Jan Zítko

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KELLOGG'S ITERATIONS FOR GENERAL COMPLEX MATRIX

JAN ZÍTKO

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1. INTRODUCTION

Among the numerical methods of finding the spectral radius and an eigenvalue of maximal modulus of a matrix, Kellogg's iterative method is very important. The detailed behaviour of iterations is known for matrices with linear elementary divisors.

The purpose of this paper is to investigate in detail Kellogg's iterative process for matrices elementary divisors of which are not linear, generally with complex elements, and to find an exact expression for the k-th iteration. The convergence in such cases is very slow. However, at the end of this paper we construct two extrapolation procedures which give very good results. One of them is analogous to the Richardson-type extrapolation.

Before summarizing the main results of this paper, we mention the notation which will be used throughout the paper.

We shall denote the field of complex numbers by C; $\mathbf{V}_n(C)$ will be the n-dimensional vector space of vectors

$$\mathbf{x} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}, \quad \text{where} \quad \xi_i \in C\,.$$

The superscript T is used for transpose and H for conjugate and transpose, and analogously for matrices. The identity matrix of the dimension k will be denoted by I_k, the index will be omitted if the dimension is obvious. The i-th column of I_k will be denoted by $\mathbf{e}_i^{(k)}$ and $\mathbf{e}^{(k)} = \sum_{i=1}^{k} \mathbf{e}_i^{(k)}$. If $\mathbf{u}_i \in \mathbf{V}_n(C)$, $i = 1, 2, \ldots, l$, $\mathbf{u}_i = (\eta_1^{(i)}, \ldots, \eta_n^{(i)})^T$, then $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_l)$ is the matrix

$$\begin{pmatrix} \eta_1^{(1)}, \eta_1^{(2)}, \ldots, \eta_1^{(l)} \\ \vdots \\ \eta_n^{(1)}, \eta_n^{(2)}, \ldots, \eta_n^{(l)} \end{pmatrix};$$

the symbol

$$(U_1, U_2, \ldots, U_l)$$

where $U_1, \ldots, U_l$ are rectangular matrices $n \times r_1, \ldots, n \times r_l$ is to be understood analogously. By J_k we shall denote the matrix

$$J_k = (\Theta, \mathbf{e}_1^{(k)}, \ldots, \mathbf{e}_{k-1}^{(k)}),$$

where $\Theta = (0, 0, \ldots, 0)^T$. Generally, the symbol Θ will be the null vector or matrix. If B_i $(i = 1, 2, \ldots, l)$ are square matrices, then $\operatorname{diag}(B_1, \ldots, B_l)$ is the matrix

$$\begin{pmatrix} B_1, & \Theta, & \ldots, & \Theta, & \Theta \\ \Theta, & B_2, & \ldots, & \Theta, & \Theta \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \Theta, & \Theta, & \ldots, & \Theta, & B_l \end{pmatrix}.$$

$\varrho(A)$ will denote the spectral radius of a matrix A; if $\mathbf{x} \in \mathbf{V}_n(C)$ then $\|\mathbf{x}\|$ denotes the spectral norm of the vector $\mathbf{x}$.

Let $A \neq \Theta$ be a matrix $n \times n$. There is a nonsingular matrix U such that

(1) $$U^{-1}AU = \operatorname{diag}(\lambda_1 I_{i_1} + J_{i_1}, \lambda_2 I_{i_2} + J_{i_2}, \ldots, \lambda_r I_{i_r} + J_{i_r})$$

(Jordan canonical form). In this paper we assume only that relations

$$\lambda_1 = \lambda_2 = \ldots = \lambda_t\,, \quad |\lambda_t| > |\lambda_l| \quad \text{for} \quad l = t + 1, \ldots, r$$

hold for an integer $t \in \langle 1, r \rangle$. For the sake of simplicity we shall assume $|\lambda_1| > |\lambda_j|$ for $j \neq 1$.

Let $\mathbf{x}_0 \in \mathbf{V}_n(C)$, $\mathbf{x}_0 \neq \Theta$. We shall denote

(2) $$\mathbf{x}_k = A^k \mathbf{x}_0\,,$$

(3) $$\mu_k = \mathbf{x}_k^H \mathbf{x}_k / \mathbf{x}_{k-1}^H \mathbf{x}_{k-1}\,,$$

(4) $$\nu_k = \mathbf{x}_{k-1}^H \mathbf{x}_k / \mathbf{x}_{k-1}^H \mathbf{x}_{k-1}\,.$$

The numbers μ_k and ν_k are studied in detail in Section 3.

Now we shall present the main results of this paper. It is proved that there exist an integer k_1, a sequence of *real numbers* $\{\omega_p\}_{p=0}^{\infty}$ and a function $\vartheta_1(k)$ such that for all $k > k_1$ the ratio (3) is defined i.e., $\mathbf{x}_{k-1}^H \mathbf{x}_{k-1} \neq 0$, the series $\sum_{p=0}^{\infty} \omega_p / k^p$ is absolutely convergent, $\omega_0 = 1$,

(5) $$\mu_k = \varrho^2(A) \left[1 + \sum_{p=1}^{\infty} \frac{\omega_p}{k^p} + \vartheta_1(k) \right],$$

and $\lim_{k \to \infty} k^s\, \vartheta_1(k) = 0$ for every integer s. The constants ω_p are determined uniquely.

Moreover, the constant ω_1 is calculated. In general case, it is $\omega_1 = 2(i_1 - 1)$, where i_1 is defined by the relation (1). Similarly there is k_2 such that

$$(6) \qquad \nu_k = \lambda_1 \left[1 + \sum_{p=1}^{\infty} \frac{\omega'_p}{k^p} + \vartheta_2(k) \right], \quad \omega'_p \in C$$

for every $k > k_2$, $\lim_{k \to \infty} k^s \, \vartheta_2(k) = 0$ for every integer s and generally $\omega'_1 = i_1 - 1$.

In Section 4 it is proved that there exist constants K, α, K', α' such that inequalities

$$(7) \qquad K > 0, \quad K' > 0, \quad \alpha > 0, \quad \alpha' > 0,$$

$$(8) \qquad |\omega_p| < K\alpha^p, \quad |\omega'_p| < K'(\alpha')^p, \quad p = 0, 1, 2, \ldots$$

hold. Numbers K and α are constructed in the course of the proof.

In section 5, special cases of matrices are investigated.

A short numerical example is enclosed in Section 6; besides we construct two extrapolation procedures. The efficiency of them shows Table 2 and the scheme at the end of this paper.

2. PRELIMINARY

We turn our attention back to the relation (1). Let us denote

$$t_l = \sum_{s=1}^{l} i_s \quad \text{for} \quad l = 0, 1, \ldots, r$$

and

$$(9) \qquad U_{i+1} = (\mathbf{u}_{t_i+1}, \ldots, \mathbf{u}_{t_{i+1}}) \quad \text{for} \quad i = 0, 1, \ldots, r-1,$$

where $\mathbf{u}_j$ is the j-th column of the matrix U. It is easy to see that

$$U = (U_1, \ldots, U_r)$$

and from (1) we obtain

$$(10) \qquad AU = U \operatorname{diag}(\lambda_1 I_{i_1} + J_{i_1}, \ldots, \lambda_r I_{i_r} + J_{i_r}),$$

and

$$(11) \qquad A^k U = U \operatorname{diag}((\lambda_1 I_{i_1} + J_{i_1})^k, \ldots, (\lambda_r I_{i_r} + J_{i_r})^k).$$

Lemma 1. *Let $\mathbf{v}_i \in \mathbf{V}_n(C)$, $i = 1, 2, \ldots, j$, $V = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j)$, i.e., a rectangular matrix $n \times j$. If $\lambda \in C$ and $k \geqq 0$ is an integer, then*

$$(12) \qquad V(\lambda I_j + J_j)^k = (\mathbf{v}_{1k}, \mathbf{v}_{2k}, \ldots, \mathbf{v}_{jk}),$$

where

$$(13)\qquad \mathbf{v}_{lk} = \sum_{i=1}^{\min(l,k+1)} \binom{k}{i-1} \lambda^{k-(i-1)} \mathbf{v}_{l-(i-1)}$$

for $l = 1, 2, \ldots, j$.

Proof. For $k = 0$ we obtain from (13) $\mathbf{v}_{l0} = \mathbf{v}_l$. Let the assertion of Lemma hold for k. Then

$$V(\lambda I_j + J_j)^{k+1} = (V(\lambda I_j + J_j)^k)(\lambda I_j + J_j) = (\mathbf{v}_{1k}, \ldots, \mathbf{v}_{jk})(\lambda I_j + J_j).$$

From this we obtain immediately (13) for $k + 1$.

