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INFERENTIAL PROCEDURES ON A GENERALIZED RAYLEIGH VARIATE (I)

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1. INTRODUCTION

Consider a random variable X with the following density function:

$$X : f(x; \theta, k) = \frac{2\theta^{k+1}}{\Gamma(k+1)} x^{2k+1} \exp\{-\theta x^2\} \tag{1}$$

with $x > 0$, $\theta > 0$, $k \geqq 0$.

This class of densities contains some important probability laws:

1) for $k = 0$ and $\theta = 1/\lambda^2$ we obtain

$$f_R(x; \lambda) = \frac{2}{\lambda^2} x \exp\{-x^2/\lambda^2\}, \quad x > 0, \quad \lambda > 0 \tag{2}$$

that is the one-parameter Rayleigh law.

2) for $k = \frac{1}{2}$ and $\theta = 1/2\lambda^2$ one obtains

$$f_M(x; \lambda) = \frac{2}{\lambda^3(2\pi)^{\frac{1}{2}}} x^2 \exp\{-x^2/2\lambda^2\}, \quad x > 0, \quad \lambda > 0 \tag{3}$$

that is the one-parameter Maxwell law.

3) if in (1) we take $\theta = 1/2\tau^2$, $\tau > 0$ and $k = \frac{1}{2}a - 1$, $a \in N$ then we shall have

$$f(x; \tau, a) = \frac{1}{2^{\frac{1}{2}a-1}\tau^a \Gamma(\frac{1}{2}a)} x^{a-1} \exp\{-x^2/2\tau^2\} \tag{4}$$

$x > 0$, $\tau > 0$, $a \in N$, that is the density function of "chi" variable with "a" degrees of freedom. (Therefore a variable with density given by (1) may be regarded as a chi-variable with certain degrees of freedom, not necessarily a natural number.)

Let us note that one can take for k also values in the interval $(-1, 0)$. For instance if $k = -\frac{1}{2}$ and $\theta = 1/2\sigma^2$ we rediscover the "half-normal" density function:

$$(5) \qquad f_{HN}(x;\sigma) = \frac{2}{\sigma(2\pi)^{\frac{1}{2}}} \exp\{-x^2/2\sigma^2\}, \quad x > 0, \quad \sigma > 0.$$

For the sake of homogenity we shall consider that $k \geqq 0$.

In the following we shall call the random variable with density function given by (1) – a generalized Rayleigh variable (GRV).

One can consider a three-parameter GRV by adding a location parameter:

$$(6) \qquad f(x;\theta, c, k) = \frac{2\theta^{k+1}}{\Gamma(k+1)} (x-c)^{2k+1} \exp[-\theta(x-c)^2]$$

$x > c > 0$, $\theta > 0$, $k \geqq 0$ but we shall restrict our attention for a while only to the case when $C = 0$.

The non-central moment of order p is given by

$$(7) \qquad E(X^p) = \frac{\Gamma(k + \frac{1}{2}p + 1)}{\Gamma(k+1)} \theta^{-\frac{1}{2}p}.$$

Therefore

$$(8) \qquad E(X) = \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+1)} \frac{1}{\theta^{\frac{1}{2}}} \quad \text{and} \quad \operatorname{Var}(X) = \left[(k+1) - \frac{\Gamma^2(k+\frac{3}{2})}{\Gamma^2(k+1)}\right] \frac{1}{\theta}.$$

Putting $E(X) = m$, we can express easily the central moment of order $2p$:

$$(9) \qquad E[(X-m)^{2p}] = \frac{m^{2p}}{\Gamma(k+1)} \sum_{l=0}^{2p} (-1)^l C_{2p}^l \theta^{-\frac{1}{2}l} m^{-l} \Gamma(k + \tfrac{1}{2}l + 1).$$

The distribution function has the expression

$$(10) \qquad F(x;\theta,k) = \frac{2\theta^{k+1}}{\Gamma(k+1)} \int_0^x t^{2k+1} e^{-\theta t^2} \, dt =$$

$$= \frac{1}{\Gamma(k+1)} \int_0^{\theta x^2} u^{(k+1)-1} e^{-u} \, du =$$

$$= \frac{\Gamma_{\theta x^2}(k+1)}{\Gamma(k+1)}$$

where we have put $\theta t^2 = u$ and the last numerator represents the incomplete Gamma function. Hence, Karl Pearson's tables can be used for various values of θ and k.

