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ON PERIODIC SOLUTIONS OF SOME EQUATIONS OF MATHEMATICAL PHYSICS

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This note is devoted to the problem of finding the 2π-periodic (in t) solutions of equations

$$P_1\left(\frac{\partial}{\partial t}\right)\frac{\partial^{2m}u}{\partial x^{2m}}(t,x) + P_2\left(\frac{\partial}{\partial t}\right)u(t,x) = f(t,x)\,, \quad x \in \langle 0, a\rangle \tag{1a}$$

$$P_1\left(\frac{\partial}{\partial t}\right)\frac{\partial^{2m}u}{\partial x^{2m}}(t,x) + P_2\left(\frac{\partial}{\partial t}\right)u(t,x) = \varepsilon f(t,x,\mathscr{D}u)\,, \quad x \in \langle 0, a\rangle \tag{2}$$

with the boundary conditions

$$\frac{\partial^{2k}u}{\partial x^{2k}}(t,0) = \frac{\partial^{2k}u}{\partial x^{2k}}(t,a) = 0\,, \quad k = 0, 1, \ldots, m-1\,, \tag{1b}$$

where f is 2π-periodic in t, $P_1(\xi)$, $P_2(\xi)$ are polynomials of the orders s_1, s_2 with complex coefficients, $P_1(i\eta) \neq 0$ for η real. By $\mathscr{D}u$ we denote the vector of certain derivatives of u (see Remark 2). Many equations of physics are included in (1a) and will be discussed in the end of the paper.

First, let the right hand side of (1a) be of the form $f(t,x) = f_n(x)\exp(int)$, $f_n \in C(\langle 0, a\rangle)$ and suppose the solution to be in the same form, i.e. $u(t,x) = u_n(x)\,.\,\exp(int)$. Then $u_n(x)$ must satisfy the equation

$$P_1(in)\,u_n^{(2m)}(x) + P_2(in)\,u_n(x) = f_n(x)\,, \quad x \in \langle 0, a\rangle \tag{3a}$$

and the boundary conditions

$$u_n^{(2k)}(0) = u_n^{(2k)}(a) = 0\,, \quad k = 0, 1, \ldots, m-1\,. \tag{3b}$$

Let us denote $b = b(n) \equiv -P_2(in)\,[P_1(in)]^{-1}$ and let $\beta_1, \beta_2, \ldots, \beta_m, \beta_{m+1}, \ldots, \beta_{2m}$ be the roots of the equation

$$\lambda^{2m} = b\,. \tag{4}$$

As $-\beta$, β are roots of (4) (if β solves (4)) it is possible to arrange $\beta_1, \beta_2, \ldots, \beta_m$, $\beta_{m+1}, \ldots, \beta_{2m}$ so that $\operatorname{Re} \beta_j \geqq 0$, $\beta_{m+j} = -\beta_j$ for $j = 1, 2, \ldots, m$. Using the notation $S_j(x) = \exp(\beta_j x) - \exp(-\beta_j x)$, $C_j(x) = \exp(\beta_j x) + \exp(-\beta_j x)$ we can formulate

Lemma 1. (a) *If $b \neq (-1)^m (k\pi/a)^{2m}$ for every integer k, then for every continuous function $f(x)$ on $\langle 0, a \rangle$ there exists a unique solution $u(x)$ of the equation*

$$u^{(2m)}(x) - b\, u(x) = f(x) \tag{5}$$

satisfying the boundary conditions (3b) and it is of the form

$$u(x) = -\sum_{j=1}^{m} \left\{ \int_0^x K_j(x, \xi) f(\xi) \, d\xi + \int_0^{a-x} K_j(a - x, \xi) f(a - \xi) \, d\xi \right\} \tag{6}$$

where

$$K_j(x, \xi) = B_j\, S_j(a - x)\, S_j(\xi)\, S_j^{-1}(a), \quad B_j^{-1} = 2\beta_j \prod_{\substack{j \neq k \\ k=1}}^{m} (\beta_j^2 - \beta_k^2). \tag{7}$$

(b) *Let k be positive integer so that $b = (-1)^m (k\pi/a)^{2m}$ and let $\beta_1 = ik\pi/a$. If $f(x)$ is continuous function on $\langle 0, a \rangle$ then the problem (5), (3b) has a solution if and only if*

$$\int_0^a f(\xi) \sin(-i\, \beta_1(n)\, \xi) \, d\xi = 0 \tag{8}$$

and it is of the form

$$\begin{aligned} u(x) = -\sum_{j=2}^{m} \left\{ \int_0^x K_j(x, \xi) f(\xi) \, d\xi + \int_0^{a-x} K_j(a - x, \xi) f(a - \xi) \, d\xi \right\} + \\ + B_1 \int_0^x S_1(x - \xi) f(\xi) \, d\xi + B \sin(-i\, \beta_1(n)\, \xi), \end{aligned} \tag{9}$$

where B is an arbitrary constant.