We shall denote by P_i ($i = 1, \ldots, r$) the projection of $\mathbf{V}_n(C)$ to the subspace generated by the columns of the matrix U_i. If $\mathbf{x}_0 \in \mathbf{V}_n(C)$ then

$$(14)\qquad \mathbf{x}_0 = \mathbf{w}_1 + \mathbf{w}_2 + \ldots + \mathbf{w}_r,$$

where $\mathbf{w}_i \in P_i \mathbf{V}_n(C)$ are uniquely determined. Let $\mathbf{w}_i = U_i \mathbf{b}_i$. The vectors $\mathbf{b}_i$ are uniquely determined and the dimension of $\mathbf{b}_i$ is equal to the number of columns of the matrix U_i.

Lemma 2. *It holds*

$$(15)\qquad A^k \mathbf{w}_j = U_j(\lambda_j I_{i_j} + J_{i_j})^k \mathbf{b}_j.$$

Lemma 3. *It holds*

$$(16)\qquad A^k \mathbf{x}_0 = \sum_{j=1}^{r} U_j(\lambda_j I_{i_j} + J_{i_j})^k \mathbf{b}_j.$$

The relations (15) and (16) follow immediately from (11) and (14).

Lemma 4. *If* $|\lambda_1| > |\lambda_j|$ *for some* j ($2 \leqq j \leqq r$), *then*

$$\lim \frac{1}{\lambda_1^k} A^k \mathbf{w}_j = \Theta.$$

Proof. We have denoted $t_l = \sum_{s=1}^{l} i_s$ and $U_j = (\mathbf{u}_{t_{j-1}+1}, \ldots, \mathbf{u}_{t_j})$. In the sequel we shall use the notation

$$(17)\qquad \mathbf{v}_i^{(j)} = \mathbf{u}_m, \text{ where } m = t_{j-1} + i \text{ and}$$
$$\mathbf{b}_j = (\beta_1^{(j)}, \ldots, \beta_{i_j}^{(j)})^T.$$

Suppose that $k > i_j$. Lemma 1 implies

$$(18)\qquad U_j(\lambda_j I_{i_j} + J_{i_j})^k \mathbf{b}_j = \sum_{s=1}^{i_j} \left\{\beta_s^{(j)} \sum_{i=1}^{s} \binom{k}{i-1} \lambda_j^{k-(i-1)} \mathbf{v}_{s-(i-1)}^{(j)}\right\} =$$

$$= \sum_{s=1}^{i_j} \sum_{i=1}^{s} \binom{k}{i-1} \lambda_j^{k-(i-1)} \beta_s^{(j)} \mathbf{v}_{s-(i-1)}^{(j)}\,.$$

Dividing this sum by λ_1^k we obtain a sum with terms

$$\text{constant} \times \binom{k}{p} \times (\lambda_j/\lambda_1)^k \times \text{vector}\,,$$

where $|\lambda_j/\lambda_1| < 1$. The rest is obvious.

From (18) we easily obtain the following relation which we shall need in our further investigation:

$$(19)\qquad U_j(\lambda_j I_{i_j} + J_{i_j})^k \mathbf{b}_j = \sum_{s=1}^{i_j} \sum_{i=1}^{s} \binom{k}{i-1} \lambda_j^{k-(i-1)} \beta_s^{(j)} \mathbf{v}_{s-(i-1)}^{(j)} =$$

$$= \sum_{i=1}^{i_j} \left\{\binom{k}{i-1} \lambda_j^{k-(i-1)} \sum_{s=i}^{i_j} \beta_s^{(j)} \mathbf{v}_{s-(i-1)}^{(j)}\right\}.$$

Lemma 5. *Let $P_j\mathbf{x}_0 \neq \Theta$, $\lambda_j \neq 0$, $k > i_j$. Then there exist an integer l_j and a sequence of matrices $\{F_j(k)\}_{k=i_j+1}^{\infty}$ such that*

$$(20)\qquad U_j(\lambda_j I_{i_j} + J_{i_j})^k \mathbf{b}_j = \lambda_j^k \binom{k}{l_j - 1} (I + F_j(k))\, \mathbf{a}_{l_j}^{(j)}\,,$$

where

$$(21)\qquad 1 \leqq l_j \leqq i_j\,, \quad \lim_{k\to\infty} F_j(k) = \Theta\,,$$

and $\mathbf{a}_{l_j}^{(j)}$ is an eigenvector which belongs to λ_j.

Proof. We use notation (17). It is

$$P_j\mathbf{x}_0 = \mathbf{w}_j = U_j\mathbf{b}_j = \beta_1^{(j)}\mathbf{v}_1^{(j)} + \ldots + \beta_{i_j}^{(j)}\mathbf{v}_{i_j}^{(j)} \neq \Theta\,.$$

Let us denote by l_j such an index that $\beta_{l_j}^{(j)} \neq 0$, but $\beta_i^{(j)} = 0$ for all $l_j < i \leqq i_j$. If we denote for $i = 1, 2, \ldots, l_j$

$$(22)\qquad \mathbf{a}_i^{(j)} = \sum_{s=i}^{l_j} \{\beta_s^{(j)} \mathbf{v}_{s-(i-1)}^{(j)} / \lambda_j^{i-1}\}$$

then from (19) we obtain

$$U_j(\lambda_j I_{i_j} + J_{i_j})^k \mathbf{b}_j = \sum_{i=1}^{l_j} \binom{k}{i-1} \lambda_j^k \mathbf{a}_i^{(j)} = \lambda_j^k \binom{k}{l_j - 1} (I + F_j(k))\, \mathbf{a}_{l_j}^{(j)}\,,$$

where

$$(23) \qquad F_j(k) = \sum_{i=1}^{l_j-1} \left\{ \frac{\binom{k}{i-1}}{\binom{k}{l_j-1}} \frac{\mathbf{a}_i^{(j)} (\mathbf{a}_{l_j}^{(j)})^H}{(\mathbf{a}_{l_j}^{(j)})^H \mathbf{a}_{l_j}^{(j)}} \right\}.$$

It is evident that $\lim_{k \to \infty} F_j(k) = \Theta$. From (22) we obtain

$$\mathbf{a}_{l_j}^{(j)} = \beta_{l_j}^{(j)} \mathbf{v}_1^{(j)} / \lambda_j^{l_j - 1},$$

but $\mathbf{v}_1^{(j)}$ is an eigenvector belonging to λ_j (see (17)). This completes the proof.

Remark. If $\lambda_j = 0$ or $P_j \mathbf{x}_0 = \Theta$, let us put, in order that equality (20) be valid, $\mathbf{a}_{l_j}^{(j)} = \Theta$, $F_j(k) = \Theta$ and $l_j = 1$.

Lemma 6. *Let $\mathbf{a}_{l_j}^{(j)}$ and $F_j(k)$ for $j = 1, 2, \ldots, r$ be vectors and matrices defined by the relations (22) and (23) or by Remark in special cases.*

Then there exists for every $j,s = 1, \ldots, r$ a sequence of numbers $\{\delta_p^{(j,s)}\}_{p=0}^{\infty}$, $\delta_p^{(j,s)} \in C$ such that for all $k > k_{\max} = \max_{j=1,\ldots,r} (i_j)$

$$(24) \qquad (\mathbf{a}_{l_j}^{(j)})^H (I + F_j^H(k)) (I + F_s(k))\, \mathbf{a}_{l_s}^{(s)} = \sum_{p=0}^{\infty} \frac{\delta_p^{(j,s)}}{k^p},$$

where the series on the right hand side of (24) is absolutely convergent for all $k > k_{\max}$.

Proof. If at least one of the equalities

$$P_j \mathbf{x}_0 = \Theta, \quad P_s \mathbf{x}_0 = \Theta, \quad \lambda_j = 0 \quad \text{and} \quad \lambda_s = 0$$

holds, then the product on the left hand side is zero and if we put $\delta_p^{(j,s)} = 0$ for all p, we obtain the assertion of Lemma 6.

Let $P_j \mathbf{x}_0 \neq \Theta$, $P_s \mathbf{x}_0 \neq \Theta$, $\lambda_j \neq 0$, $\lambda_s \neq 0$, $l_j > 1$ and $l_s > 1$. (The cases $l_j = 1$ or $l_s = 1$ will be evident.) Let $k > k_{\max}$. For $i = 1, 2, \ldots, l_j - 1$ it is

$$\binom{k}{i-1} \Big/ \binom{k}{l_j-1} = \frac{(l_j-1)!}{(i-1)!} \frac{1}{k^{l_j-i}} \frac{1}{1-(i-1)/k} \cdots \frac{1}{1-(l_j-2)/k} =$$

$$= \frac{(l_j-1)!}{(i-1)!} \frac{1}{k^{l_j-i}} \left(\sum_{s=0}^{\infty} \left(\frac{i-1}{k} \right)^s \right) \left(\sum_{s=0}^{\infty} \left(\frac{i}{k} \right)^s \right) \cdots \left(\sum_{s=0}^{\infty} \left(\frac{l_j-2}{k} \right)^s \right)$$

and all series are absolutely convergent when $k > k_{\max}$. If we multiply these series and rearrange terms, we obtain that

$$\binom{k}{i-1} \Big/ \binom{k}{l_j-1} = \frac{1}{k^{l_j-i}} \sum_{p=0}^{\infty} \frac{\gamma_p^{(i,j)}}{k^p}$$

where the series is absolutely convergent and

$$\gamma_p^{(i,j)} = \frac{(l_j - 1)!}{(i-1)!} \sum_{\substack{s_1+s_2+\ldots+s_t=p \\ s_1 \geqq 0, s_2 \geqq 0, \ldots, s_t \geqq 0}} (i-1)^{s_1} i^{s_2} \ldots (l_j - 2)^{s_t} \geqq 0\,, \quad \text{where} \quad t = l_j - i\,.$$