Using the usual transformation $Y = (X - E(X))/\sqrt{\operatorname{Var}(X)}$ we obtain the standardized density function as follows:

$$(11) \qquad f_0(x; k) = \frac{2A}{\Gamma(k+1)} (Ax + B)^{2k+1} \exp\{-(Ax+B)^2\}, \quad x, k > 0$$

where

$$(12) \qquad A = \left[(k+1) - \frac{\Gamma^2(k+\frac{3}{2})}{\Gamma^2(k+1)}\right]^{\frac{1}{2}} \quad \text{and} \quad B = \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+1)}.$$

The corresponding standardized distribution function is

$$(13) \qquad F_0(x; k) = \frac{1}{\Gamma(k+1)} \left[\Gamma_{(Ax+B)^2}(k+1) - \Gamma_{B^{1/2}}(k+1)\right].$$

The GR distribution is one-modal and asymmetric. The mode is

$$(14) \qquad x_M = \frac{(2k+1)^{\frac{1}{2}}}{(2\theta)^{\frac{1}{2}}}.$$

If $\theta = 1$, then the Mellin transform (Epstein [3]) associated with X is

$$(14) \qquad h(s) = E(X^{s-1}) = \int_0^\infty x^{s-1} f(x; 1, k)\,\mathrm{d}x = \frac{\Gamma(k + \frac{1}{2}(1+s))}{\Gamma(k+1)}.$$

2. PRELIMINARIES

We shall prove first

Lemma 1. *If X is a GRV, the X^α, $\alpha > 0$ has the following density function:*

$$(15) \qquad f_{X^\alpha}(x; \theta, k) = \frac{2\theta^{k+1}}{\alpha\,\Gamma(k+1)} x^{(2/\alpha)(k+1)-1} \exp\{-\theta x^{2/\alpha}\}, \quad x, \theta, k, \alpha > 0.$$

Proof. Taking the derivative of

$$(16) \qquad F(y) = \operatorname{Prob}\{X < y^{1/\alpha}\} = \frac{2\theta^{k+1}}{\Gamma(k+1)} \int_0^{y^{1/\alpha}} x^{2k+1} e^{-\theta x^2}\,\mathrm{d}x$$

we obtain immediately (15).

It is important to note that for $\alpha = 2$, the variable X^2 obeys a Gamma law

$$(17) \qquad X^2 : f_{X^2}(x; \theta, k) = \frac{\theta^{k+1}}{\Gamma(k+1)} x^k e^{-\theta x}, \quad x > 0, \quad \theta > 0, \quad k \geqq 0.$$

The density function of X^{-1} can be obtained formally from

$$X^{\alpha} : f_{X^{\alpha}}(x; \theta, k) = \frac{2\theta^{k+1}}{|\alpha|\, \Gamma(k+1)} x^{(2/\alpha)(k+1)-1} \exp\{-\theta x^{2/\alpha}\} \tag{18}$$

$x, \theta > 0$, $k \geqq 0$, $\alpha \in R$ by taking $\alpha = -1$. We have:

$$X^{-1} : f_{X^{-1}}(x; \theta, k) = \frac{2\theta^{k+1}}{\Gamma(k+1)} x^{-(2k+3)} e^{-\theta/x_2}, \quad \theta, x > 0, \quad k \geqq 0. \tag{19}$$

If we take in (19) $k = 0$ we have

$$f_{X^{-1}}(x; \theta, 0) = \frac{2\theta}{x^3} \exp\{-\theta/x^2\}, \quad x, \theta > 0 \tag{20}$$

which is just the density function of the "inverse Rayleigh" variable introduced for the first time in the literature by Treyer [17] and studied later by Iliescu-Vodă [7] and Vodă [19].

Now, if we take in (19) $\alpha = -2$ we have

$$X^{-2} : f_{X^{-2}}(x; \theta, k) = \frac{\theta^{k+1}}{\Gamma(k+1)} x^{-(k+2)} e^{-\theta/x}, \quad x, \theta > 0, \quad k \geqq 0. \tag{21}$$

For $k = -\frac{1}{2}$, $\theta = \frac{1}{2}\lambda$ we obtain

$$X^{-2} : f_{X^{-2}}(x; \tfrac{1}{2}\lambda, -\tfrac{1}{2}) = \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \exp(-\lambda/2x), \quad x, \lambda > 0 \tag{22}$$

which is just the density function which arises in standard Brownian motion problems (Roy-Wasan [14]).

If we take in (17) $\theta = 1$ we obtain a Gamma variate with scale 1 and shape $(k+1)$.

