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ALGORITMY

32. SOMMERFELD COX

COMPUTATION OF SOMMERFELD'S ATTENUATION FUNCTION

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This procedure computes the complex-valued Sommerfeld attenuation function, $G(p)$, which appears within the theory of propagation of electromagnetic waves [11]:

$$G(p) = 1 + i \sqrt{(\pi p)} e^{-p} \operatorname{erfc}(-i \sqrt{p})$$

where

$$\operatorname{erfc}(-i \sqrt{p}) = \frac{2}{\sqrt{\pi}} \int_{-i\sqrt{p}}^{\infty} e^{-t^2} dt,$$

provided that $0 \leq \arg(p) \leq \pi/2$. This function has been tabulated [7].

By means of the function $w(z)$ [6], defined by

$$w(z) = e^{-z^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2} dt \right) = e^{-z^2} \operatorname{erfc}(-iz)$$

the function $G(p)$ can be expressed as

$$G(p) = 1 + i \sqrt{(\pi p)} w(\sqrt{p}).$$

The function $w(z)$ can be approximated by means of [4], and a way to find $G(p)$ for a given value of p could simply comprise a determination of $w(\sqrt{p})$. But due to the structure of the approximation of $w(z)$ the connection between $w(\sqrt{p})$ and $G(p)$ can be taken into account in a more efficient way.

Given a value of $p = pr + ipi$ then $\sqrt{p} = \operatorname{sqr}t(p) = \operatorname{sqr}t(pr + ipi) = x + iy = z$ is computed according to a method [8], which is used in [3]. Then depending on the value of z the approximation of $G(p)$ is performed by one of two different methods:

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1) Small values of $|z|$:

The function $w(z)$ is written as

$$w(z) = e^{-z^2} + \frac{2i}{\sqrt{\pi}} z f\left(\frac{z^2}{5}\right)$$

where $f(t)$ is approximated using Lanczos' τ -method ([9], p. 489, ex. 5), but instead of using Chebyshev polynomials in the error term, it turns out to be better to use Legendre polynomials. In [2] section 3 the formulas are derived, and $f(t)$ is approximated as the ratio between two polynomials with real coefficients (of degree 10) in the complex variable $t = z^2/5$: $f(t) \approx T(t)/N(t)$.

This means that the function $G(p)$ can be written

$$G(p) = 1 + i \sqrt{(\pi p)} e^{-p} - 10 \frac{\frac{p}{5} T\left(\frac{p}{5}\right)}{N\left(\frac{p}{5}\right)},$$

where $p/5$ $T(p/5)$ and $N(p/5)$ are polynomials (with complex variable) which can be evaluated as in [4] using a procedure *PK* which is a simplified version of [1]; the method is given in [9], p. 16. In the ALGOL-text this part begins with the comment: Legendre approximation.

2) Large values of $|z|$:

The value of $w(z)$ is found as shown in [4] section 2.2 using a Gauss-Hermite quadrature, from which the function $G(p)$ is computed. In the ALGOL-text this part begins with the comment: Hermite quadrature.

Depending on the value of p , the following approximate execution times are obtained in the GIER ALGOL 4 system (where – for comparison – a call of the procedure *exp(x)* takes 4.4 msec ([10], p. 76)):

$$\begin{aligned} 0 \leq \arg(p) \leq \pi/2 : \text{small } |p| &: \text{approx. } 100 \text{ msec} \\ 0 \leq \arg(p) \leq \pi/2 : \text{large } |p| &: \text{approx. } 50 \text{ msec} \\ \arg(p) \text{ not in the interval} &: \text{approx. } 10 \text{ msec} . \end{aligned}$$

No many-decimal table of the function $G(p)$ seems to exist, and consequently no direct test of the approximation has been possible. However, the accuracy can be estimated using the information about the computation of the function $w(z)$ ([4], section 4): $\text{Re}(w(z))$ and/or $\text{Im}(w(z))$ can have an absolute error up to 1.5×10^{-6} , when z is in the neighbourhood of $1.5 + i 1.5$, i.e. p near $5i$. When $G(p)$ is determined from $w(z)$ (as shown above) the absolute error in $G(p)$ should not be greater than 10×10^{-6} when p is near $5i$. For smaller values of $|p|$ the absolute error is smaller. For larger values of $|p|$ (or $|z|$) the relative error in $w(z)$ has not been determined,

and the absolute error in $G(p)$ has been estimated as shown below. When $|p|$ is very small or very large the function $G(p)$ can easily be computed with high accuracy by means of simple formulas [11]. For $p = 0.01, 0.1, 50, 0.01i, 0.1i, 50i$ there was an error up to 2×10^{-8} in the results obtained by the procedure. This is in accordance with the fact that [4] is very accurate when $|z|$ is very small or very large. The procedure has also been tested in other ways (for example by comparing 441 pairs of values with the table [7]; for details, see [5] section 4.2.3), but the results of these tests can not change the following estimate of the accuracy of the approximation:

The absolute error in $G(p)$ is about $1 \times 10^{-5} - 1 \times 10^{-8}$.

boolean procedure *Sommerfeld cox* (*pr*, *pi*, *gr*, *gi*);

value *pr*, *pi*;

real *pr*, *pi*, *gr*, *gi*;

comment This procedure computes the value of the Sommerfeld attenuation function: $G(p)$.

