proved by a well-known topological method (namely the Montgomery operation (cf. [5]) is used), enable us to use the analogy to the indirect proof described above also for proofs that a set is "σ-porous" in a sense (e.g. cone-small or σ-cone supported). The important fact is that Corollary 1 and Corollary 2 hold also in non-separable spaces where the direct method of [8] does not work.

Lemma 1. *Let X be a Banach space and let $T\colon X \mapsto X^*$ be a monotone operator with an arbitrary domain $D(T) = \{x\colon T(x) \neq \emptyset\}$. Let $0 < 3a < A$, $x \in X$ and*

$N \subset D(T)$ be given such that

(1) $$\lim_{\delta \to 0+} \operatorname{diam} T(B(x, \delta) \cap N) < a$$

and

(2) $$\lim_{\delta \to 0+} \operatorname{diam} T(B(x, \delta)) > A .$$

Then N is $3a/A$ – cone porous at x.

Proof. On account of (1) we can choose $R > 0$ such that

(3) $$\operatorname{diam} T(B(x, R) \cap N) < a .$$

If $T(B(x, R) \cap N) = \emptyset$, then the assertion of the lemma is obviously satisfied. In the opposite case choose $f \in T(B(x, R) \cap N)$ and consider an arbitrary $r > 0$. On account of (2) we can find $z \in B(x, r)$ and $T_z \in T(z)$ such that $\|T_z - f\| > A/2$. To show that N is $3a/A$ – cone porous at x it is sufficient to prove that

$$B(x, R) \cap N \cap \{y \in X: \langle y - z, T_z - f \rangle >$$
$$> (3a/A) \|T_z - f\| \|y - z\|\} = \emptyset .$$

Suppose on the contrary that there exists $y \in N \cap B(x, R)$ for which

$$\langle y - z, T_z - f \rangle > (3a/A) \|T_z - f\| \|y - z\|$$

and choose $T_y \in T(y)$. Since (3) implies $\|T_y - f\| < a$ we obtain, using the monotonicity of T, the following inequalities:

$$a\|y - z\| \geqq \langle y - z, T_y - f \rangle = \langle y - z, T_y - T_z \rangle +$$
$$+ \langle y - z, T_z - f \rangle \geqq \langle y - z, T_z - f \rangle >$$
$$> (3a/A) \|T_z - f\| \|y - z\| \geqq (3a/A)(A/2) \|y - z\| \geqq a\|y - z\| .$$

This is a contradiction which completes the proof.

Lemma 2. *Let X be a Banach space and let $T: X \to X^*$ be a monotone operator with an arbitrary domain $D(T) = \{x: T(x) \neq \emptyset\}$. Let $H \subset D(T)$, $x \in H$, $v \in X$, $\|v\| = 1$, $c \in R$, $\varepsilon > 0$, $K > 0$, $y \in T(x)$,*

(i) *$\langle v, y \rangle > c + \varepsilon$ and*

(ii) $\lim_{\delta \to 0+} \operatorname{diam} T(B(x, \delta) \cap H) < K$.

Then there exists $\varrho > 0$ such that for every

$$\tilde{x} \in B(x, \varrho) \cap H \cap (x + A(v, \varepsilon/K)) \quad \textit{and} \quad \tilde{y} \in T(\tilde{x})$$

the inequality $\langle v, \tilde{y} \rangle > c$ holds.

Proof. By (ii) we can choose $\varrho > 0$ such that

(4) $$\|y - \tilde{y}\| \leqq K \quad \text{whenever} \quad \tilde{x} \in B(x, \varrho) \cap H \quad \text{and} \quad \tilde{y} \in T(\tilde{x}) .$$

Suppose that $\tilde{x} \in B(x, \varrho) \cap H \cap (x + A(v, \varepsilon/K))$ and $\tilde{y} \in T(\tilde{x})$ are given. By Definition 3 we can find $\lambda > 0$ and $w \in X$, $\|w\| < \lambda\varepsilon/K$ such that $\tilde{x} = x + \lambda v + w$. Monotonicity of T implies

(5) $$0 \leqq \langle \tilde{x} - x, \tilde{y} - y \rangle = \langle \lambda v + w, \tilde{y} \rangle - \langle \lambda v + w, y \rangle .$$

Using (5), (i) and (4) we obtain $\langle \lambda v, \tilde{y} \rangle \geqq \langle \lambda v, y \rangle + \langle w, y - \tilde{y} \rangle > \lambda(c + \varepsilon) - \|w\| K > \lambda(c + \varepsilon) - \lambda\varepsilon = \lambda c$. Consequently $\langle v, \tilde{y} \rangle > c$.

