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ON THE MOMENTS IN NONHOMOGENEOUS BIRTH, IMMIGRATION AND DEATH PROCESSES

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1. INTRODUCTION

This paper is devoted to a study of some internal relations between general moments $\alpha_i(t) = \mathsf{E}[\xi^i(t)]$, $i = 1, 2, 3$ of the distribution of the integer-valued random variable $\xi(t)$ which assumes the values x of the population size at the time t. We suppose that the change of population size is controlled by one of these nonhomogeneous processes: the birth-death process, the birth process, the death process, the birth-immigration-death process and the immigration-death process. This paper links up with the results published by KENDALL [4], BARTLETT [1] and by the author of this paper [2, 3] in the domain of introduced types of the nonhomogeneous processes.

When studying several types of processes in biology, physics, technical sciences etc. we meet with actions the result of which is a change of the population size at the time t. The nucleation by crystallisation or by solid-solid transformation can be used as an example of such a process in technical applications. For example when we study the graphitization of malleable cast iron we can determine the number of graphite particles in the unit specimen volume with the aid of the observation of several polished plane sections prepared on the metal samples corresponding to a time t. On the basis of these measurements the number of which can be very large we can estimate the values of the general moments $\alpha_i(t)$ of the distribution of the population size at the time t. Using these informations an analysis of graphitization of malleable cast iron can be made.

Under the assumption that the values $\alpha_i(t_j)$, $i = 1, 2, 3$, are known for a sequence of time instants $\{t_j\}_0^k$ the question arises how to employ this information for the determination of the type of the process controlling the change of the population size in the time. In the general case this change is conceived as the function of the birth rate $\lambda(t)$, the death rate $\mu(t)$ and the immigration rate $\nu(t)$, all being positive functions continuous in the open interval $T = (0, \infty)$. The first approach to the solution of this problem is the derivation of several criteria based on the knowledge of the first three general moments $\alpha_i(t)$, $i = 1, 2, 3$. In the main it is the problem of finding the necessary conditions which had to be fulfilled in order that the studied type of process had the required properties or in other words was of a certain type. Such conditions in the form of equalities or inequalities for the introduced types of processes are given in this paper. As far as possible the functions of the first three moments were chosen from the point of view to reach a mutually comparability between several types of processes.

In conclusion a short note about the notation: for the birth-death process with a birth rate $\lambda(t)$ and a death rate $\mu(t)$ we shall use a short form: the process $\{\lambda(t), \mu(t)\}$, for the birth-immigration-death process with a birth rate $\lambda(t)$, the immigration rate $\nu(t)$ and the death rate $\mu(t)$ the short form: the process $\{\lambda(t), \nu(t), \mu(t)\}$ etc.

2. NONHOMOGENEOUS BIRTH-DEATH PROCESSES

Consider the birth-death process $\xi(t)$ with states E_x $(x = 0, 1, \dots)$ the birth and death rates $\lambda(t)$ and $\mu(t)$.

Assumptions:

a) if at time t the system is in state E_x, then the probability of the transition $E_x \rightarrow$ $\rightarrow E_{x+1}$ in the interval $(t, t + \Delta t)$ is $x\,\lambda(t)\,\Delta t + o(\Delta t)$ for $x = 1, 2, \dots$;

b) if at time t the system is in state E_x, then the probability of the transition $E_x \rightarrow$ $\rightarrow E_{x-1}$ in the interval $(t, t + \Delta t)$ is $x\,\mu(t)\,\Delta t + o(\Delta t)$ for $x = 1, 2, \dots$;

c) the probability of a transition to a state other than the neighbouring one is $o(\Delta t)$;

d) if at time t the system is in state E_x, then the probability of no change in the interval $(t, t + \Delta t)$ is $1 - \{\lambda(t) + \mu(t)\}\,x\,\Delta t + o(\Delta t)$;

e) at time $t = 0$ the system is in state E_1.