Now we shall calculate the product $F_j(k)\,\mathbf{a}_{l_j}^{(j)}$. From (23),

$$F_j(k)\,\mathbf{a}_{l_j}^{(j)} = \sum_{i=1}^{l_j-1} \left\{ \sum_{p=0}^{\infty} \frac{\gamma_p^{(i,j)} \mathbf{a}_i^{(j)}}{k^{p+l_j-i}} \right\},$$

where the inner series is absolutely convergent. If $(\xi_1, \ldots, \xi_n)^T$ are components of the vector $\mathbf{a}_i^{(j)}$, then an infinite series $\sum_{p=0}^{\infty} \gamma_p^{(i,j)}\, \mathbf{a}_i^{(j)}/k^{p+l_j-i}$ is said to be (absolutely) convergent if $\sum_{p=0}^{\infty} \gamma_p^{(i,j)} \xi_s / k^{p+l_j-i}$ is (absolutely) convergent for $s = 1, \ldots, n$.

If we denote

$$N_0 = \{0, 1, 2, 3, \ldots\}\,, \quad N_1 = \{1, 2, \ldots, l_j - 1\} \quad \text{and} \quad M = N_0 \times N_1$$

then for every finite $M_1 \subset M$,

$$\sum_{M_1} \frac{\gamma_p^{(i,j)} \left|\mathbf{a}_i^{(j)}\right|}{k^{p+l_j-i}} < \sum_{i=1}^{l_j-1} \left(\sum_{p=0}^{\infty} \frac{\gamma_p^{(i,j)} \left|\mathbf{a}_i^{(j)}\right|}{k^{p+l_j-i}} \right) < +\infty\,,$$

i.e., the generalized series $\sum_M \left(\gamma_p^{(i,j)} \mathbf{a}_i^{(j)} / k^{p+l_j-i}\right)$ is absolutely convergent. The theory of generalized series is studied in [4]. Let us rearrange this series successively according to the non-decreasing powers of $1/k$ and sum its terms with the same power of $1/k$. Obviously we obtain

$$F_j(k)\,\mathbf{a}_{l_j}^{(j)} = \sum_{m=1}^{\infty} \frac{\mathbf{d}_m^{(j)}}{k^m}\,, \tag{25}$$

where

$$\mathbf{d}_m^{(j)} = \sum_{s=0}^{m-1} \gamma_s^{(l_j-m+s,j)}\, \mathbf{a}_{l_j-m+s}^{(j)} \quad \text{for} \quad m \leqq l_j - 1\,, \tag{26}$$

$$\mathbf{d}_m^{(j)} = \sum_{s=0}^{l_j-2} \gamma_{s+m-l_j+1}^{(1+s,j)}\, \mathbf{a}_{1+s}^{(j)} \quad \text{for} \quad m \geqq l_j - 1$$

and the series are absolutely convergent.

Now we calculate

$$(\mathbf{a}_{l_j}^{(j)})_H \left(I + F_j^H(k)\right)\left(I + F_s(k)\right) \mathbf{a}_{l_s}^{(s)} =$$

$$= (\mathbf{a}_{l_j}^{(j)})^H\, \mathbf{a}_{l_s}^{(s)} + \sum_{m=1}^{\infty} \frac{(\mathbf{d}_m^{(j)})^H\, \mathbf{a}_{l_s}^{(s)}}{k^m} + \sum_{m=1}^{\infty} \frac{(\mathbf{a}_{l_j}^{(j)})^H\, \mathbf{d}_m^{(s)}}{k^m} + \sum_{m_1, m_2 = 1}^{\infty} \frac{(\mathbf{d}_{m_1}^{(j)})^H\, \mathbf{d}_{m_2}^{(s)}}{k^{m_1+m_2}}\,.$$

If we put

$$(27)\qquad \delta_0^{(j,s)} = (\mathbf{a}_{l_j}^{(j)})^H \mathbf{a}_{l_s}^{(s)}\,,$$

$$\delta_p^{(j,s)} = (\mathbf{d}_p^{(j)})^H \mathbf{a}_{l_s}^{(s)} + (\mathbf{a}_{l_j}^{(j)})^H \mathbf{d}_p^{(s)} + \sum_{\substack{m_1,m_2 \geqq 1\\ m_1+m_2=p}} (\mathbf{d}_{m_1}^{(j)})^H \mathbf{d}_{m_2}^{(s)}\,,$$

then we obtain assertion (24) of Lemma 6, which completes the proof.

Lemma 7. *Let $k > 1$, $p \geqq 0$ be integers. Then*

$$\left(\sum_{s=0}^{\infty}\left(\frac{1}{k}\right)^s\right)^p = \sum_{s=0}^{\infty}\frac{\eta_{sp}}{k^s}\,,$$

where $\eta_{00} = 1$, $\eta_{s0} = 0$ for $s > 0$, $\eta_{sp} = \binom{s+p-1}{p-1}$ for $p \geqq 1$ and the series $\sum_{s=0}^{\infty} \eta_{sp}/k^s$ is absolutely convergent.

The proof is evident.

Lemma 6 and Lemma 7 immediately imply:

Lemma 8. *Let the assumptions from Lemma 6 be valid. Then there is a sequence of numbers $\{\delta_{p,1}^{(j,s)}\}_{p=1}^{\infty}$, $\delta_{p,1}^{(j,s)} \in C$ such that for all $k > k_{\max}$*

$$(28)\qquad (\mathbf{a}_{l_j}^{(j)})^H (I + F_j^H(k-1))(I + F_s(k))\,\mathbf{a}_{l_s}^{(s)} = \sum_{p=0}^{\infty}\frac{\delta_{p,1}^{(j,s)}}{k^p}\,,$$

where the series on the right hand side of (28) is absolutely convergent for all $k > k_{\max}$.

Lemma 9. *Let the series $\sum_{p=0}^{\infty} \delta_p/k^p$ be absolutely convergent with $\delta_0 \neq 0$. Then there exist an integer k' and a sequence of numbers $\{\delta'_p\}_{p=0}^{\infty}$ such that $\sum_{p=0}^{\infty} \delta'_p/k^p$ is absolutely convergent for $k > k'$, $\sum_{p=0}^{\infty} \delta_p/k^p \neq 0$ for $k > k'$ and*

$$\left(\sum_{p=0}^{\infty}\frac{\delta_p}{k^p}\right)^{-1} = \sum_{p=0}^{\infty}\frac{\delta'_p}{k^p}\,.$$

Proof. Let us consider in the complex plane the series $\sum_{p=0}^{\infty} \delta_p z^p$. Since $\delta_0 \neq 0$, this implies that there is a neighbourhood $\mathsf{U}_\varepsilon(0)$ such that $\sum_{p=0}^{\infty} \delta_p z^p \neq 0$ in $\mathsf{U}_\varepsilon(0)$, which further implies that $(\sum_{p=0}^{\infty} \delta_p z^p)^{-1}$ is a holomorphic function in $\mathsf{U}_\varepsilon(0)$. There exists $\{\delta'_p\}_{p=0}^{\infty}$ such that

$$(\sum_{p=0}^{\infty} \delta_p z^p)^{-1} = \sum_{p=0}^{\infty} \delta'_p z^p$$

and, for some k', $1/k \in \mathsf{U}_\varepsilon(0)$ for all $k > k'$. The rest is obvious.

Lemma 10. *Let $l \geqq 1$. Then a sequence $\{\tau_p\}_{p=0}^{\infty}$ exists such that*

$$\binom{k}{l-1}^{-2} = \left(\frac{(l-1)!}{k^{l-1}}\right)^2 \sum_{p=0}^{\infty} \frac{\tau_p}{k^p},$$

where the series is absolutely convergent.

The proof is analogous to that of Lemma 6.

3. BEHAVIOUR OF μ_k AND ν_k

Let the following assumptions be satisfied.

Assumption 1. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of the matrix A defined in (1) are mutually different.

Assumption 2. The inequalities

$$|\lambda_1| > |\lambda_2| \geqq \ldots \geqq |\lambda_r|$$

hold.

Assumption 3. For the initial vector $\mathbf{x}_0$ it holds:

$$P_1 \mathbf{x}_0 \neq \Theta .$$

Remark. We suppose, only to simplify the following formulas, that eigenvalues $\lambda_1, \ldots, \lambda_r$ are mutually different. We shall see easily that all results hold if instead of assumptions 1 and 2 we put $\lambda_1 = \ldots = \lambda_t$ and $|\lambda_t| > |\lambda_j|$ for all $j = t+1, \ldots, r$.