Consider now two GR variables, say X and Y with scales 1 and shape parameters k_1 and k_2.

Therefore $X^2 + Y^2$ is a Gamma variate with shape $k_1 + k_2 + 2$. Now, a well-known property (Jambunathan [10] or IMT [9] page 3.26) may be restated as follows:

Lemma 2. *If X and Y are two independent GR variates with scales* 1 *and shapes k_1, k_2, respectively, then the variable*

$$U = \frac{X^2}{X^2 + Y^2} \tag{23}$$

obeys a Beta law with parameters $(k_1 + 1, k_2 + 1)$.

Proof. The proof is very simple if we use an idea of Emiliana Ursianu [18], namely to apply the Mellin transform:

$$(24) \qquad h_U(s) = h_{X^2}(s)\, h_{(X^2+Y^2)^{-1}}(s) = \frac{\Gamma(k_1 + s)}{\Gamma(k_1 + 1)} \frac{\Gamma(k_1 + k_2 + 3 - s)}{\Gamma(k_1 + k_2 + 2)}$$

since $h_{2^{-1}}(s) = h_2(-s + 2)$.

The basic inversion formula yields

$$(25) \quad f(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{-s} h_u(s)\, \mathrm{d}s = \frac{1}{B(k_1 + 1;\ k_2 + 1)}\, u_{k_1}(1 + u) - (k_1 + k_2 + 2),$$

$$k_1, k_2 > 0$$

where

$$(26) \qquad B(m; n) = \Gamma(m)\, \Gamma(n) / \Gamma(m + n)\,.$$

ESTIMATION PROBLEMS

A. *Maximum likelihood estimates*: consider first the simplest case, namely when k is assumed to be known. Let $x_1, x_2, \ldots, x_n$ be an independent sample on X and let L be the likelihood function. Putting $\vartheta = 1/\theta$, the log-likelihood equation

$$(27) \qquad \frac{\partial \ln L}{\partial \hat{\theta}} = n(k + 1) \frac{1}{\hat{\theta}} - \sum_{i=1}^{n} x_i^2 = 0$$

provides the solution

$$(28) \qquad \hat{\vartheta} = \frac{1}{n(k + 1)} \sum_{i=1}^{n} x_i^2\,.$$

Now, due to Lemma 1 (in the case $\alpha = 2$) we find that

$$(29) \qquad E(\hat{\vartheta}) = \vartheta \quad \text{and} \quad \operatorname{Var}(\hat{\vartheta}) = \frac{1}{n(k + 1)}\, \vartheta^2\,,$$

that is, the maximum likelihood estimate of ϑ is unbiased. This is an expected result since X^2 obeys a Gamma law and (28) involves a sum of Gamma variates. It follows therefore that all known results regarding some inferences on Gamma variable are valid.

We shall state only some of them without detailed computations:

1) the statistic $\hat{\vartheta}$ is an estimation with minimum variance; indeed, we have

$$(30) \qquad \frac{\partial \ln f}{\partial \vartheta} = -(k + 1)\, \theta + \theta^2 x^2$$

and the Rao-Cramer bound is therefore

$$\mathscr{B}_0 = \left\{nE\left(\frac{\partial \ln f}{\partial \vartheta}\right)\right\}^{-1} = \frac{1}{n(k+1)\,\theta^2}. \tag{31}$$

2) The statistic $\hat{\vartheta}$ is sufficient since it is an efficient one.

3) The density function of $\hat{\vartheta}$ is

$$\varphi^*(\hat{\vartheta};\theta) = \frac{\theta^{nk+n}(nk+n)^{nk+n}}{\Gamma(nk+n)}\,\hat{\vartheta}^{nk+n-1}\exp\{-\theta(nk+n)\,\hat{\vartheta}\}. \tag{32}$$

4) The likelihood function can be written as

$$L = \varphi^* e^{q(x_1,x_2,\ldots,x_n)} \tag{33}$$

where $q(x_1, \ldots, x_n)$ is a function which depends only on the sample values.