We find from the fundamental differential–difference equations which the functions

$$P_x(t) = \mathscr{P}\{\xi(t) = x\}$$

must satisfy with initial conditions

$$P_1(0) = 1 \quad \text{and} \quad P_x(0) = 0\,, \quad x \neq 1\,, \tag{1}$$

that the probability generating function

$$v = \varphi(z, t) = \sum_{x=-\infty}^{+\infty} P_x(t)\,z^x\,, \tag{2}$$

where we define $P_x(t) = 0$ for $x < 0$, satisfies the linear partial differential equation

$$\frac{\partial \varphi}{\partial t} - (z - 1)\,[z\,\lambda(t) - \mu(t)]\,\frac{\partial \varphi}{\partial z} = 0\,, \tag{3}$$

where

$$v = z \quad \text{for} \quad t = 0 \quad \text{and} \quad v = 1 \quad \text{for} \quad z = 1\,. \tag{4}$$

Theorem 1. *Let $\lambda(t)$ and $\mu(t)$ be positive functions, continuous in the open interval $T = (0, \infty)$. Then in the process $\{\lambda(t), \mu(t)\}$ the relation*

$$\frac{\alpha_1(t)\,[\alpha_3(t) - \alpha_1(t)]}{\alpha_2^2(t) - \alpha_1^2(t)} = \frac{3}{2} \tag{5}$$

holds for every $t \in T$.

Proof. The integral surface $v = \varphi(z, t)$, which fulfils the conditions (4), is according to [4]

$$v = 1 + \frac{z - 1}{e^{\varrho(t)} - (z - 1)\displaystyle\int_0^t \lambda(\tau)\, e^{\varrho(\tau)}\,\mathrm{d}\tau}\,, \tag{6}$$

where

$$\varrho(t) = \int_0^t [\mu(\tau) - \lambda(\tau)]\,\mathrm{d}\tau\,. \tag{7}$$

From the well known relations between the probability generating function $\varphi(z, t)$ and the general moments $\alpha_i(t)$, $i = 1, 2, 3$, there follow according to (6) the expressions

$$\alpha_1(t) = \frac{\partial\varphi}{\partial z}\bigg|_{z=1} = e^{-\varrho(t)}\,, \tag{8}$$

$$\alpha_2(t) = \left[\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial\varphi}{\partial z}\right]_{z=1} = e^{-\varrho(t)}\left[1 + 2e^{-\varrho(t)}\int_0^t \lambda(\tau)\, e^{\varrho(\tau)}\,\mathrm{d}\tau\right], \tag{9}$$

and

$$\alpha_3(t) = \left[\frac{\partial^3\varphi}{\partial z^3} + 3\,\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial\varphi}{\partial z}\right]_{z=1} = \tag{10}$$

$$= 6e^{-2\varrho(t)}\int_0^t \lambda(\tau)\, e^{\varrho(\tau)}\,\mathrm{d}\tau\left\{1 + e^{-\varrho(t)}\int_0^t \lambda(\tau)\, e^{\varrho(\tau)}\,\mathrm{d}\tau\right\} + e^{-\varrho(t)}\,.$$

The derivation of (5) from (8), (9) and (10) is evident.

Note. The process in which the ratio

$$\frac{\mu(t)}{\lambda(t)} = c \tag{11}$$

is constant everywhere in T, in paper [3] was investigated. The arranged corresponding necessary condition is given by equations (19) and (20) in the quoted paper [3].

3. NONHOMOGENEOUS BIRTH PROCESS

Consider the birth process $\xi(t)$ with states E_x $(x = 1, 2, \ldots)$ and the birth rate $\lambda(t)$. The changes in the assumptions (see Chap. 2) and in the differential equation (3) are evident. The conditions (1) and (4) still hold.

Theorem 2. *Let $\lambda(t)$ be positive function, continuous in T. Then in process $\{\lambda(t)\}$ the relations*

$$\frac{\alpha_1(t)}{\alpha_2(t)}\left[2\alpha_1(t) - 1\right] = 1 \tag{12}$$

and

$$\frac{\alpha_3(t) - \alpha_1(t)}{\left[\alpha_2(t) + \alpha_1(t)\right]\left[\alpha_1(t) - 1\right]} = 3 \tag{13}$$

hold for every $t \in T$.

Proof. In view of the preliminary note we obtain from (8) to (10) that

$$\alpha_1(t) = e^{\varrho_b(t)}\,, \tag{14}$$

$$\alpha_2(t) = e^{\varrho_b(t)}\left[2e^{\varrho_b(t)} - 1\right] \tag{15}$$

and

$$\alpha_3(t) = 6e^{2\varrho_b(t)}\left[e^{\varrho_b(t)} - 1\right] + e^{\varrho_b(t)}\,, \tag{16}$$

where

$$\varrho_b(t) = \int_0^t \lambda(\tau)\,\mathrm{d}\tau\,. \tag{17}$$

Hence there follow the relations to be proved.

4. NONHOMOGENEOUS DEATH PROCESS

Consider the death process $\xi(t)$ with states E_x $(x = 0, 1, 2, \ldots)$ and the death rate $\mu(t)$.