(c) *If $b = 0$, $f(x)$ is continuous on $\langle 0, a \rangle$ then there exists a unique solution $u(x)$ of (5), (3b) and it is of the form*

$$u(x) = \int_0^x Q_1(x - \xi) f(\xi) \, d\xi + \int_0^a Q_2(x, \xi) f(\xi) \, d\xi, \tag{10}$$

where Q_1, Q_2 are polynomials of the order $2m - 1$. In all cases $u(x)$ has $2m$ continuous derivatives.

Proof. It is known from the theory of ordinary differential equations that the solution $u(x)$ of (5) and (3b) is of the form (for $b \neq 0$)

$$u(x) = \int_0^x \left\{ \sum_{j=1}^{2m} B_j \exp(\beta_j(x - \xi)) f(\xi) \right\} d\xi + \sum_{j=1}^{2m} \tilde{B}_j \exp(\beta_j x), \tag{11}$$

where the vector $B = (B_1, B_2, \ldots, B_{2m})$ solves the system of equations

$$\sum_{j=1}^{2m} \beta_j^l B_j = \delta_{l(2m-1)}\,, \quad l = 0, 1, \ldots, 2m - 1 \tag{12}$$

(δ_{ls} is Kronecker delta) and the vector $\tilde{B} = (\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_{2m})$ is determined so that $u(x)$ satisfies the boundary conditions (3b), i.e. $\tilde{B}$ solves the system of linear equations

$$\sum_{j=1}^{2m} \beta_j^{k2} \tilde{B}_j = 0\,, \tag{13}$$

$$\sum_{j=1}^{2m} \beta_j^{2k} \exp(\beta_j a)\, \tilde{B}_j = -\int_0^a \{\sum_{j=1}^{2m} B_j \beta_j^{2k} \exp[\beta_j(a - \xi)]\} f(\xi)\, d\xi$$

$$k = 0, 1, \ldots, m - 1\,.$$

As $\beta_{m+j} = -\beta_j$ $(j = 1, 2, \ldots, m)$, m equations from (12) for $l = 0, 2, \ldots, 2m - 2$ can be reduced to the system

$$\sum_{j=1}^{m} (B_j + B_{m+j})\, \beta_j^{2k} = 0\,, \quad k = 0, 1, \ldots, m - 1\,, \tag{12a}$$

which implies $B_{m+j} = -B_j$ $(j = 1, 2, \ldots, m)$. Then the system of m equations for $l = 1, 3, \ldots, 2m - 1$ from (12) assumes the form

$$\sum_{j=1}^{m} \beta_j^{2k}(2\beta_j B_j) = \delta_{k,m-1}\,, \quad k = 0, 1, \ldots, m - 1\,.$$

This system has the unique solution

$$2\beta_j B_j = (-1)^{j+m}\, W(\beta_1^2, \ldots, \beta_{j-1}^2, \beta_{j+1}^2, \ldots, \beta_m^2)\, W^{-1}(\beta_1^2, \ldots, \beta_m^2) = [\prod_{\substack{j \neq k\\ k=1}}^{m} (\beta_j^2 - \beta_k^2)]^{-1}$$

where W is Van der Monde determinant. Further, the first m equations of (13) are (due to $\beta_{m+j} = -\beta_j$) of the same form as (12a) and hence $-\tilde{B}_j = \tilde{B}_{m+j}$. Substituting these results into the last m equations of (13) we obtain the system

$$\sum_{j=1}^{m} \beta_j^{2k}[\tilde{B}_j\, S_j(a)] = -\int_0^a \sum_{j=1}^{m} \beta_j^{2k}[B_j\, S_j(a - \xi)]\, f(\xi)\, d\xi$$