The parameters are:

pr: real part of input p ,

pi: imaginary part of input p ,

gr: real part of output $G(p)$,

gi: imaginary part of output $G(p)$,

Sommerfeld cox: is **true** when $0 \leq \arg(p) \leq \pi/2$, otherwise it is **false**;

if $pr < 0 \vee pi < 0$

then *Sommerfeld cox* := **false**

else

begin

real x, y, M ;

Sommerfeld cox := **true**;

$M := pr^2 + pi^2$;

$x := \text{sqrt}(\text{sqrt}(M) + pr)/2$;

$y := \text{if } x = 0 \text{ then } 0 \text{ else } pi/2/x$;

if $y > 1.7 - 0.2 \times x \vee y > 3.9 - x$

then

begin comment Hermite quadrature;

real $p1, p2, p3, p4, p5, p6, n1, n2, n3, n4, n5, n6, a, b, T$;

$M := y^2$;

$a := b := 0$;

for $T := -x, x$ **do**

begin

$p1 := 0.31424\ 03763 + T$;

$p2 := 0.94778\ 83912 + T$;

$p3 := 1.59768\ 26352 + T$;

```

    p4 := 2.27950 70805 + T;
    p5 := 3.02063 70251 + T;
    p6 := 3.88972 48979 + T;
    n1 := 0.18147 96822/(p1↑2 + M);
    n2 := 0.08291 72776 3/(p2↑2 + M);
    n3 := 0.01642 73320 3/(p3↑2 + M);
    n4 := 0.00124 31244 32/(p4↑2 + M);
    n5 := 0.00002 72908 9347/(p5↑2 + M);
    n6 := 0.00000 00846 24328 41/(p6↑2 + M);
    a := a + n1 + n2 + n3 + n4 + n5 + n6;
    b := -b + p1 × n1 + p2 × n2 + p3 × n3
        + p4 × n4 + p5 × n5 + p6 × n6
end T;
gr := 1 - 1.77245 38509 × (x × b + M × a);
gi := 1.77245 38 509 + (x × a - b) × y
end Hermite quadrature
else
begin comment Legendre approximation;
real p1, p2, p3, n1, n2, t1, t2, T;
procedure PK(pa, pb, a0, a1, a2, a3, a4, a5, a6, a7, a8, a9, a10);
value a0, a1, a2, a3, a4, a5, a6, a7, a8, a9, a10;
real pa, pb, a0, a1, a2, a3, a4, a5, a6, a7, a8, a9, a10;
begin
p3 := a9 + T × a10;
p2 := a8 + T × p3 + M × a10;
p1 := a7 + T × p2 + M × p3;
p3 := a6 + T × p1 + M × p2;
p2 := a5 + T × p3 + M × p1;
p1 := a4 + T × p2 + M × p3;
p3 := a3 + T × p1 + M × p2;
p2 := a2 + T × p3 + M × p1;
p1 := (a1 + T × p2 + M × p3)/5;
pa := a0 + pr × p1 + M × p2;
pb := pi × p1
end PK;
T := 0.4 × pr;
M := -0.04 × M;
PK(t1, t2,
0, 12096.51250, -8488.78070,
14448.00988, -4495.93759, 3287.20821,
-519.3045, 210.21, -14.3,
3.3, 0);

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PK(n1, n2,
12096·51250, 31832·92763, 39914·35198,
31537·26576, 17481·0636 , 7151·3442 ,
2207·205 , 514·8 , 89·1 ,
11 , 1 );
p3 := 10/(n1↑2 + n2↑2);
p2 := cos(pi);
p1 := sin(pi);
T := 1·77245 38509 × exp(-pr);
gr := 1 + T × (x × p1 - y × p2) - p3 × (n1 × t1 + n2 × t2);
gi := T × (x × p2 + y × p1) - p3 × (n1 × t2 - n2 × t1)
end Legendre approximation
end 0 ≤ arg(p) ≤ phi/2
finis Sommerfeld cox;

```

Test values.

pr	pi	gr	gi
0·01	0	0·980 132 803	0·175 481 762
0·1	0	0·812 814 910	0·507 160 572
50	0	-0·010 316 145	0·000 000 000
0	0·01	0·875 794 815	0·106 578 972
0	0·1	0·631 896 434	0·234 452 957
0	50	0·000 298 977	0·009 985 086
1	0	-0·076 159 008	0·652 049 327
10	0	-0·060 75	0·000 25
0	1	0·190 47	0·232 20
0	10	0·006 96	0·048 35
10	10	-0·024 34	0·029 16

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