Assumptions:

a) at time $t = 0$ the system is in state E_n, $n \geqq 1$ is an integer;

b) if at time t the system is in state E_x, then the probability of the transition $E_x \to E_{x-1}$ in the interval $(t, t + \Delta t)$ is $x\,\mu(t)\,\Delta t + o(\Delta t)$ for $x = 1, 2, \ldots, n$;

c) the probability of a transition to a state other than the neighbouring lower one is $o(\Delta t)$;

d) if at time t the system is in state E_x, then the probability of no change in the interval $(t, t + \Delta t)$ is $1 - x\,\mu(t)\,\Delta t + o(\Delta t)$.

We shall first look for the appropriate generating function

$$v = \varphi(z, t),$$

where

$$\begin{aligned} P_x(t) &= 0 \quad \text{for} \quad x < 0 \quad \text{and} \quad x > n, \\ P_x(0) &= 0 \quad \text{for} \quad x \neq n, \\ P_n(0) &= 1. \end{aligned} \tag{18}$$

The corresponding linear partial differential equation is of the form

$$\frac{\partial \varphi}{\partial t} + \mu(t)(z - 1)\frac{\partial \varphi}{\partial z} = 0, \tag{19}$$

where

$$v = z^n \quad \text{for} \quad t = 0 \quad \text{and} \quad v = 1 \quad \text{for} \quad z = 1. \tag{20}$$

Theorem 3. *Let $\mu(t)$ be positive function, continuous in T. Then in the process $\{\mu(t)\}$ the relation*

$$\frac{\alpha_1(t)}{\alpha_2(t)}\left[\alpha_1(t) - \frac{1}{n}\alpha_1(t) + 1\right] = 1 \tag{21}$$

holds for every $t \in T$. In addition to this equation simultaneously two inequalities

$$\frac{\alpha_1(t)\left[\alpha_1(t) + 1\right]}{\alpha_2(t)} > 1 \quad \textit{and} \quad \frac{\alpha_3(t) - \alpha_1(t)\left[1 + \alpha_1^2(t)\right]}{\alpha_1^2(t)} < 3 \tag{22}$$

must hold.

Proof. The integral surface $v = \varphi(z, t)$ fulfilling the condition (20) is evidently equal to

$$v = \left[(z - 1)\, e^{-\int_0^t \mu(\tau) d\tau} + 1\right]^n. \tag{23}$$

Similarly as in Chap. 2 we get

$$\alpha_1(t) = n e^{-\varrho_d(t)}, \tag{24}$$

$$\alpha_2(t) = n e^{-\varrho_d(t)}\left[(n - 1)\, e^{-\varrho_d(t)} + 1\right] \tag{25}$$

and

$$\alpha_3(t) = \left[(n - 1)(n - 2)\, e^{-2\varrho_d(t)} + 3(n - 1)\, e^{-\varrho_d(t)} + 1\right] n e^{-\varrho_d(t)}, \tag{26}$$

where

$$\varrho_d(t) = \int_0^t \mu(\tau)\, d\tau. \tag{27}$$

Hence we obtain the relations (21) and (22). These forms of inequalities will appear as very interesting in comparison with the results in Chapters 5 and 6.

Consider now the birth-immigration-death process $\xi(t)$ with states E_x $(x = 0, 1, 2, \ldots)$ and the birth rate $\lambda(t)$, the death rate $\mu(t)$ and the immigration rate $\nu(t)$.

Assumptions:

a) if at time t the system is in state E_x, then the probability of the transition $E_x \to \to E_{x+1}$ in the interval $(t, t + \Delta t)$ is $\{\nu(t) + x\,\lambda(t)\}\,\Delta t + o(\Delta t)$ for $x = 0, 1, \ldots;$

b) if at time t the system is in state E_x, then the probability of transition $E_x \to E_{x-1}$ in the interval $(t, t + \Delta t)$ is $x\,\mu(t)\,\Delta t + o(\Delta t)$ for $x = 1, 2, \ldots;$

c) the probability of a transition to a state other than the neighbouring one is $o(\Delta t)$;

d) if at time t the system is in state E_x, then the probability of no change in the interval $(t, t + \Delta t)$ is $1 - \{\nu(t) + x[\lambda(t) + \mu(t)]\}\,\Delta t + o(\Delta t)$;

e) at time $t = 0$ the system is in state E_0.