Since we need some facts both for the σ-ideal of cone-small sets and for the σ-ideal of σ-cone supported sets, we shall formulate our lemmas in an abstract way using the following notion of $\sigma - V$-porous sets.

Definition 4. (cf. [11], p. 333.) Let P be a metric space and let $V = V(x, A)$ be a relation between points $x \in P$ and sets $A \subset P$. We say that V is a *porosity relation* if:

(6) if $A \subset B$ and $V(x, B)$, then $V(x, A)$;

(7) $V(x, M)$ iff there is $r > 0$ such that $V(x, M \cap B(x, r))$;

(8) $V(x, A)$ iff $V(x, \bar{A})$.

We say that A is *V-porous at x* if $V(x, A)$ holds. The notions of V-porous and *$\sigma - V$-porous* sets are defined in the obvious way.

The following lemma states that a set is $\sigma - V$-porous whenever it is locally $\sigma - V$-porous.

Lemma 3. *Let (P, ϱ) be a metric space and let V be a porosity relation. Let $M \subset P$ and let for each $x \in M$ there exists $\delta_x > 0$ such that $M \cap B(x, \delta_x)$ is $\sigma - V$-porous. Then M is $\sigma - V$-porous.*

Proof. Let us order the points of M to a transfinite sequence $\{x_\alpha; 0 \leqq \alpha < \beta\}$. Put $G_\alpha = B(x_\alpha, \delta_{x_\alpha})$ and $K_\alpha = G_\alpha - \bigcup_{\eta<\alpha} G_\eta$. For each natural number n define $G_{\alpha,n} = \{x: \operatorname{dist}(x, P - G_\alpha) \geqq 1/n\}$ and $M_{\alpha,n} = M \cap K_\alpha \cap G_{\alpha,n}$. Obviously $M \subset \bigcup_{\alpha<\beta} G_\alpha = \bigcup_{\alpha<\beta} K_\alpha$ and $G_\alpha = \bigcup_{n=1}^{\infty} G_{\alpha,n}$. Consequently $\bigcup_{n=1}^{\infty} M_{\alpha,n} = M \cap K_\alpha$ and $\bigcup_{\alpha<\beta} \bigcup_{n=1}^{\infty} M_{\alpha,n} = \bigcup_{n=1}^{\infty} \bigcup_{\alpha<\beta} M_{\alpha,n} = M$. Thus it is sufficient to prove that, for each fixed n, the set $M_n := \bigcup_{\alpha<\beta} M_{\alpha,n}$ is $\sigma - V$-porous. By the definition of G_α and (6), each $M_{\alpha,n}$ is $\sigma - V$-porous. Choose V-porous sets $M_{\alpha,n,k}$, $k = 1, 2, \ldots$, such that $M_{\alpha,n} = \bigcup_{k=1}^{\infty} M_{\alpha,n,k}$. Consequently we have $M_n = \bigcup_{k=1}^{\infty} \bigcup_{\alpha<\beta} M_{\alpha,n,k}$. Now observe that $\operatorname{dist}(K_\eta \cap G_{\eta,n}, K_\xi \cap G_{\xi,n}) \geqq 1/n$ and consequently also

(9) $$\operatorname{dist}(M_{\eta,n,k}, M_{\xi,n,k}) \geqq 1/n \quad \text{whenever} \quad \eta \neq \xi .$$

In fact, suppose that $\eta < \xi$, $x \in K_\eta \cap G_{\eta,n}$ and $y \in K_\xi \cap G_{\xi,n}$. Since $y \in K_\xi$ and $x \in G_{\eta,n}$, we obtain $y \notin G_\eta$ and $\operatorname{dist}(x, P - G_\eta) \geqq 1/n$. Therefore $\varrho(x, y) \geqq 1/n$. Using (9) and (7) we obtain that $\bigcup_{\alpha<\beta} M_{\alpha,n,k}$ is V-porous. Consequently M_n is $\sigma - V$-porous.