From Lemma 3 and Lemma 5 we easily obtain

$$\mathbf{x}_k^H \mathbf{x}_k = |\lambda_1|^{2k} \binom{k}{l_1 - 1}^2 (\mathbf{a}_{l_1}^{(1)})^H (I + F_1^H(k))(I + F_1(k))\, \mathbf{a}_{l_1}^{(1)} (1 + \sigma(k)),$$

where

$$(29) \quad \sigma(k) = |\lambda_1|^{-2k} \binom{k}{l_1 - 1}^{-2} \{(\mathbf{a}_{l_1}^{(1)})^H (I + F_1^H(k))(I + F_1(k))\, \mathbf{a}_{l_1}^{(1)}\}^{-1} \,.$$

$$\cdot \sum_{j,s=1}^{r}{}' \left\{ \bar{\lambda}_j^k \lambda_s^k \binom{k}{l_j - 1} \binom{k}{l_s - 1} (\mathbf{a}_{l_j}^{(j)})^H (I + F_j^H(k))(I + F_s(k))\, \mathbf{a}_{l_s}^{(s)} \right\},$$

and $\sum'$ denotes that we sum over all j, s with the exception of $j = s = 1$. Lemma 6 implies

$$\mathbf{x}_k^H \mathbf{x}_k = |\lambda_1|^{2k} \binom{k}{l_1 - 1}^2 \left(\sum_{p=0}^{\infty} \frac{\delta_p^{(1,1)}}{k^p} \right) (1 + \sigma(k)) \tag{30}$$

and

$$\sigma(k) = \binom{k}{l_1 - 1}^{-2} \left(\sum_{p=0}^{\infty} \frac{\delta_p^{(1,1)}}{k^p} \right)^{-1} \sum_{j,s=1}^{r}{}' \left\{ \binom{k}{l_j - 1} \binom{k}{l_s - 1} \left(\sum_{p=0}^{\infty} \frac{\delta_p^{(j,s)}}{k^p} \right) \bar{q}_j^k q_s^k \right\},$$

where we put $q_j = \lambda_j / \lambda_1$ for $j = 1, 2, \ldots, r$. Using Lemma 9 and Lemma 10, after a small arrangement of the last series, we obtain

$$\sigma(k) = \sum_{j,s=1}^{r}{}' \left\{ \bar{q}_j^k q_s^k k^{l_j + l_s - 2l_1} \sum_{p=0}^{\infty} \frac{\xi_p^{(j,s)}}{k^p} \right\}, \tag{31}$$

where $\sum_{p=0}^{\infty} \xi_p^{(j,s)}/k^p$ is absolutely convergent, $\xi_p^{(j,s)} = \bar{\xi}_p^{(s,j)}$ since $\delta_p^{(j,s)} = \bar{\delta}_p^{(s,j)}$ and $\xi_p^{(j,s)} = 0$ for all p if $\lambda_j = 0$ or $\lambda_s = 0$ or $P_j \mathbf{x}_0 = \Theta$ or $P_s \mathbf{x}_0 = \Theta$ since the same holds for $\delta_p^{(j,s)}$.

Because $\delta_p^{(j,s)} = \bar{\delta}_p^{(s,j)}$ for $p = 0, 1, 2, \ldots$, it is also $\xi_p^{(j,s)} = \bar{\xi}_p^{(s,j)}$.

Definition 1. *Let $\mathbf{W}_j \subset \mathbf{V}_n(C)$ be a subspace generated by all principal vectors which belong to the submatrix $\lambda_j I_{i_j} + J_{i_j}$ (see (1)). Let $\mathbf{y} \in \mathbf{V}_n(C)$. Let $\mathbf{H}_j \subset \mathbf{W}_j$ be an invariant subspace (with respect to the matrix A) of $\mathbf{W}_j$ of minimal dimension, which we denote l_j, such that $P_j \mathbf{y} \in \mathbf{H}_j$ (projection of $\mathbf{y}$ to $\mathbf{W}_j$).*

Then we shall say that the dimension of the vector $\mathbf{y}$ in $\mathbf{W}_j$ is equal to l_j and write $\dim_j \mathbf{y} = l_j$.

It is easy to see that the numbers l_j from Definition 1 are the same as those from Lemma 5.

Theorem 1. *Let A be an $n \times n$ matrix, $\mathbf{x}_0 \in \mathbf{V}_n(C)$, $\mathbf{x}_0 \neq \Theta$ and let the assumptions 1–3 be fulfilled. Let us denote $l_j = \dim_j \mathbf{x}_0$ and $q_j = \lambda_j / \lambda_1$.*

Then there exist an integer k_1, a uniquely determined sequence of real numbers $\{\omega_p\}_{p=0}^{\infty}$ and sequences of complex numbers $\{\xi_p^{(j,s)}\}_{p=0}^{\infty}$; $j, s = 1, \ldots, r$ except $j = s = 1$ such that for $k \geqq k_1$ the ratio $\mathbf{x}_k^H \mathbf{x}_k / \mathbf{x}_{k-1}^H \mathbf{x}_{k-1}$ is defined, the series

$$\sum_{p=0}^{\infty} \frac{\omega_p}{k^p} \quad \text{and} \quad \sum_{p=0}^{\infty} \frac{\xi_p^{(j,s)}}{k^p}$$

are absolutely convergent for $k \geqq k_1$, $\omega_0 = 1$, $\omega_1 = 2(l_1 - 1)$, the expression

$$\sigma(k) = \sum_{j,s=1}^{r}{}' \left\{ \bar{q}_j^k q_s^k k^{l_j + l_s - 2l_1} \sum_{p=0}^{\infty} \frac{\xi_p^{(j,s)}}{k^p} \right\} \tag{32}$$

is less than 1 *for* $k \geqq k_1$ *and if we put for* $k > k_1$

(33) $$\vartheta_1(k) = (\sigma(k) - \sigma(k-1))\left(\sum_{p=0}^{\infty} \frac{\omega_p}{k^p}\right) \left(\sum_{p=0}^{\infty}(-1)^p \sigma^p(k-1)\right)$$

then

(34) $$\mu_k = \varrho^2(A)\left[1 + \sum_{p=1}^{\infty} \frac{\omega_p}{k^p} + \vartheta_1(k)\right]$$

and

(35) $$\lim_{k \to \infty} k^s \vartheta_1(k) = 0$$

for every integer s.

The sequences $\{\xi_p^{(j,s)}\}_{p=0}^{\infty}$ *can be constructed so that*

(36) $$\xi_p^{(s,j)} = \bar{\xi}_p^{(j,s)} \quad \textit{for all} \quad p = 0, 1, 2, \ldots .$$

There exist sequences of complex numbers $\{\tau_p^{(j,s)}\}_{p=0}^{\infty}$ *such that the series* $\sum_{p=0}^{\infty} \tau_p^{(j,s)}/k^p$ *absolutely converges for* $k > k_1$, $\tau_p^{(j,s)} = \bar{\tau}_p^{(s,j)}$ *and*

(37) $$(\sigma(k) - \sigma(k-1)) \sum_{p=0}^{\infty} \frac{\omega_p}{k^p} = \sum_{j,s=1}^{r}{}' \left\{\bar{q}_j^k q_s^k k^{l_j + l_s - 2l_1} \sum_{p=1}^{\infty} \frac{\tau_p^{(j,s)}}{k^p}\right\}.$$

Proof. From Lemma 7 it follows that for $k > \max(i_j) + 1$

(38) $$\sum_{p=0}^{\infty} \frac{\delta_p^{(1,1)}}{(k-1)^p} = \sum_{p=0}^{\infty} \frac{\zeta_p}{k^p}$$

where

(39) $$\zeta_p = \sum_{\substack{m+s=p \\ m \geqq 0, s \geqq 0}} \delta_m^{(1,1)} \eta_{sm}.$$

From this we obtain

(40) $$\zeta_0 = \delta_0^{(1,1)} = (\mathbf{a}_{l_1}^{(1)})^H \mathbf{a}_{l_1}^{(1)} \neq 0 \quad \text{and} \quad \zeta_1 = \delta_1^{(1,1)}.$$

According to Lemma 9 there exist an integer k_3 and a sequence $\{\zeta'_p\}_{p=0}^{\infty}$ such that the series $\sum_{p=0}^{\infty} \zeta'_p/k^p$ is absolutely convergent and

(41) $$\left(\sum_{p=0}^{\infty} \frac{\zeta_p}{k^p}\right)^{-1} = \sum_{p=0}^{\infty} \frac{\zeta'_p}{k^p} \quad \text{for} \quad k > k_3.$$

It can be verified immediately that for $k > i_1$

$$\binom{k}{l_1 - 1} \Big/ \binom{k-1}{l_1 - 1} = \sum_{s=0}^{\infty} \left(\frac{l_1 - 1}{k}\right)^s.$$

From (31) it follows that $\lim_{k\to\infty} \sigma(k) = 0$, which implies that there exists such an integer k_4 that for $k \geqq k_4$ it is $|\sigma(k)| < 1$.