We find from the relevant fundamental differential-difference equations with the initial conditions

$$P_0(0) = 1 \quad \text{and} \quad P_x(0) = 0 \quad \text{for} \quad x = 1, 2, \ldots, \tag{28}$$

that the probability generating function $v = \varphi(z, t)$ satisfies the linear partial differential equation

$$\frac{\partial \varphi}{\partial t} - (z - 1)\,[z\,\lambda(t) - \mu(t)]\,\frac{\partial \varphi}{\partial z} = \nu(t)\,(z - 1)\,v\,, \tag{29}$$

where

$$v = 1 \quad \text{for} \quad t = 0 \quad \text{and} \quad v = 1 \quad \text{for} \quad z = 1\,. \tag{30}$$

Theorem 4. *Let $\lambda(t)$, $\nu(t)$ and $\mu(t)$ be positive functions, continuous in T. Then in the process $\{\lambda(t), \nu(t), \mu(t)\}$ two inequalities*

$$\frac{\alpha_1(t)\,[\alpha_1(t) + 1]}{\alpha_2(t)} < 1 \quad \textit{and} \quad \frac{\alpha_3(t) - \alpha_1(t)\,[1 + \alpha_1^2(t)]}{\alpha_1^2(t)} > 3 \tag{31}$$

hold simultaneously for every $t \in T$.

Proof. As it is known in a general case, it is impossible to obtain the integral surface $v = \varphi(z, t)$ fulfilling conditions (30) as a direct solution of differential equation (29). However, the generating function v can be found either with the aid of generating function (6) in the way introduced in [1] or direct as the probability generating function of the filtered nonhomogeneous Poisson's process (see Theorem 5A in PARZEN's book [5]) in the form

$$v = \exp\left\{\int_0^t \frac{(z-1)\,\nu(\tau)\,\mathrm{d}\tau}{e^{\varrho(\tau,t)} - (z-1)\int_\tau^t \lambda(s)\,e^{\varrho(\tau,s)}\,\mathrm{d}s}\right\}, \tag{32}$$

where

$$\varrho(\tau, t) = \int_\tau^t [\mu(s) - \lambda(s)]\,\mathrm{d}s\,. \tag{33}$$

Using now the same procedure as in Chapter 1 we get

$$\alpha_1(t) = \int_0^t \nu(\tau)\,e^{-\varrho(\tau,t)}\,\mathrm{d}\tau\,, \tag{34}$$

$$\alpha_2(t) = \alpha_1(t)\,[\alpha_1(t) + 1] + 2\int_0^t \nu(\tau)\,e^{-2\varrho(\tau,t)}\int_\tau^t \lambda(s)\,e^{\varrho(\tau,s)}\,\mathrm{d}s\,\mathrm{d}\tau \tag{35}$$

and

$$\begin{aligned}\alpha_3(t) = {} & \alpha_1^3(t) + 6\,\alpha_1(t)\int_0^t \nu(\tau)\,e^{-2\varrho(\tau,t)}\int_\tau^t \lambda(s)\,e^{\varrho(\tau,s)}\,\mathrm{d}s\,\mathrm{d}\tau\, + \\ & + 6\int_0^t \nu(\tau)\,e^{-3\varrho(\tau,t)}\left[\int_\tau^t \lambda(s)\,e^{\varrho(\tau,s)}\,\mathrm{d}s\right]^2 \mathrm{d}\tau\, + \\ & + 3\left[\alpha_1^2(t) + 2\int_0^t \nu(\tau)\,e^{-2\varrho(\tau,t)}\int_\tau^t \lambda(s)\,e^{\varrho(\tau,s)}\,\mathrm{d}s\,\mathrm{d}\tau\right] + \alpha_1(t)\,.\end{aligned} \tag{36}$$

From the last three expressions it may be concluded after some evident modifications the validity of the inequalities (31).

Note. The process in which the ratios

$$\frac{\nu(t)}{\lambda(t)} = b \quad \text{and} \quad \frac{\mu(t)}{\lambda(t)} = c \tag{37}$$

are constant everywhere in T, was investigated in paper [2]. Using the equations (18) and (19) of the quoted paper [2] and the relation

$$\alpha_3(t) = \alpha_1(t)\left[1 + 3\left(1 + \frac{1}{b}\right)\alpha_1(t) + \left(1 + \frac{1}{b}\right)\left(1 + \frac{2}{b}\right)\alpha_1^2(t)\right], \tag{38}$$

derived from the corresponding generating function v the shape of which is given by (15) in [2], the validity of the inequalities (31) in the process $\{\lambda(t), \nu(t), \mu(t)\}$ with the properties (37) can be shown.