Lemma 4. *Let (P, ϱ) be a metric space and let V be a porosity relation. Suppose that $M \subset P$ is not $\sigma - V$-porous. Then there exists $\emptyset \neq N \subset M$ such that N is V-porous at no point of N.*

Proof. Let S be the set of all points $x \in M$ for which there exists $r > 0$ such that $M \cap B(x, r)$ is $\sigma - V$-porous. Using (6) we obtain that S is locally $\sigma - V$-porous and consequently it is, by Lemma 3, $\sigma - V$-porous. Put $M^* = M - S$. Obviously $M^* \neq \emptyset$ and

(10) $\qquad M^* \cap B(x, r)$ is not $\sigma - V$-porous whenever $x \in M^*$ and $r > 0$.

Now let T be the set of all points $x \in M^*$ at which M^* is V-porous. By (6) T is a V-porous set and consequently $N: = M^* - T$ si nonempty by (10). Moreover,

(11) $\qquad N$ is dense in M^*.

To prove (11), suppose that there is an open set $G \subset P$ such that $G \cap N = \emptyset$ and $G \cap M^* \neq \emptyset$. Then $\emptyset \neq G \cap M^* \subset T$ which contradicts to (10). Now consider an arbitrary $x \in N$. By the definition of N, M^* is not V-porous at x. Consequently (11) and (8) imply that also N is not V-porous at x.

Corollary 1. *Let X be a Banach space. Suppose that $M \subset X$ is not σ-cone supported. Then there exists $\emptyset \neq N \subset M$ such that N is cone supported at no point of N.*

Corollary 2. *Let X be a Banach space and let $0 < \alpha < 1$. Suppose that $M \subset X$ is not $\sigma - \alpha$-cone porous. Then there exists $\emptyset \neq N \subset M$ such that N is α-cone porous at no point of N.*

THEOREMS

Theorem 1. *Let X be an Asplund space and let $T: X \to X^*$ be a locally bounded monotone operator with an arbitrary domain $D(T) = \{x: T(x) \neq \emptyset\}$. Then there exists a σ-cone supported set $A \subset D(T)$ such that T is single-valued at each point of $D(T) - A$.*

Proof. Suppose on the contrary that

$$A := \{x \in D(T): T \text{ is not single-valued at } x\}$$

is not σ-cone supported. We can obviously write $A = \bigcap_{n=1}^{\infty} A_n$ where

$$A_n := \{x \in D(T): \operatorname{diam} T(x) > 1/n\} .$$

Consequently we can choose a positive integer n such that A_n is not σ-cone supported. By Corollary 1 there exists a set $\emptyset \neq N \subset A_n$ which is cone supported at no its point. Choose $x \in N$. Since T is locally bounded, there exists $r > 0$ such that $T(B(x, r))$ is bounded. Putting $H := N \cap B(x, r)$, we easily see that $\emptyset \neq H$ is cone supported at no point of H and $T(H)$ is a bounded subset of X^*. Choose $K > 0$

such that $\|y\| < K$ for each $y \in T(H)$. Since X is an Asplund space, every nonempty bounded subset of X^* admits weak* slices of arbitrary small diameters (see e.g. [6]). Consequently there exist $v \in X$, $\|v\| = 1$ and $c > 0$ such that the weak* slice of $T(H)$

$$S := \{x^* \in T(H) : \langle v, x^* \rangle > c\}$$

is nonempty and has diameter less than $1/n$. Since $S \neq \emptyset$, we can choose $x \in H$ and $y \in S \cap T(x)$. Choose $\varepsilon > 0$ such that $\langle v, y \rangle > c + \varepsilon$. By Lemma 2 there exists $\varrho > 0$ such that for each $\tilde{x} \in B(x, \varrho) \cap H \cap (x + A(v, \varepsilon/2K))$ and $\tilde{y} \in T(\tilde{x})$ the inequality $\langle v, \tilde{y} \rangle > c$ holds. Since H is not cone supported at x we can choose $\tilde{x} \in \in B(x, \varrho) \cap H \cap (x + A(v, \varepsilon/2K))$. Since $H \subset A_n$ we have $\operatorname{diam} T(\tilde{x}) > 1/n$. But $T(\tilde{x}) \subset S$ and $\operatorname{diam} S < 1/n$, a contradiction.

As an immediate corollary we obtain the following theorem.