Lemma 3. *The maximum likelihood estimate of ϑ is consistent and asymptotically efficient in Kuldorff's sense.*

Proof. It is enough to check the conditions of Kuldorff's theorem (Kuldorff [11]) which reduces to compute

$$\partial^2 \ln f/\partial\vartheta^2\,, \quad \partial^3 \ln f/\partial\vartheta^3$$

and to verify that

$$\int_0^\infty \frac{\partial f}{\partial \vartheta}\,\mathrm{d}x = 0\,, \quad \int_0^\infty \frac{\partial \vartheta^2}{\partial^2 f}\,\mathrm{d}x = 0\,, \quad -\infty < \int_0^\infty \frac{\partial^2 \ln f}{\partial \vartheta^2} f\,\mathrm{d}x < 0\,.$$

Finally, it is necessary to construct a positive function $u = u\vartheta$ such that there exist $\mathrm{d}u/\mathrm{d}\vartheta$, $\mathrm{d}^2u/\mathrm{d}\vartheta^2$ for every $\theta > 0$ and a positive function H which does not depend on θ such that

$$\left|\frac{\partial^2}{\partial\vartheta^2}\left(u\vartheta\frac{\partial \ln f}{\partial \vartheta}\right)\right| < H \quad \text{and} \quad \int_0^\infty H f(x)\,\mathrm{d}x < +\infty\,.$$

For instance the last condition may be easily proved if we choose $u\vartheta = \vartheta^2$ and H constant and positive.

Asymptotic normality of $\hat{\vartheta}$: Let us denote

$$a_i = E(\hat{\vartheta}_i) = \vartheta\,, \quad \sigma_i^2 = \mathrm{Var}(\hat{\vartheta}_i) = \frac{1}{i(k+1)}\,\vartheta^2\,,$$

$$\varrho_i^3 = E(|\hat{\vartheta}_i - a_i|^3)\,, \quad \sigma_2^{(n)} = \sum_{i=1}^n \sigma_i^2 = \frac{1}{\theta^2(k+1)}\sum_{i=1}^n \frac{1}{i}\,, \quad \varrho_{(n)}^3 = \sum_{i=1}^n \varrho_i^3$$

$$\hat{\vartheta}_{(n)} = \frac{1}{\sigma_{(n)}}\sum_{i=1}^n (\hat{\vartheta}_i - a_i)\,, \quad \Phi_0(x) = \int_0^x (2\pi)^{-\frac{1}{2}}\,e^{-\frac{1}{2}t^2}\,\mathrm{d}t\,.$$

Let now F_i be the distribution function $\hat{\vartheta}_i$ and $F_{(n)}$ the distribution function of $\hat{\vartheta}_{(n)}$. A well-known corollary of Lyapunoff's theorem (IMT [9]) asserts that if ϱ_i^3 exists for every $i = 1, 2, \ldots, n$ and $\lim_n \varrho_{(n)}/\sigma_{(n)} = = 0$ then $\lim_n F_{(n)}(x) = \Phi_0(x)$.

In our case

$$\varrho_{(n)}^3 = \sum_{i=1}^{n} \varrho_i^3 = \frac{2}{\theta^3(k+1)^2} \sum_{i=1}^{n} \frac{1}{i^2} + \frac{6}{\theta^2(k+1)} \sum_{i=1}^{n} \frac{1}{i} \tag{33}$$

but

$$\sum_{i=1}^{n} \frac{1}{i} = \psi(n+1) + c = \psi(n) + \frac{1}{n} + U$$

where $c \cong 0{\cdot}57715664$ (Euler-Mascheroni's constant) and

$$\psi(n) = \frac{\mathrm{d} \ln \Gamma(n)}{\mathrm{d}n}.$$

We have also

$$\psi(n) = \ln n + \int_0^\infty e^{-nt}\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right) \mathrm{d}t \tag{34}$$

(Ryshyk-Gradstein [15]).

On the other hand,

$$\sum_{i=1}^{\infty} \frac{1}{i^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots = \xi(p) \tag{35}$$

where $\xi(p)$ is the Riemann function, defined as

$$\xi(i) = \frac{1}{(1-2^{1-i})\,\Gamma(i)} \int_0^\infty \frac{t^{i-1}}{1+e^t}\, \mathrm{d}t\,. \tag{36}$$

For $p = 2$ we obtain $\xi(2) = \frac{1}{6}\pi^2$ (Ryshyk-Gradstein [15]) and noticing that $\lim_n \psi(n+1) = +\infty$ we obtain immediately the condition from Lyapunoff's corollary; therefore the variable

$$Y = \frac{\hat{\vartheta} - \vartheta}{\vartheta} \sqrt{[(k+1)\, n]} \tag{37}$$

is of class $N(0, 1)$ for large n.

It is interesting to obtain some information on the speed of convergence of $\hat{\vartheta}_n$ to $N\,(\vartheta 1/n(k+1)\,\theta^2)$.