which has the solution

$$\tilde{B}_j = -S_j^{-1}(a)\, W^{-1}(\beta_1^2, \ldots, \beta_m^2) \int_0^a \sum_{k=0}^{m-1} \sum_{l=1}^{m} (-1)^{j+k+1}\, B_l \beta_l^{2k}\, S_l(a - \xi)\, f(\xi)\,.$$

$$.\, W_{k+1\,j}(\beta_1^2, \ldots, \beta_m^2)\, d\xi = -S_j^{-1}(a) \int_0^a \sum_{l=1}^{m} B_l\, S_l(a - \xi)\, \delta_{jl}(\xi)\, d\xi =$$

$$= -S_j^{-1}(a) \int_0^a B_j\, S_j(a - \xi)\, f(\xi)\, d\xi$$

if $S_j(a) \neq 0$ while $\tilde{B}_j$ may be chosen arbitrarily if $S_j(a) = 0$ and (8) holds (W_{kj} is the minor of W). Using this expression in (11) we get

$$u(x) = \sum_{j=1}^{m} B_j \left\{ \int_0^x S_j(x - \xi) f(\xi) \,\mathrm{d}\xi - \int_0^a S_j(a - \xi) \, S_j(x) \, S_j^{-1}(a) f(\xi) \,\mathrm{d}\xi \right\}$$

in the case (a) and

$$u(x) = \sum_{j=2}^{m} B_j \left\{ \int_0^x S_j(x - \xi) f(\xi) \,\mathrm{d}\xi - \int_0^a S_j(a - \xi) \, S_j(x) \, S_j^{-1}(a) f(\xi) \,\mathrm{d}\xi + \right.$$
$$+ B_1 \int_0^x S_1(x - \xi) f(\xi) \,\mathrm{d}\xi + B \, S_1(x)$$

for (b). Dividing the second integral in these expressions into two parts $\int_0^x + \int_x^a$ and adding the integral $\int_0^x$ to the first one we get the formulas (6) and (9). (c) follows easily from the theory of ordinary differential equations.

For $b = b(n)$ we denote by N_1 the set of integers n such that there exists $j \in \{1, 2, \ldots, m\}$ which satisfies $S_j(a) = 0$.

Lemma 2. *Let the polynomials P_1, P_2 satisfy one of the following conditions*:

(α) $s_1 > s_2$;

(β) *there exists a constant $K > 0$ so that*

$$[\mathrm{Re}\,(a\,\beta_j(n))]^2 + [\sin\{\mathrm{Im}\,(a\,\beta_j(n))\}]^2 \geqq K$$

for $j = 1, 2, \ldots, m$ and for $n \notin N_1$ sufficiently large;

(γ) *there exist constants $\tilde{C} > 0$, $0 \leqq \alpha < +\infty$ so that either*

$$|b(n)|^{1/2m} \min |\arg \beta_j(n) - (l + \tfrac{1}{2})\,\pi| \geqq \tilde{C}|n|^{-\alpha}$$

or

$$\min\,(a\,|m\,\beta_j(n) - l\pi|) \geqq \tilde{C}|n|^{-\alpha} \quad \textit{for every} \quad j = 1, 2, \ldots, m$$

and for $n \notin N_1$ large enough (the minimum is taken over all integers l).

Let $f_n \in C(\langle 0, a \rangle)$, $n = 1, 2, \ldots$ satisfy in the case (b) *of Lemma* 1 *the assumption* (8).

Then there exists a constant C such that for at least one solution u_n of (3) *the inequality*

$$|u_n^{(k)}(x)| \leqq C|n|^{\alpha - s_2 + (s_2 - s_1)(k+1)/2m} \int_0^a |f_n(\xi)| \,\mathrm{d}\xi \tag{14}$$

holds for $x \in \langle 0, a \rangle$, $k = 0, 1, \ldots, 2m - 1$ with $\alpha = 0$ in the cases (α), (β).

Proof. By Lemma 1 (with $b = b(n) = -P_2(in)\,[P_1(in)]^{-1}$, $f = [P_1(in)]^{-1} f_n$) the solution $u_n(x)$ of (3) exists and is of the form (6), (9), or (10). As $|P_i(in)| \sim |n|^{s_i}$

for $n \to +\infty$, $i = 1, 2$, $B_j = B_j(n)$ from (7) may be estimated as follows

(15) $|B_j(n)|^{-1} \geqq C|b(n)|^{(2m-1)/2m} \geqq C|n|^{(s_2-s_1)(1-1/2m)}$ (c positive constant).