Let us put $k_1 = \max(\max(i_j), k_3, k_4) + 1$. Then for $k > k_1$ the inequality

$$\mathbf{x}_{k-1}^H \mathbf{x}_{k-1} = |\lambda_1|^{2k-2} \binom{k-1}{l_1-1}^2 \left(\sum_{p=0}^{\infty} \frac{\zeta_p}{k^p}\right)(1 + \sigma(k-1)) \neq 0$$

holds and for μ_k we obtain a product of absolutely convergent series, i.e.,

$$(42) \quad \mu_k = \varrho^2(A)\left(\sum_{s=0}^{\infty}\left(\frac{l_1-1}{k}\right)^s\right)^2 \left(\sum_{p=0}^{\infty} \frac{\delta_p^{(1,1)}}{k^p}\right)\left(\sum_{p=0}^{\infty}\frac{\zeta_p'}{k^p}\right)\frac{1+\sigma(k)}{1+\sigma(k-1)}\,.$$

The product of the series is an absolutely convergent series which we write $\sum_{p=0}^{\infty} \omega_p/k^p$.

It is evident that ω_p are real, $\omega_0 = 1$ and

$$\omega_1 = 2(l_1 - 1) + \zeta_1'\delta_0^{(1,1)} + \delta_1^{(1,1)}\zeta_0' =$$
$$= 2(l_1 - 1) - (\delta_1^{(1,1)}/\zeta_0^2)\,\delta_0^{(1,1)} + \delta_1^{(1,1)}/\zeta_0 = 2(l_1 - 1)\,.$$

The relations (33) and (34) follow immediately from (42) and from the fact that

$$\frac{1+\sigma(k)}{1+\sigma(k-1)} = 1 + \frac{\sigma(k)-\sigma(k-1)}{1+\sigma(k-1)} =$$
$$= 1 + (\sigma(k) - \sigma(k-1))\sum_{p=0}^{\infty}(-1)^p\,\sigma^p(k-1)\,.$$

We have shown that

$$\sigma(k) = \sum_{j,s=1}^{r}{}' \left\{\bar{q}_j^k q_s^k k^{l_j+l_s-2l_1}\left(\sum_{p=0}^{\infty}\frac{\xi_p^{(j,s)}}{k^p}\right)\right\},$$

$\bar{\xi}_p^{(j,s)} = \xi_p^{(s,j)}$ and $\xi_p^{(j,s)} = 0$ if $q_j q_s = 0$ or $P_j\mathbf{x}_0 = \Theta$ or $P_s\mathbf{x}_0 = \Theta$. From this we easily obtain that there exists a sequence of numbers $\{{}^{(1)}\xi_p^{(j,s)}\}_{p=0}^{\infty}$ such that the series $\sum_{p=0}^{\infty} {}^{(1)}\xi_p^{(j,s)}/k^p$ is absolutely convergent and

$$\sigma(k-1) = \sum_{j,s=1}^{r}{}' \left\{\bar{q}_j^k q_s^k k^{l_j+l_s-2l_1}\left(\sum_{p=0}^{\infty}\frac{{}^{(1)}\xi_p^{(j,s)}}{k^p}\right)\right\}.$$

Here

${}^{(1)}\xi_p^{(j,s)} = 0$ if $\bar{q}_j q_s = 0$, ${}^{(1)}\xi_0^{(j,s)} = \xi_0^{(j,s)}\bar{q}_j q_s$ if $\bar{q}_j q_s \neq 0$, ${}^{(1)}\xi_p^{(j,s)} = {}^{(1)}\bar{\xi}_p^{(s,j)}$ for all j, $s = 1, 2, \ldots, r$ except $j = s = 1$. Hence

$$(43) \quad \sigma(k) - \sigma(k-1) = \sum_{j,s=1}^{r}{}' \left\{\bar{q}_j^k q_s^k k^{l_j+l_s-2l_1}\left(\sum_{p=1}^{\infty}\frac{\xi_p^{(j,s)} - {}^{(1)}\xi_p^{(j,s)}}{k^p}\right)\right\}.$$

Assertions (36) and (37) follow immediately from the fact that $\delta_p^{(j,s)} = \bar{\delta}_p^{(s,j)}$ and from (43).

The uniqueness of ω_p follows from (34) and (35), which completes the proof.

It is not difficult to show that the number of mutually different summands in the sum

$$\sum_{\substack{s_1+s_2+\ldots+s_t=p\\ s_1\geqq 0,\ldots,s_t\geqq 0}} (i-1)^{s_1} i^{s_2} \ldots (l_j-2)^{s_t}\,, \quad \text{where} \quad t = l_j - i$$

is equal to $\binom{p+l_j-3}{l_j-2}$. Constructing successively the bounds for $\delta_p^{(1,1)}$, ζ_p and ζ'_p according to Lemma 6 and Lemma 9, we easily calculate an estimate for $|\omega_p|$ according to (42). In this way we obtain immediately the following lemma, which will be used in the next section.

Lemma 11. *Let the assumptions of Theorem 1 be fulfilled. Let $l_1 > 1$. Let us define the following functions:*

1) $\xi_0(0) = \|\mathbf{a}_{l_1}^{(1)}\|$,

$$\xi_0(p) = (l_1-1)!\binom{p+l_1-3}{l_1-2}(l_1-2)^{p-1}\sum_{s=0}^{l_1-2}\|\mathbf{a}_{1+s}^{(1)}\|$$

for $p = 1, 2, \ldots$, where $\mathbf{a}_i^{(1)}$ is defined by (22).

2) $\xi_1(p) = \sum_{s=0}^{p} \xi_0(s)\,\xi_0(p-s)$ *for* $p = 0, 1, 2, \ldots$.

3) $\xi_2(p) = \sum_{s=0}^{p} \xi_1(s)\,\eta_{p-s,s}$ *for* $p = 0, 1, 2, \ldots$,

where η_{ij} are the numbers defined in Lemma 7.

4) $\xi_3(0) = (\xi_2(0))^{-1}$,

$$\xi_3(p) = (\xi_2(0))^{-1}\sum_{s=1}^{p}\xi_2(s)\,\xi_3(p-s) \quad \textit{for} \quad p = 1, 2, 3, \ldots.$$

5) $\xi_4(p) = \sum_{s=0}^{p} \xi_3(s)\,\xi_1(p-s)$ *for* $p = 0, 1, 2, \ldots$.

6) $\xi_5(p) = \sum_{s=0}^{p} \xi_4(s)\,(l_1-1)^{p-s}(p-s+1)$ *for* $p = 0, 1, 2, \ldots$.

Then the coeficients ω_p from Theorem 1 satisfy

(44) $$|\omega_p| < \xi_5(p) \quad \textit{for} \quad p = 0, 1, 2, \ldots.$$

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled. Then there is an integer k_2 and a uniquely determined sequence of complex numbers $\{\omega'_p\}_{p=0}^{\infty}$ such* *that for $k > k_2$ the series* $\sum_{p=0}^{\infty} \omega'_p / k^p$ *is absolutely convergent.* $\omega'_0 = 1$, $\omega'_1 = l_1 - 1$,

(45) $$\nu_k = \lambda_1 \left[1 + \sum_{p=1}^{\infty} \frac{\omega'_p}{k^p} + \vartheta_2(k) \right]$$

and

$$\lim k^s \vartheta_2(k) = 0$$

for every integer s.

The proof is analogous to that of Theorem 1. (We use only Lemma 8 instead of Lemma 6.)

Remark. The numbers $\tau_k = \mu_k^{1/2}$ satisfy

(46) $$\tau_k = \varrho(A) \left[1 + \sum \frac{\omega''_p}{k^p} + \vartheta_3(k) \right],$$

where the series is absolutely convergent, ω''_p are real and $\lim_{k \to \infty} k^s \vartheta_3(k) = 0$.

4. BOUNDS FOR ω_p AND ω'_p

From Theorem 1 and Theorem 2 it is easy to see that the coefficients $|\omega^p|$ and $|\omega'_p|$ cannot increase more quickly than α^p for a positive number α. In this section a constructive proof of this assertion will be given, i.e., we shall construct positive numbers K and α for which $|\omega_p| < K\alpha^p$ and analogously for ω'_p.

Theorem 3. *Let the assumptions of Theorem* 1 *be fulfilled. Let $l_1 > 1$. Then there exist positive numbers K and α such that*

(47) $$|\omega_p| < K\alpha^p \quad \textit{for} \quad p = 0, 1, 2, \ldots.$$

Proof. We find by easy computation that for $p \geqq 1$

$$\binom{p + l_1 - 3}{l_1 - 2} (l_1 - 2)^{p-1} \leqq e^{(l_1 - 2)^2} [(l_1 - 2) e^{l_1 - 2}]^{p-1} / (l_1 - 2)!\,.$$

(a) If we denote

$$K_0 = (l_1 - 1) e^{(l_1 - 2)^2} \sum_{s=0}^{l_1 - 2} \|\mathbf{a}_{1+s}^{(1)}\|, \quad \alpha_0 = (l_1 - 2) e^{l_1 - 2},$$

$$\eta_0(0) = \|\mathbf{a}_{l_1}^{(1)}\| \quad \text{and} \quad \eta_0(p) = K_0 \alpha_0^{p-1} \quad \text{for} \quad p = 1, 2, \ldots,$$

then

(48) $$\xi_0(p) \leqq \eta_0(p)\,.$$

(Throughout this proof, $\xi_0(p), \xi_1(p), \ldots, \xi_5(p)$ are the functions from Lemma 11.)