With regard to the fact that the statistic $T = \theta(nk + n)\,\hat{\vartheta}$ is distributed as a Gamma variable with scale 1 and shape $nk + n$, let γ_{α_1} be the α_1-quantile of the above Gamma variable for fixed θ, k, and n.

Compute therefore $\theta(nk + n)\,\gamma_{\alpha_1}$. This quantity is just an α_2-quantile of $\hat{\vartheta}_n$ distribution: let it be $[\hat{\vartheta}]_{\alpha_2}$. To this one there corresponds a value Y_{α_2} which is just the α_2-quantile of Y, that is

$$(38)\qquad Y_{\alpha_2} = \frac{\hat{\vartheta}_{\alpha_2} - \vartheta}{\vartheta} \sqrt{[(k + 1)\, n]}\,.$$

It follows therefore that the difference $\alpha_2 - \alpha_1$ where α_1 is derived by entries in tables for Gamma distribution and computed on the basis of $N(0, 1)$ tables, represents a measure of the convergence speed of $\hat{\vartheta}_n$ to $N(\vartheta, 1/(n(k + 1)\,\theta^2)$. If we have no special tables for Gamma distribution, χ^2 tables may be used, taking into account that if $G(u, p) = [\Gamma(p)]^{-1} \int_0^u x^{p-1} e^{-x}\, dx$, then for $2x = y$ we have

$$(39)\qquad G(u, p) = \frac{1}{2^p\, \Gamma(p)} \int_0^{2u} y^{p-1} e^{-\frac{1}{2}y}\, dy = \text{Prob}\,\{\chi^2_{2p}\,(2u)\}$$

(Bolshev-Smirnov [1]).

Asymptotic normality proved with the aid of entropy ratio: a tool which avoids the use of a characteristic function or the existence of certain moments is that introduced by Scala [16]. Namely, in our case we have to prove that

$$(40)\qquad \lim_{n\to\infty} R = \lim_{n\to\infty} \frac{-\int_0^\infty \hat{\vartheta}_n \ln \hat{\vartheta}_n\, d\hat{\vartheta}}{\ln \sqrt{(2\pi e \operatorname{Var} \hat{\vartheta}_n)}} = 1$$

in order to establish the asymptotic normality of $\hat{\vartheta}_n$.

The quantity R is called "entropy ratio" and this approach does not require the independence of variables $\hat{\vartheta}_n$.

It is easy to recognize that the numerator represents the entropy of $\hat{\vartheta}_n$.

After some tedious algebra, using some formulas from Gröbner and Hoffreiter [6] we get:

$$(41)\qquad \int_0^\infty \hat{\vartheta}_n \ln \hat{\vartheta}_n\, d\hat{\vartheta} = \tfrac{1}{2} \ln \frac{n^2}{(k + 1)\, n - 1} + \ln\,[2\pi e^{-1} \theta(k + 1)]$$

$$(42)\qquad \ln \sqrt{(2\pi e \operatorname{Var} \hat{\vartheta}_n)} = -\tfrac{1}{2} \ln n + \tfrac{1}{2} \ln \frac{2\pi e}{\theta^2(k + 1)}\,.$$

Applying L'Hospital's rule we obtain easily (40); therefore $\hat{\vartheta}_n \sim N(\vartheta, 1/(k + 1)\,\theta^2 n)$ for large n.

We shall point out now an interesting property of the maximum likelihood estimate for $\vartheta^{\frac{1}{2}}$ in the case k-known.

Lemma 4. *The maximum likelihood estimate of $\vartheta^{\frac{1}{2}}$, namely $\vartheta^{\frac{1}{2}}_{ML}$ has the property that $A_0 = \hat{\vartheta}^{\frac{1}{2}}_{ML}/\vartheta^{\frac{1}{2}}_t$ is distributed independently of $\vartheta^{\frac{1}{2}}$.*

Here $\vartheta^{\frac{1}{2}}_t$ is the true value of the parameter $\vartheta^{\frac{1}{2}}$.

Proof. If we put in (1) $y = x/\vartheta^{\frac{1}{2}}$ we obtain the reduced GRV, that is

$$X_r : f(y; k) = \frac{2}{\Gamma(k+1)} y^{2k+1} e^{-y^2}, \quad y > 0, \quad k \geqq 0 \tag{43}$$

which does not depend on $\vartheta^{\frac{1}{2}}$.