(α) If $s_1 > s_2$ then $|b(n)| \to 0$ for $|n| \to +\infty$ which implies $|b(n)|\, a < \pi/2$ for n large enough and

$$|S_j(\xi)| = 2\{[\operatorname{sh}(\operatorname{Re}(\beta_j\xi))]^2 + [\sin(\operatorname{Im}(\beta_j\xi))]^2\}^{1/2} \leqq |S_j(a)| \quad \text{for} \quad \xi \in \langle 0, a\rangle .$$

As $|S_j(a-x)|$, $|C_j(a-x)| \leqq$ const. for $b(n)$ bounded we can write

$$\left|\frac{\partial^k K_j}{\partial x^k}(x, \xi)\right| \leqq C|b|^{-1+1/2m}\,|b|^{k/2m} \leqq C|n|^{-(s_2-s_1)(1-(k+1)/2m)}$$

which implies (14) for $s_1 > s_2$.

(β) Let $s_1 \leqq s_2$, $M_j = \{n$ integer; $[a \operatorname{Re}(\beta_j(n))]^2 + [\sin(a \operatorname{Im}\beta_j(n))]^2 \geqq K\}$ for $j = 1, 2, \ldots, m$. Now, the ratios $S_j(a-x)\, S_j(\xi)/S_j(a)$, $C_j(a-x)\, S_j(\xi)/S_j(a)$ are bounded for $n \in M_j$, $x, \xi \in \langle 0, a\rangle$, $\xi < x$.

Then the derivatives $\partial^k K_j/\partial x^k(x, \xi)$ of the kernel $K_j(x, \xi)$ can be estimated (using (15)) by $C|n|^{-s_2+(s_2-s_1)(k+1)/2m}$, where C does not depend on j, n.

Let condition (γ) be fulfilled and $s_1 \leqq s_2$, $n \notin N_1$. As $S_j(a-x)\, S_j(\xi)$ and $C_j(a-x)\, S_j(\xi)$ are bounded the following inequalities hold:

$$\begin{aligned}
2|S_j(a)| &\geqq a \operatorname{Re}\beta_j(n) + |\sin(a \operatorname{Im}\beta_j(n))| \geqq \\
&\geqq |b(n)|^{1/2m}\, a|\cos(\arg\beta_j(n))| + |\sin(a \operatorname{Im}\beta_j(n))| \geqq \\
&\geqq |b(n)|^{1/2m}\, a|\sin(\arg\beta_j(n) + \pi/2 - l\pi)| + |\sin(a \operatorname{Im}\beta_j(n) - l\pi)| \geqq \\
&\geqq |b(n)|^{1/2m}\, a \min|\sin(\arg\beta_j(n) + \pi/2 - l\pi)| + \min|\sin(\operatorname{Im} a\, \beta_j(n) - l\pi)| \geqq \\
&\geqq \tfrac{1}{2}\{|b(n)|^{1/2m}\, a|\arg\beta_j(n) + \pi/2 - l_j(n)\,\pi| + |\operatorname{Im}(a\,\beta_j(n)) - \bar{l}_j(n)\,\pi|\} \geqq \\
&\geqq C|n|^{-\alpha}
\end{aligned}$$

for l integer, n large enough, $l_j(n)$, $\bar{l}_j(n)$ being integers which minimize the expressions in (γ).

By (γ) the lower bound of the last term is $C|n|^{-\alpha}$ and hence the derivatives $\partial^k K_j/\partial x^k(x, \xi)$ of the kernel $K_j(x, \xi)$ are estimated by $C|B_j(n)|\, |b(n)|^{k/2m}\, |n|^{\alpha}$, where C does not depend on j, n, x, ξ, k ($k = 0, 1, \ldots, 2m-1$). Putting $B = 0$ in (9) the estimate of $K_j(x, \xi)$ may be obtained for $n \in N_1$. Lemma 2 follows from the estimations given above and the following formula

$$u^{(k)}(x) = -[P_1(in)]^{-1} \sum_{j=1}^{m} \left\{\int_0^x \frac{\partial^k K_j}{\partial x^k}(x, \xi)\, f(\xi)\, \mathrm{d}\xi + (-1)^k \,.\right.$$

$$\left. . \int_0^{a-x} \frac{\partial^k K_j}{\partial x^k}(a-x, \xi)\, f(a-\xi)\, \mathrm{d}\xi\right\}, \quad k = 0, 1, \ldots, 2m-1 .$$

Finally, the $2m$-th derivatives can be estimated by means of the equation (3a).