(b) For $p > 1$,

$$\xi_1(p) = \sum_{s=0}^{p} \xi_0(s)\, \xi_0(p-s) \leqq 2\|\mathbf{a}_{l_1}^{(1)}\|\, \eta_0(p) + \sum_{s=1}^{p-1} \eta_0(s)\, \eta_0(p-s) \leqq$$

$$\leqq 2\|\mathbf{a}_{l_1}^{(1)}\|\, K_0 \alpha_0^{p-1} + K_0^2 \alpha_0^{p-2}\, e^{(p-1)/2}\,.$$

Hence putting

$$K_1 = 2\|\mathbf{a}_{l_1}^{(1)}\|\, K_0 + \alpha_0^{-1} K_0^2\,, \quad \alpha_1 = \alpha_0 e^{1/2} \quad \text{if} \quad \alpha_0 \neq 0 \ (\text{i.e. } l_1 \neq 2)$$

and

$$K_1 = \max\left(2\|\mathbf{a}_{l_1}^{(1)}\|\, K_0, K_0^2\right), \quad \alpha_1 = 1 \quad \text{if} \quad \alpha_0 = 0\,,$$

$$\eta_1(0) = \xi_0^2(0)\,, \quad \eta_1(p) = K_1 \alpha_1^{p-1}\,,$$

we obtain

(49) $$\xi_1(p) \leqq \eta_1(p)\,.$$

(c) According to 3) from Lemma 11 and (b) it follows for $p \geqq 1$

$$\xi_2(p) = \sum_{s=0}^{p} \xi_1(s)\, \eta_{p-s,s} \leqq \sum_{s=0}^{p} \eta_1(s)\, \eta_{p-s,s} =$$

$$= \binom{p-1}{0} \eta_1(1) + \binom{p-1}{1} \eta_1(2) + \ldots + \binom{p-1}{p-1} \eta_1(p) = K_1(1+\alpha_1)^{p-1}\,.$$

Writing

$$K_2 = K_1/(1+\alpha_1)\,, \quad \alpha_2 = 1 + \alpha_1\,,$$

$$\eta_2(0) = \eta_1(0)\,, \quad \eta_2(p) = K_2 \alpha_2^p \quad (p \geqq 1)\,,$$

it is

(50) $$\xi_2(p) \leqq \eta_2(p)\,.$$

(d) Now we show by induction that if we define

$$\eta_3(0) = 1\,, \quad \eta_3(p) = \frac{1}{\xi_2(0)} \sum_{s=1}^{p} \eta_2(s)\, \eta_3(p-s)\,,$$

then for $p = 1, 2, \ldots$

(51) $$\eta_3(p) = K_3 \alpha_3^p\,,$$

where

$$K_3 = \frac{K_2}{\xi_2(0)} \left(1 + \frac{K_2}{\xi_2(0)}\right)^{-1} \quad \text{and} \quad \alpha_3 = \alpha_2 \left(1 + \frac{K_2}{\xi_2(0)}\right).$$

(I) For $p = 1$,

$$\eta_3(1) = \frac{1}{\xi_2(0)} \sum_{s=1}^{1} \eta_2(1)\,\eta_3(0) = \frac{K_2}{\xi_2(0)}\,\alpha_2 = K_3\alpha_3 .$$

(II) If (51) holds for an integer p then

$$\eta_3(p + 1) = \frac{1}{\xi_2(0)} \sum_{s=1}^{p} K_2\alpha_2^s\, K_3\alpha_3^{p+1-s} + \frac{K_2}{\xi_2(0)}\,\alpha_2^{p+1} =$$

$$= \frac{K_2K_3}{\xi_2(0)} \sum_{s=1}^{p} \alpha_2^s\alpha_2^{p+1-s}\left(1 + \frac{K_2}{\xi_2(0)}\right)^{p+1-s} + \frac{K_2}{\xi_2(0)}\,\alpha_2^{p+1} =$$

$$= K_3\alpha_2^{p+1}\left(\frac{K_2}{\xi_2(0)} \sum_{s=1}^{p}\left(1 + \frac{K_2}{\xi_2(0)}\right)^{p+1-s}\right) + K_3\left(1 + \frac{K_2}{\xi_2(0)}\right)\alpha_2^{p+1} = K_3\alpha_3^{p+1} .$$

which shows the validity of (51) for $p + 1$.

Evidently

(52) $$\xi_3(p) \leqq \eta_3(p) \quad \text{for} \quad p = 1, 2, \ldots .$$

(e) By easy calculation we obtain

$$\sum_{s=0}^{p} \eta_3(s)\,\eta_1(p - s) \leqq K_4'\alpha_3^p ,$$

where

$$K_4' = \frac{K_1K_3\,\xi_2(0)}{K_2\alpha_2} + K_1/\alpha_3 + K_3\,\xi_0^2(0) .$$

Writing

$$K_4 = \max(K_4', 1), \quad \alpha_4 = \alpha_3 \quad \text{and} \quad \eta_4(p) = K_4\alpha_4^p ,$$

we have

(53) $$\xi_4(p) \leqq K_4\alpha_4 \quad \text{for} \quad p = 0, 1, 2, \ldots .$$

(For $p \geqq 1$ we have used (52) and (49), for $p = 0$ it is $\xi_4(0) = 1 \leqq K_4$.)

(f) It is

$$\sum_{s=0}^{p} \eta_4(s)\,(l_1 - 1)^{p-s}\,(p - s + 1) < K_4(1 + l_1 - 1 + \alpha_4)^{p+1} < K\alpha^p ,$$

where we put

$$K = K_4(l_1 + \alpha_4) \quad \text{and} \quad \alpha = l_1 + \alpha_4 .$$

This implies

(54) $$\xi_5(p) < K\alpha^p$$

and consequently from Lemma 11

(55) $$|\omega_p| < K\alpha^p$$

which completes the proof.

Theorem 4. *Let the assumptions of Theorem 3 be fulfilled. Then there exist real positive numbers K' and α' such that*

$$|\omega'_p| < K'(\alpha')^p . \tag{56}$$

The proof is analogous to that of Theorem 3.

5. SOME SPECIAL CASES OF THEOREM 1

In this section we shall suppose that the matrix A is normalizable, i.e.,

$$U^{-1}AU = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

(if $i_j = 1$ for $j = 1, \ldots, n$; see assumptions 1–3 in Section 4), or more generally

$$U^{-1}AU = \operatorname{diag}(\lambda_1 I_{i_1}, \lambda_2 I_{i_2}, \ldots, \lambda_r I_{ir}) .$$

In both cases we shall suppose

$$|\lambda_1| > |\lambda_j| \quad \text{for} \quad j \neq 1 . \tag{57}$$

Hence using the theory from Section 4, we obtain $\delta_p^{(i,j)} = 0$ for $p > 0$ (Lemma 6). Continuing in this way we easily obtain $\omega_p = 0$, $\xi_p^{(j,s)} = 0$ and $\tau_p^{(j,s)} = 0$ for $p \geqq 1$, so that the following theorem immediately follows from Theorem 1.

Theorem 5. *Let A be a normalizable matrix $n \times n$, $(n > 1)$ and suppose, if*

$$U^{-1}AU = \operatorname{diag}(\lambda_1 I_{i_1}, \lambda_2 I_{i_2}, \ldots, \lambda_r I_{i_r}) ,$$

that $|\lambda_1| > |\lambda_j|$ for all $j \neq 1$. Let $\mathbf{x}_0 \in \mathbf{V}_n(C)$, $\mathbf{x}_0 \neq \Theta$ and $P_1\mathbf{x}_0 \neq \Theta$ (projection to the proper subspace of $\mathbf{V}_n(C)$ which belongs to λ_1).