Theorem 1*. *Let X be an Asplund space, $G \subset X$ an open convex set and let f be a continuous convex function on G. Then there exists a σ-cone supported set $A \subset G$ such that f is Gateaux differentiable at each point of $G - A$.*

Theorem 2. *Let X be a Banach space which admits an equivalent norm whose dual norm is strictly convex. Let $T: X \to X^*$ be a maximal monotone operator with a domain $D(T) = \{x: T(x) \neq \emptyset\}$ which has a nonempty interior $G = \operatorname{Int} D(T)$. Then there exists a σ-cone supported set $A \subset G$ such that T is single-valued at each point of $G - A$.*

Proof. Assume that the norm in X^* is strictly convex. Since T is maximal, we know that $T(x)$ is convex for each $x \in D(T)$. Therefore, for each $x \in G$ at which T is not single-valued, there exist $y, z \in T(x)$ such that $\|y\| \neq \|z\|$. Consequently

$$A = \bigcup \{A_c\colon c\text{-rational}\},$$

where $A = \{x \in G\colon T \text{ is not single-valued at } x\}$ and $A_c = \{x \in G\colon \|z\| < c < \|y\| \text{ for some } y, z \in T(x)\}$. It is sufficient to prove that each A_c is cone supported. To this end, consider an arbitrary $x \in A_c$. Choose $y, z \in T(x)$ such that $\|z\| < c < \|y\|$ and $v \in X$, $\|v\| = 1$, $\varepsilon > 0$ for which $\langle v, y \rangle > c + \varepsilon$. Further choose $K > 0$ such that $\lim_{\delta \to 0_+} \operatorname{diam} T(B(x, \delta)) < K$ (it is possible since T is monotone and consequently locally bounded in G) and apply Lemma 2 with $H = D(T)$. Thus we obtain $\varrho > 0$ such that for each $\tilde{x} \in B(x, \varrho) \cap D(T) \cap (x + A(v, \varepsilon/K))$ and $\tilde{y} \in T(\tilde{x})$ the inequality $\langle v, \tilde{y} \rangle > c$ holds. It is now sufficient to prove that $B(x, \varrho) \cap A_c \cap (x + A(v, \varepsilon/K)) = \emptyset$. Suppose on the contrary that there exists

$$\tilde{x} \in B(x, \varrho) \cap A_c \cap (x + A(v, \varepsilon/K)).$$

Since $\tilde{x} \in A_c$, we can choose $\tilde{y} \in T(\tilde{x})$ such that $\|\tilde{y}\| < c$. But $\langle v, \tilde{y} \rangle > c$, which is a contradiction, since $\|v\| = 1$.

As an immediate corollary we obtain the following theorem.

Theorem 2*. *Let X be a Banach space which admits an equivalent norm whose dual norm is strictly convex. Let f be a continuous convex function defined on an* *open convex set G. Then there exists a σ-cone supported set $A \subset G$ such that f is Gateaux differentiable at each point of $G - A$.*

Theorem 3. *Let X be an Asplund space and let $T: X \to X^*$ be a maximal monotone operator with a domain $D(T) = \{x: T(x) \neq \emptyset\}$ which has a nonempty interior $G = \text{Int}\, D(T)$. Then the set A of all points $x \in G$ at which T is single-valued but is not norm-to-norm upper semicontinuous is cone-small.*

Proof. Suppose on the contrary that A is not cone-small. Then there exists $0 < \alpha < 1$ such that A is not $\sigma - \alpha$-cone porous. Obviously $A = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \{x \in A: \lim_{\delta \to 0+} \text{diam}\, T(B(x, \delta)) > 1/n\} .$$

Consequently we can choose a positive integer n such that A_n is not $\sigma - \alpha$-cone porous. By Corollary 2 there exists a set $\emptyset \neq N \subset A_n$ which is α-cone porous at no point of N. Choose $x \in N$. Since T is locally bounded at each point of G we can choose $r > 0$ such that $T(B(x, r))$ is bounded. Since X is an Asplund space, every nonempty bounded subset of X^* admits weak* slices of arbitrary small diameters. Consequently there exist $v \in X$, $\|v\| = 1$ and $c > 0$ such that the weak* slice of $T(B(x, r) \cap N)$