Remark 1. The growth $u_n(x)$ (if $|n| \to +\infty$) is given by the distance of the set $\{\beta_j(n), j = 1, 2, \ldots, m\}$ from the sequence $\{il\pi/a;\ l \text{ integrer}\}$, where $(il\pi/a)^{2m}$ are the eigenvalues of the operator d^{2m}/dx^{2m} with the boundary conditions (3b). Let H_l be the space of 2π-periodic functions $v(t)$ whose derivatives (in the sense of distributions) up to the order l $(l = 0, 1, \ldots)$ are square integrable on $\langle 0, 2\pi\rangle$ with the norm

$$\|v\|_l = \left[\sum_{j=0}^{l} \int_0^{2\pi} |v^{(j)}(t)|^2 \, dt\right]^{1/2} .$$

As the system $\{(2\pi)^{-1/2} \exp(int)\}_{n=-\infty}^{+\infty}$ is complete in H_l the function $v(t)$ belongs to H_l if and only if the coefficients

$$v_n = (2\pi)^{-1/2} \int_0^{2\pi} v(t) \exp(int) \, dt$$

satisfy

$$\sum_{n=-\infty}^{+\infty} |n|^{2l} |v_n|^2 < +\infty .$$

Then

$$\|v\|_l^2 = \sum_{n=-\infty}^{+\infty} |n|^{2l} |v_n|^2 .$$

Now, denoting

$$\frac{du}{dx}(\cdot, x) = \lim_{h\to 0} \frac{u(\cdot, x+h) - u(\cdot, x)}{h}$$

in the norm H_l we define the spaces $C^k(\langle 0, a\rangle, H_l) = \{u(t, x);\ d^j u/dx^j$ is a continuous function on $\langle 0, a\rangle$ in the norm of H_l, $0 \leqq j \leqq k\}$ with the norm

$$\|u\|_{k,l} = \max_{0\leqq j\leqq k} \max_{x\in\langle 0,a\rangle} \left\|\frac{d^j u}{dx^j}(\cdot, x)\right\|_l .$$

Proposition 1. *The function $u(t, x)$ belongs to $C^k(\langle 0, a\rangle, H_l)$ if and only if the Fourier coefficients $u_n(x)$ of the function $u(t, x)$ have continuous derivatives up to the order k on $\langle 0, a\rangle$ and*

$$\sum_{n=-\infty}^{+\infty} |n|^{2l} |u_n^{(j)}(x)|^2$$

converges uniformly with respect to $x \in \langle 0, a\rangle$ for $j = 0, 1, \ldots, k$.

Sufficiency of this proposition follows from the definition of H_l. Theorem of Dini and the formula

$$u_n^{(k)}(x) = (2\pi)^{-1/2} \int_0^{2\pi} \frac{d^k u}{dx^k}(t, x) \exp(int) \, dt$$

imply the necessity of the above condition.

By the imbedding theorems, if $u \in C^k(\langle 0, a\rangle, H_{l+1})$ then

$$\frac{\mathrm{d}^j u}{\mathrm{d}x^j}(t, x) = \frac{\partial^j u}{\partial x^j}(t, x) \quad (j = 0, 1, \ldots, k)$$

and all derivatives

$$\frac{\partial^{j+j'} u}{\partial x^j \, \partial t^{j'}}(t, x) \quad (j = 0, 1, \ldots, k, j' = 0, 1, \ldots, l)$$

are continuous on $\langle 0, 2\pi\rangle \times \langle 0, a\rangle$.

Theorem. *Let the polynomials P_1, P_2 satisfy the assumptions of Lemma 2 and $s_1 \leqq s_2$. Let $f \in C(\langle 0, a\rangle, H_r)$, where r is the smallest integer such that $r \geqq \alpha + (s_2 - s_1)/2m + 1$.*

If $N_1 = \emptyset$ then there exists a unique solution $u(t, x)$ of the problem (1),

$$u \in C^{2m}(\langle 0, a\rangle, H_{s_1+1}) \cap C(\langle 0, a\rangle, H_{s_2+1}) .$$

If $N_1 \neq \emptyset$ then the solution $u(t, x)$,

$$u \in C^{2m}(\langle 0, a\rangle, H_{s_1+1}) \cap C(\langle 0, a\rangle, H_{s_2+1})$$

exists if and only if

$$(16) \qquad \int_0^a \int_0^{2\pi} f(t, x) \sin\left(-i\, \beta_1(n)\, x\right) \exp\left(int\right) \mathrm{d}t \, \mathrm{d}x = 0 \quad \textit{for every} \quad n \in N_1 ,$$

where $\beta_1(n) = i\, k(n)\, \pi/a$ (see Lemma 1).