Then there is such an integer k_4 that for all $k > k_4$

$$\mu_k = \varrho^2(A)\,[1 + \vartheta_4(k)] , \tag{58}$$

where the function $\vartheta_4(k)$ is obtained from $\vartheta_1(k)$ if we put in the relation (37) $\omega_p = \tau_p^{(j,s)} = \xi_p^{(j,s)} = 0$ for $p \geqq 1$, $l_j = 1$ for $j = 1, \ldots, r$, i.e., there exists numbers $\tau_0^{(j,s)}$, $\xi_0^{(j,s)}$, $j, s = 1, 2, \ldots, r$ except $j = s = 1$ such that

$$\vartheta_4(k) = \Big(\sum_{j,s=1}^{r}{}' \tau_0^{(j,s)} \bar{q}_j^k q_s^k\Big)\Big(1 + \sum_{p=1}^{\infty} (-1)^p \sigma^p(k-1)\Big) , \tag{59}$$

where

$$\sigma(k) = \sum_{j,s=1}^{r}{}' \xi_0^{(j,s)} \bar{q}_j^k q_s^k . \tag{60}$$

Moreover, if A is normal, then $\tau_0^{(j,s)} = 0$ and $\xi_0^{(j,s)} = 0$ for $j \neq s$ and

$$\vartheta_4(k) = \Big(\sum_{j=2}^{r} \tau_0^{(j,j)} |q_j|^{2k}\Big)\Big(1 + \sum_{p=1}^{\infty} \Big\{(-1)^p \Big(\sum_{j=2}^{r} \xi_0^{(j,j)} |q_j|^{2k-2}\Big)^p\Big\}\Big) . \tag{61}$$

The rate of convergence is in this case twice as quick as in the case of general normalizable matrices.

Proof. The first part of the theorem follows immediately from Lemma and Theorem 1. If A is normal, i.e., unitarly similar to a diagonal matrix, then the proper vectors are orthogonal. The rest is evident.

Remark. An analogous theorem holds for $\mu_k^{1/2}$ and ν_k.

Remark. Let us return to the general case in Theorem 1. If the assumption $|\lambda_1| > |\lambda_j|$ does not hold, for example,

$$|\lambda_1| = |\lambda_2| > |\lambda_j| \quad \text{for} \quad j \neq 1, 2, \quad \lambda_1 \neq \lambda_2,$$

then Kellogg's iterations converge if $l_1 > l_2$ or $l_1 < l_2$. This follows from the theory in Section 3. We do not study this problem in this paper.

6. EXTRAPOLATION AND NUMERICAL RESULTS

Let $U^{-1}AU = \operatorname{diag}(I_3 + J_3, 0{\cdot}95 I_1)$, i.e.,

$$U^{-1}AU = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0{\cdot}95 \end{pmatrix},$$

$$\mathbf{x}_0 = (1, 1, 1, 1)^T$$

and

$$U^{-1} = \begin{pmatrix} 2 + i, & 2 + 8i, & 3 - 2i, & -1 + 7i \\ 5 + 3i, & -2 + 2i, & 3 + 4i, & 0{\cdot}5 \\ 11, & 8 + i, & 5 - i, & -3 + 5i \\ -4 + 6i, & -5, & 6 - 3i, & -1 + 2i \end{pmatrix}.$$

Now we calculate A and $\omega_1, \omega_2, \ldots, \omega_6$ according to the proof of Theorem 1. We obtain

$$\begin{aligned} \omega_0 &= 1 & \omega_4 &= -\ 71{\cdot}07088246\ldots \\ \omega_1 &= 4 & \omega_5 &= -117{\cdot}18223994\ldots \\ \omega_2 &= 9{\cdot}35645366\ldots & \omega_6 &= 465{\cdot}03609659\ldots \\ \omega_3 &= -0{\cdot}84973996\ldots \end{aligned}$$

Table 1 shows the differences

$$\mu_k - \sum_{p=0}^{l} \frac{\omega_p}{k^p}$$

for $l = 1, \ldots, 6$ and $k = 50, 100, 200, 500, 900$. It is seen from Table 1 that the sum of several first terms of the series gives us a good approximation for μ_k and the error is small relative to this sum.

Table 1

k	l	$\mu_k - \sum_{p=0}^{l} (\omega_p/k^p)$
	1	0·003724581230736
50	2	−0·000018000235765
	3	−0·000011202316064
	4	0·000000169025136
	1	0·000934080732889
100	2	−0·000001564633711
	3	−0·000000714893811
	4	−0·000000004185011
	1	0·000233760354689
	2	−0·000000150986913
200	3	−0·000000044769506
	4	−0·000000000350106
	5	0·000000000016094
	6	0·000000000008889
	1	0·000037417875911
	2	−0·000000007938889
500	3	−0·000000001140988
	4	−0·000000000003887
	5	−0·000000000000185
	6	−0·000000000000217
	1	0·000011549903325
	2	−0·000000001273980
900	3	−0·000000000108378
	4	−0·000000000000076

The convergence of μ_k is very slow. At the end of this paper we show two procedures which accelerate the calculation of $\varrho^2(A)$.

We have proved that

$$\mu_k = \varrho^2(A)\left[1 + \sum_{p=1}^{l} \frac{\omega_p}{k^p} + \vartheta(k)\right],$$

where

$$\vartheta(k) = \mathrm{o}\left(\frac{1}{k^l}\right) \quad \text{for} \quad k \to \infty$$

and l is a fixed positive integer. Let

(62) $$k_1 \leqq m_1 < m_2 < \ldots < m_{l+1}$$

be a strongly increasing sequence of integers. For every pair (m_i, m_{i+1}) $i = 1, \ldots, l$, let us perform the following procedure:

$$\mu_{m_i} = \varrho^2(A) \left[1 + \sum_{p=1}^{l} \frac{\omega_p}{m_i^p} + \vartheta(m_i) \right],$$

$$\mu_{m_{i+1}} = \varrho^2(A) \left[1 + \sum_{p=1}^{l} \frac{\omega_p}{m_{i+1}^p} + \vartheta(m_{i+1}) \right].$$

Cross-multiplying these equations we obtain

(63) $$\mu_{m_{i+1}} - \mu_{m_i} = \sum_{p=1}^{l} \omega_p \left[\frac{\mu_{m_i}}{m_{i+1}^p} - \frac{\mu_{m_{i+1}}}{m_i^p} \right] + z(m_i, m_{i+1}),$$

where

$$z(m_i, m_{i+1}) = \mu_{m_i} \vartheta(m_{i+1}) - \mu_{m_{i+1}} \vartheta(m_i).$$

Writing (63) for all pairs (m_i, m_{i+1}) without $z(m_i, m_{i+1})$, we obtain a system of linear algebraic equations for $\omega_1, \ldots, \omega_l$.

Let us denote

$$\varkappa_{ip} = \log_{10} \left\{ \left| \frac{\mu_{m_i}}{m_{i+1}^p} - \frac{\mu_{m_{i+1}}}{m_i^p} \right| \right\}$$

for $i = 1, 2, \ldots, l$ and $p = 0, 1, \ldots, l$. It is not difficult to show that in numerical calculation, in which every real number is correctly rounded to d decimal places, the number

$$\varkappa_{(l)} = \max_{i=1,\ldots,l} \{\varkappa_{i0} - \varkappa_{il}\}$$

cannot be greater then $d - 1$. This condition gives the upper bound for l. (This depends on the computer used.) In practice it is better to take $\varkappa_{(l)} < d - 2$ or $\varkappa_{(l)} < d - 3$.

Let us denote for some admissible l

(64) $$\varrho_{ex}^2(k) = \mu_k \Bigg/ \left[1 + \sum_{p=1}^{l} \frac{\Omega_p}{k^p} \right],$$

where $\Omega_1, \Omega_2, \ldots, \Omega_l$ is a solution of a linear algebraic system for $\omega_1, \omega_2, \ldots, \omega_l$.

Four our matrix 4×4 it was possible to take $l = 4$. Table 2 compares the numbers $\varrho_{ex}^2(k)$ and μ_k for four cases.

Case 1:	$(m_1, m_2, m_3, m_4, m_5) = (40, 41, 42, 43, 44)$
Case 2:	$(m_1, m_2, m_3, m_4, m_5) = (54, 55, 56, 57, 58)$
Case 3:	$(m_1, m_2, m_3, m_4, m_5) = (74, 75, 76, 77, 78)$
Case 4:	$(m_1, m_2, m_3, m_4, m_5) = (100, 101, 102, 103, 104)$

Remark. Analogous calculations for ν_k and $\mu_k^{1/2}$ were made; they show similar behaviour.

Table 2

Case	$\varrho_{ex}^2(k)$	μ_{m_5}
1	0·99999931482	1·095713367851625
2	0·999999681996	1·071736295510767
3	0·9999999098124	1·052816220440535
4	1·0000000063206	1·039325228250185

The second procedure is analogous to the Richardson-type extrapolation (see [2]). Let us assume

$$\mu_k = \varrho^2(A)\left[1 + \sum_{p=1}^{\infty} \frac{\omega_p}{k^p}\right].$$

From this it follows

$$\varrho^2(A) = \mu_k - \sum_{p=1}^{\infty} \frac{\Omega_p}{k^p},$$

where we put $\Omega_p = \omega_p \, \varrho^2(A)$. Let

$$k_1 < k_2 < k_3 < \dots$$

be an increasing sequence of integers such that

$$\frac{k_{i+1}}{k_i} = \alpha = \text{const} \quad (>1)$$

for $i = 1, 2, \dots$.