We come to the same conclusions for other types of processes $\{\lambda(t), \nu(t), \mu(t)\}$ in which only one ratio of two arbitrary rates is constant everywhere in T and the third rate is independent of the first two.

Consider the immigration-death process $\xi(t)$ with states E_x $(x = 0, 1, 2, \ldots)$ and the immigration and death rates $\nu(t)$ and $\mu(t)$. The changes in the assumptions (see Chap. 5) and in the differential equation (29) are evident and follow after application of identity $\lambda(t) \equiv 0$ in them.

Theorem 5. *Let $\nu(t)$ and $\mu(t)$ be positive functions, continuous in T. Then in the process $\{\nu(t), \mu(t)\}$ two equalities*

$$(39) \qquad \frac{\alpha_1(t)\,[\alpha_1(t) + 1]}{\alpha_2(t)} = 1 \quad \textit{and} \quad \frac{\alpha_3(t) - \alpha_1(t)\,[1 + \alpha_1^2(t)]}{\alpha_1^2(t)} = 3$$

hold simultaneously for every $t \in T$.

Proof. In view of the preliminary note in this chapter we obtain the final expressions of the moments $\alpha_i(t)$, $i = 1, 2, 3$, from (34) to (36) in the forms

$$(40) \qquad \alpha_1(t) = \int_0^t \nu(\tau)\, e^{-\int_\tau^t \mu(s) \mathrm{d}s}\, \mathrm{d}\tau\,,$$

$$(41) \qquad \alpha_2(t) = \alpha_1(t)\,[\alpha_1(t) + 1]$$

and

$$(42) \qquad \alpha_3(t) = [\alpha_1^2(t) + 3\,\alpha_1(t) + 1]\,\alpha_1(t)\,.$$

Hence the equalities to be proved are evident.

Note. In paper [2] the probability generating functions were derived for other types of immigration-death processes especially for the case when

$$\frac{\nu(t)}{\mu(t)} = \int_0^t \nu(\tau)\,\mathrm{d}\tau \qquad \text{everywhere in } T$$

and for the case when

$$\frac{\mu(t)}{\nu(t)} = a = \text{const.} \quad \text{everywhere in } T\,.$$

It can easily be shown that in these cases, too, the two equalities given in (39) remain valid.

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Souhrn

O MOMENTECH V NEHOMOGENNÍCH PROCESECH ROZENÍ, IMIGRACE A UMÍRÁNÍ

VRATISLAV HORÁLEK

V práci jsou odvozeny vztahy mezi prvními třemi obecnými momenty $\alpha_i(t) = \mathsf{E}[\xi^i(t)]$, $i = 1, 2, 3$ nehomogenních procesů: procesu rození-umírání, procesu rození, procesu umírání, procesu rození-imigrace-umírání a procesu imigrace-umírání. Předpokládá se, že intenzita rození $\lambda(t)$, intenzita umírání $\mu(t)$ a intenzita imigrace $\nu(t)$ jsou positivními a spojitými funkcemi v intervalu $T = (0, \infty)$.

Je ukázáno, že

a) v každém procesu rození-umírání platí pro každé $t \in T$

$$\frac{\alpha_1(t)\,[\alpha_3(t) - \alpha_1(t)]}{\alpha_2^2(t) - \alpha_1^2(t)} = \frac{3}{2}\,;$$

b) v každém procesu rození platí pro každé $t \in T$

$$\frac{\alpha_1(t)}{\alpha_2(t)}\,[2\,\alpha_1(t) - 1] = 1$$

a

$$\frac{\alpha_3(t) - \alpha_1(t)}{[\alpha_2(t) + \alpha_1(t)]\,[\alpha_1(t) - 1]} = 3\,;$$

c) v každém procesu umírání platí pro každé $t \in T$

$$A(t) > 1 \quad \text{a} \quad B(t) < 3\,;$$

d) v každém procesu rození-imigrace-umírání platí pro každé $t \in T$

$$A(t) < 1 \quad \text{a} \quad B(t) > 3\,;$$

e) v každém procesu imigrace-umírání platí pro každé $t \in T$

$$A(t) = 1 \quad \text{a} \quad B(t) = 3\,,$$

kde

$$A(t) = \frac{\alpha_1(t)\,[\alpha_1(t) + 1]}{\alpha_2(t)}$$

a

$$B(t) = \frac{\alpha_3(t) - \alpha_1(t)\,[1 + \alpha_1^2(t)]}{\alpha_1^2(t)}\,.$$

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