The solution u is of the form

$$u(t, x) = (2\pi)^{-1/2} \sum_{n=-\infty}^{+\infty} u_n(x) \exp\left(int\right)$$

where $u_n(x)$ is obtained from (6) for $n \notin N_1$, $b(n) \neq 0$, from (9) with $B = 0$ for $n \in N_1$ and from (10) for $b(n) = 0$ with

$$f(x) = \left[P_1(in)\right]^{-1} \int_0^{2\pi} f(t, x) \exp\left(int\right) \mathrm{d}t \,, \quad b = -P_2(in) \left[P_1(in)\right]^{-1} .$$

Moreover, the following estimate holds:

$$(17) \qquad \sum_{k,l} \max_{t,x} \left| \frac{\partial^{l+k} u}{\partial t^l \, \partial x^k}(t, x) \right| \leqq C |f|_{0,r} ,$$

where

$$l + (s_2 - s_1)(k + 1)/2m \leqq r + s_2 - \alpha \,, \quad t \in \langle 0, 2\pi\rangle \,, \quad x \in \langle 0, a\rangle .$$

Proof. For $N_1 = \emptyset$ the existence of the solution follows from Lemmas 1 and 2 and Proposition 1. Let u_1, u_2 be two solutions of (1). $u = u_1 - u_2$ is a solution of (1) for $f \equiv 0$ and

$$u \in C^{2m}(\langle 0, a\rangle, H_{s_1+1}) \cap C(\langle 0, a\rangle, H_{s_2+1}) .$$

By Proposition 1 $u(t, x)$ is of the form

$$u(t, x) = \sum_{n=-\infty}^{+\infty} (2\pi)^{-1/2} u_n(x) \exp(int) . \tag{18}$$

This series and all the series obtained by the formal differentiation of (18) involved in (1a) converge uniformly on $\langle 0, 2\pi\rangle \times \langle 0, a\rangle$. Putting (18) into equation (1a) we get (due to the completeness of the orthonormal system $\{(2\pi)^{-1/2} \exp(int)\}_{n=-\infty}^{+\infty}$ in the spaces H_l, l being positive integer) that $u_n(x)$ is a solution of the problem (3) with $f_n \equiv 0$. By Lemma 1, $u_n(x) \equiv 0$ for n integer. Hence $u(t, x) \equiv 0$. Let $N_1 \neq 0$ and let the function $f(t, x)$ satisfy (16). Due to the estimates (14) for $u_n(x)$ from Lemma 2 the series of the form (18) and those obtained by the formal differentiation of (18) involved in (1a) are convergent uniformly on $\langle 0, 2\pi\rangle \times \langle 0, a\rangle$ and hence $u(t, x)$ solves (1). If $u(t, x)$ is a solution of (1) and

$$u \in C^{2m}(\langle 0, a\rangle, H_{s_1+1}) \cap C(\langle 0, a\rangle, H_{s_2+1})$$

then the n-th Fourier coefficient $u_n(x)$ of $u(t, x)$ solves (3). By Lemma 1 (8) holds for every $n \in N_1$, which implies (16). The estimate (17) follows from those of Lemma 2 and from Proposition 1 and imbedding theorems.

Remark 2. The solution u of the weakly nonlinear problem (2), (1a) may be found using the theorem given above and either the fixed point theorem for $N_1 = \emptyset$ or the theorem by O. Vejvoda and M. Sova ([1], [2]). $\mathscr{D}u$ is a vector of all derivatives $\partial^{l+k}u/\partial t^l\, \partial x^k(t, x)$ of $u(t, x)$ such that $l + (s_2 - s_1)(k + 1)/2m \leqq s_2 - \alpha$.