Let us write

(65) $$\varrho^2(A) = \mu_{k_i} - \sum_{p=1}^{\infty} \frac{\Omega_p}{k_i^p}$$

and

(66) $$\varrho^2(A) = \mu_{k_{i+1}} - \sum_{p=1}^{\infty} \frac{\Omega_p}{k_{i+1}^p}.$$

Multiplying (65) by k_i and (66) by k_{i+1}, substracting and then solving for $\varrho^2(A)$, we get

$$\varrho^2(A) = \mu_{k_i}^{(1)} - \sum_{p=2}^{\infty} \frac{\Omega_p^{(1)}}{k_i^p},$$

where

$$\mu_{k_i}^{(1)} = \frac{\mu_{k_{i+1}} k_{i+1} - \mu_{k_i} k_i}{k_{i+1} - k_i}$$

and

$$\Omega_p^{(1)} = \Omega_p \left(\frac{1}{\alpha^{p-1}} - 1 \right) \left(\frac{1}{\alpha - 1} \right).$$

Continuing analogously this procedure we obtain easily for $l = 2, 3, 4, \ldots$

(67) $$\varrho^2(A) = \mu_{k_i}^{(l)} - \sum_{p=l+1}^{\infty} \frac{\Omega_p^{(l)}}{k_i^p},$$

where

(68) $$\mu_{k_i}^{(l)} = \frac{\mu_{k_{i+1}}^{(l-1)} k_{i+1}^l - \mu_{k_i}^{(l-1)} k_i^l}{k_{i+1}^l - k_i^l}$$

and

(69) $$\Omega_p^{(l)} = \Omega_p^{(l-1)} \left(\frac{1}{\alpha^{p-l}} - 1 \right) \left(\frac{1}{\alpha^l - 1} \right)$$

for $p = l + 1, l + 2, \ldots$.

If we put $\mu_{k_i}^{(0)} = \mu_{k_i}$ and $\Omega_p^{(0)} = \Omega_p$ then (68) and (69) hold also for $l = 1$. It is convenient to introduce a scheme of $\mu_k^{(l)}$ in the following way:

$$\begin{array}{llll}
\mu_{k_1} = \mu_{k_1}^{(0)} & & & \\
 & \mu_{k_1}^{(1)} & & \\
\mu_{k_2} = \mu_{k_2}^{(0)} & & \mu_{k_1}^{(2)} & \\
 & \mu_{k_2}^{(1)} & & \mu_{k_1}^{(3)} \\
\mu_{k_3} = \mu_{k_3}^{(0)} & & \mu_{k_2}^{(2)} & \\
 & \mu_{k_3}^{(1)} & & \\
\mu_{k_4} = \mu_{k_4}^{(0)} & & & \\
\ldots\ldots & & &
\end{array}$$

For the matrix 4×4 from the beginning of this section we have obtained for $k_1 = 40$, $k_2 = 60$, $k_3 = 90$ and $k_4 = 135$ (the scheme cannout be continued further):

1·105807272897			
	0·996154400089		
1·069256315294		0·999979216309	
	0·998279297989		1·000002354722
1·045597309526		0·999995498896	
	0·999232742937		
1·030142453996			

We have obtained again a very good result. We cannot continue this process because $\alpha k_4 = \frac{3}{2} \times 135$ is not an integer.

The situation in this extrapolation procedure is this: the nearer α is to one, the more slowly the sequence $k_1, k_2, \ldots$ increases. But if α is for example 1.1 then for $k_1 = 40$ we get $k_2 = 44$ and k_3 is not an integer. In this case we can calculate only $\mu_{40}^{(1)}$. The choice $\alpha = 2$ makes it possible to calculate $\mu_k^{(l)}$ for every integer l, but to evaluate $\mu_k^{(l)}$ we need to know μ_s, where $s = k \,.\, 2^{l-1}$.

The second procedure is easier than the first one, but generally requires to calculate more iterations of μ_k than in the first case.

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Souhrn

KELLOGGOVA ITERAČNÍ METODA PRO OBECNOU KOMPLEXNÍ MATICI

JAN ZÍTKO

Nechť $A \neq \Theta$ je komplexní matice $n \times n$, $\mathbf{x}_0 \in \mathbf{V}_n(C)$, $\mathbf{x}_0 \neq \Theta$. Definujme

$$\mathbf{x}_k = A^k \mathbf{x}_0 \,,$$
$$\mu_k = \mathbf{x}_k^H \mathbf{x}_k / \mathbf{x}_{k-1}^H \mathbf{x}_{k-1} \,,$$
$$\nu_k = \mathbf{x}_{k-1}^H \mathbf{x}_k / \mathbf{x}_{k-1}^H \mathbf{x}_{k-1} \,.$$

Nechť U je regulární transformace, která převádí matici A na Jordanův kanonický tvar, tj.

$$U^{-1} A U = \operatorname{diag}\left(\lambda_1 I_{i_1} + J_{i_1}, \lambda_2 I_{i_2} + J_{i_2}, \ldots, \lambda_r I_{i_r} + J_{i_r}\right)$$

a předpokládejme, že

$$\lambda_1 = \lambda_2 = \ldots = \lambda_t, \quad |\lambda_t| > |\lambda_l| \quad (l = t + 1, \ldots, r)$$

pro nějaké přirozené číslo $t \in \langle 1, r \rangle$. Pro zjednodušení zápisu můžeme bez újmy na obecnosti předpokládat, že $|\lambda_1| > |\lambda_j|$ pro $j \neq 1$.

V této práci jsou úplně studována čísla μ_k a v_k. Je dokázáno, že existuje přirozené číslo k_1, posloupnost *reálných čísel* $\{\omega_p\}_{p=0}^{\infty}$ a funkce $\vartheta_1(k)$ tak, že pro všechna $k > k_1$ je $\mathbf{x}_{k-1}^H \mathbf{x}_{k-1} \neq 0$, řada $\sum_{p=0}^{\infty} \omega_p/k^p$ absolutně konverguje, $\omega_0 = 1$,

$$\mu_k = \varrho^2(A) \left[1 + \sum_{p=1}^{\infty} \frac{\omega_p}{k^p} + \vartheta_1(k) \right]$$

a $\lim_{k \to \infty} k^s \vartheta_1(k) = 0$ pro libovolné celé číslo s. Konstanty ω_p jsou určeny jednoznačně. Navíc je spočítána funkce $\vartheta_1(k)$ a číslo ω_1, které je v obecném případě rovno $2(i_1 - 1)$. Podobně existuje přirozené číslo k_2 tak, že

$$v_k = \lambda_1 \left[1 + \sum_{p=1}^{\infty} \frac{\omega'_p}{k^p} + \vartheta_2(k) \right] \qquad (\omega'_p \in C)$$

pro všechna $k > k_2$, $\lim_{k \to \infty} k^s \vartheta_2(k) = 0$ pro libovolné celé s a obecně je $\omega_1 = i_1 - 1$. Z asymptotického chování aproximací μ_k a v_k je zřejmé, že čísla $|\omega_p|$ resp. $|\omega'_p|$ nemohou růst rychleji než α^p pro některé reálné číslo $\alpha > 0$. V práci jsou navíc sestrojena čísla K, α, K', α' tak, že platí

$$K > 0, \quad \alpha > 0, \quad K' > 0, \quad \alpha' > 0,$$
$$|\omega_p| < K\alpha^p, \quad |\omega'_p| < K'(\alpha')^p \quad \text{pro} \quad p = 0, 1, 2, \ldots.$$

Kromě obecného případu $i_j > 1$ se v článku zabýváme také speciálními případy a sice normalisovatelnými a normálními maticemi. Je-li matice A normalisovatelná, pak $\omega_p = 0$ pro $p \geqq 1$ a

$$\vartheta_1(k) = \Big(\sum_{j,s=1}^{r}{}' \tau_0^{(j,s)} \bar{q}_j^k q_s^k \Big) \Big(\sum_{p=0}^{\infty} (-1)^p \sigma^p(k-1) \Big),$$

kde

$$\sigma(k) = \sum_{j,s=1}^{r}{}' \xi_0^{(j,s)} \bar{q}_j^k q_s^k,$$

$q_j = \lambda_j/\lambda_1$ a $\tau_0^{(j,s)}$ a $\xi_0^{(j,s)}$ jsou komplexní čísla. $\sum'$ značí, že se nesčítá pro $j = s = 1$. V případě, že A je normální, pak $\tau_0^{(j,s)} = 0$ a $\xi_0^{(j,s)} = 0$ pro $j \neq s$.

V závěru práce je ukázáno, jak je možno obecné formule pro μ_k použít k extrapolaci $\varrho^2(A)$. V tabulce 2 je ukázáno, jak podstatně se liší extrapolované hodnoty od iterací pro některá k. Zvolili jsme jako kontrolní příklad matici 4×4, $i_1 = 3$ a $i_2 = 1$. Přesná hodnota $\varrho^2(A)$ je číslo 1.

Author's address: RNDr. *Jan Zítko* CSc., Katedra numerické matematiky na MFF KU, Malostranské náměstí 25, 118 00 Praha 1.