$$S := \{x^* \in T(B(x, r) \cap N) : \langle v, x^* \rangle > c\}$$

is nonempty and has diameter less than $\alpha/3n$. Since $S \neq \emptyset$, we can choose $\tilde{x} \in B(x, r) \cap \cap N$ such that $\langle v, T(\tilde{x})\rangle > c$. (Here $T(\tilde{x}) \in X^*$, since T is single-valued at $\tilde{x} \in N \subset A$). Since $\{x^*: \langle v, x^* \rangle > c\}$ is weak* open and since T is norm-to-weak* upper semicontinuous, there exists $d > 0$ such that $B(\tilde{x}, d) \subset B(x, r)$ and $T(B(\tilde{x}, d)) \subset \subset \{x^*: \langle v, x^* \rangle > c\}$. Consequently $T(B(\tilde{x}, d) \cap N) \subset S$ and therefore

$$\lim_{\delta \to 0+} \text{diam}\, T(B(\tilde{x}, \delta) \cap N) \leqq \text{diam}\, S < \alpha/3n .$$

Since $\tilde{x} \in N \subset A_n$, we have $\lim_{\delta \to 0+} \text{diam}\, T(B(\tilde{x}, \delta)) > 1/n$. Using Lemma 1 with $x = \tilde{x}$, $A = 1/n$, $a = \alpha/3n$ we obtain that N is α-cone porous at $\tilde{x}$, which is a contradiction.

As immediate consequences of Theorem 1 and Theorem 3 we obtain the following results.

Theorem 4. *Let X be an Asplund space and let $T: X \to X^*$ be a maximal monotone operator with a domain $D(T)$ which has a nonempty interior G. Then there exist a σ-cone supported set $A \subset G$ and a cone-small set $B \subset G$ such that T is single-valued and norm-to-norm upper semicontinuous at each point of $G - (A \cup B)$.*

Theorem 4*. *Let X be an Asplund space and let f be a continuous convex function defined on an open convex set G. Then there exist a σ-cone supported set $A \subset G$ and a cone-small set $B \subset G$ such that f is Frechet differentiable at each of $G - - (A \cup B)$.*

Note 5. If X is a separable Asplund space, then we can [8] choose $A = \emptyset$ in Theorem 4. It is probable that Theorem 4 can be improved in this way also in the general case.

References

[1] *N. Aronszajn:* Differentiability of Lipschitzian mappings between Banach spaces, Studia Math. *57* (1976), 147—190.

[2] *P. S. Kenderov:* The set-valued monotone mappings are almost everywhere single-valued, C. R. Acad. Bulgare Sci. *27* (1974), 1173—1175.

[3] *P. S. Kenderov:* Monotone operators in Asplund spaces, C. R. Acad. Bulgare Sci. *30* (1977), 963—964.

[4] *S. V. Konjagin:* On the points of the existence and nonunicity of elements of the best approximation (in Russian), in Teorija funkcij i ee prilozhenija, P. L. Uljanov ed., Izdatelstvo Moskovskovo Universiteta, pp. 38—43, Moscow (1986).

[5] *K. Kuratowski:* Topology, Vol. I (transl.), Academic Press, New York (1966).

[6] *R. R. Phelps:* Convex functions, monotone operators and differentiability, Lect. Notes in Math., Nr. 1364, Springer-Verlag, (1989).

[7] *D. Preiss* and *L. Zajíček:* Fréchet differentiation of convex functions in a Banach space with a separable dual, Proc. Amer. Math. Soc. *91* (1984), 202—204.

[8] *D. Preiss* and *L. Zajíček:* Stronger estimates of smallness of sets of Fréchet nondifferentiability of convex functions, Proc. 11th Winter School, Suppl. Rend. Circ. Mat. Palermo, Ser. II, No. 3 (1984), 219—223.

[9] *L. Zajíček:* On the differentiation of convex functions in finite and infinite dimensional spaces, Czechoslovak Math. J. *29* (*104*) (1979), 340—348.

[10] *L. Zajíček:* Differentiability of the distance function and points of multi-valuedness of the metric projection in Banach space, Czechoslovak Math. J. *33* (*108*), (1983), 292—308.

[11] *L. Zajíček:* Porosity and σ-porosity, Real Analysis Exchange *13* (1987-88), 314—350.

[12] *L. Zajíček:* On the points of multivaluedness of metric projections in separable Banach spaces, Comment. Math. Univ. Carolinae *19* (1978), 513—523.

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