Examples:

$m = 1$:

(1) The *heat conduction equation*

$$u_{xx} - u_t + cu = f$$

with boundary conditions

$$u(t, 0) = u(t, \pi) = 0 . \tag{19}$$

In this case $P_1(\xi) \equiv 1$, $P_2(\xi) = -\xi + c$, $b(n) = in - c$. Then $N_1 = \emptyset$ for c, a satisfying $ca^2/\pi^2 \neq k^2$ (k integer) and $N_1 = \{0\}$ for $c = k^2\pi^2/a^2$ (k integer). As $\frac{1}{2} \arg b(n) = \frac{1}{2} \operatorname{arctg}(-n/c)$ tends to $\mp\pi/2$ for $n \to \pm\infty$ then α from Lemma 2 is equal to 0 and (17) holds for $r = 2$, $k/2 + l \leqq \frac{3}{2}$. The necessary and sufficient condition for the existence of the solution of this problem is given by the condition

$$\int_0^a \int_0^{2\pi} f(t, x) \sin (x \sqrt{c}) \, dx \, dt = 0 \tag{20}$$

(if $ca^2/\pi^2 = k^2$ – k-integer).

(2) *The telegraph equation*

$$u_{xx} - u_{tt} + 2au_t + cu = f, \quad a \neq 0 \tag{21}$$

with the boundary conditions (19).

For $c \neq k^2\pi^2/a^2$ it is $\alpha = 0$, $N_1 = \emptyset$ and (17) holds with $r = 2$, $k + l \leqq 3$. If $c = k^2\pi^2/a^2$ then $N_1 = \{0\}$ and the condition (16) is again of the form (20).

(3) The equation (21) for $a = 0$, $c = 0$ is the wave equation, i.e.

$$u_{xx} - u_{tt} = f$$

(the boundary conditions are of the form (19)). As $P_1(\xi) \equiv 1$, $P_2(\xi) = -\xi^2$ we have $b(n) = -n^2$, $\beta_1(n) = i|n|$. Hence

$$|S_1(a)| = 2|\sin (na)| \geqq \tfrac{1}{2} \min_l |na - l\pi| \, , \quad N_1 = \{n, na/\pi \text{ is integer}\} \, , \quad \alpha = 0$$

for a such that a/π is a rational number and $N_1 = \emptyset$, α may be positive for a such that a/π is an irrational number. (17) holds with $r \geqq 2 + \alpha$, $k + l \leqq 3$.

(4) The *equation of vibrations with inner friction*

$$u_{xx} - u_{tt} + au_{txx} = f, \quad a \neq 0$$

with the boundary conditions (19).

In this case $P_1(\xi) = a\xi + 1$, $P_2(\xi) = -\xi^2$, $b(n) = -n^2/(1 + ina)$, $N_1 = \emptyset$, $\alpha = 0$ and (17) holds with $r = 2$, $k/2 + l \leqq \frac{7}{2}$.

$m = 2$: The *vibrations of the bar* of the length a with fixed ends is described by the equation

$$u_{xxxx} + u_{tt} = f$$

and by the boundary conditions

$$u(t, 0) = u(t, a) = u_{xx}(t, 0) = u_{xx}(t, a) = 0 \, .$$

In this case

$$P_1(\xi) \equiv 1 \, , \quad P_2(\xi) = \xi^2 \, , \quad b(n) = n^2 \, , \quad \beta_1(n) = \sqrt{|n|} \, , \quad \beta_2(n) = i \sqrt{|n|} \, ,$$
$$N_1 = \{n; a \sqrt{|n|}/\pi \text{ is integer}\} \, ,$$

α is defined by the growth of $\min_l |a\sqrt{|n|} - l\pi|$ for $|n| \to +\infty$, (17) holds with $r = [\alpha + \frac{3}{2}] + 1$ if $\alpha + \frac{3}{2}$ is not integer and $r = \alpha + \frac{3}{2}$ if $\alpha + \frac{3}{2}$ is integer, $k/2 + l \leqq \leqq r + \frac{3}{2} - \alpha$.

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Souhrn

O PERIODICKÝCH ŘEŠENÍ JEDNOHO TYPU ROVNIC MATEMATICKÉ FYZIKY

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Na základě příkladů z fyziky, které jsou uvedeny na konci článku je vyšetřována úloha najít periodické řešení obecné rovnice

$$P_1\left(\frac{\partial}{\partial t}\right)\frac{\partial^{2m}u}{\partial x^{2m}} + P_2\left(\frac{\partial}{\partial t}\right)u = f(t, x)\,, \quad x \in \langle 0, a\rangle\,,\ t \in \langle 0, 2\pi\rangle$$

s homogenními okrajovými podmínkami Dirichletova typu.

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