Since $\vartheta^{\frac{1}{2}}_{ML}$ is a solution of the equation

$$\frac{\partial L(x_i; \vartheta^{\frac{1}{2}}, k)}{\partial \vartheta^{\frac{1}{2}}} = 0 \tag{44}$$

we can write immediately

$$\max_{\vartheta^{1/2}} L(x_i; \vartheta^{\frac{1}{2}}, k) = L(x_i; \hat{\vartheta}^{\frac{1}{2}}_{ML}, k) \tag{45}$$

which can be expressed in terms of the reduced variables as follows:

$$\max_{\vartheta^{1/2}} L(x_i; \vartheta^{\frac{1}{2}}, k) = \frac{2^n}{\Gamma^n(k+1)} A_0^{-n} \prod_{i=1}^{n} \left(\frac{y_i}{A_0}\right)^{2k+1} \exp\left\{-\sum_{i=1}^{n} \left(\frac{y_i}{A_0}\right)^2\right\} \tag{46}$$

and which is in fact $\max_{\vartheta^{1/2}} L(y_i, k)$. Therefore A_0 corresponds to the estimation of $\vartheta^{\frac{1}{2}}$ when the sample is drawn from the reduced variable. Since the likelihood function on the reduced variable does not depend on $\vartheta^{\frac{1}{2}}$ it follows that A_0 is distributed independently of $\vartheta^{\frac{1}{2}}$.

B. *Maximum likelihood estimation in the truncation case*: it is known (Cohen [2]) that the density function of the left-truncated variable is given as

$$X_T : f_T(x; \theta, k) = \frac{1}{1 - F(x_T; \theta, k)} f(x; \theta, k) \tag{47}$$

where $x > x_T > 0$, $\theta > 0$, $k \geqq 0$, x_T being the truncation point.

Let $x_1, x_2, \ldots, x_n$, $x_i > x_T$ $(i = 1, 2, \ldots, n)$ be a sample on the truncated GRV.

The log-likelihood equation is

$$\frac{\partial \ln L}{\partial \hat{\vartheta}} = \sum_{i=1}^{n} \frac{1}{f(x_i; \hat{\theta}, k)} \cdot \frac{\partial f(x_i; \hat{\theta}, k)}{\partial \hat{\vartheta}} + n \frac{(\partial/\partial\hat{\vartheta}) F(x_T; \hat{\theta}, k)}{1 - F(x_T; \hat{\theta}, k)} = 0 \tag{48}$$

which in our case becomes after some calculations

$$\frac{\partial \ln L}{\partial \hat{\vartheta}} = -\frac{n}{\hat{\vartheta}} + \frac{1}{k+1} \frac{1}{\hat{\vartheta}^2} \sum_{i=1}^{n} x_i^2 - \frac{1}{\hat{\vartheta}} \frac{n}{2} x_T \Gamma(k+1) \lambda(x_T) = 0 \tag{49}$$

where $\lambda(x_T)$ is the hazard rate (see section C) computed at the truncation point.

If x_T is assumed to be known, in particular if $k = 0$ (truncated Rayleigh variable, X_{TR}) we obtain

$$\hat{\vartheta}_{ML} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - x_T^2 . \tag{50}$$

It is to note that in this case $\hat{\vartheta}_{ML}$ is not unbiased. We have

$$E(X_{TR}) = \frac{\pi^{\frac{1}{2}}}{2\theta^{\frac{1}{2}}} + x_T e^{-\theta x_T^2} - \Phi_L(x_T(2\pi)^{\frac{1}{2}} \quad \text{and} \quad E(X_{TR}^2) = \tag{51}$$

$$= \vartheta e^{-\theta x_T^2} + x_T^2 e^{-\theta x_T^2}$$

where $\Phi = \pi^{-\frac{1}{2}} \int_0^u e^{-\frac{1}{2}t^2} \, dt$.

It follows that

$$E(\hat{\vartheta}_{ML}) = \vartheta e^{-\theta x_T^2} + (e^{-\theta x_T^2} - 1)\, x_T^2 . \tag{52}$$

For $x_T = 0$ the usual situation occurs.

C. *Maximum likelihood estimation from censored samples*: an application to life testing. Starting from an engineering problem which concerns the durability testing of a certain kind of cutting tools, we develop in this section an estimation of the expected life of tools when time to failure obeys the GR law.

Experimental data are considered to be subjected to a general kind of censorization. The method applied is that of the maximum likelihood in the case of censored samples.

In engineering practice, the following situation occurs: we have to test the durability of some grinding tools of a batch containing a large number of items.

From this lot, a random sample of n tools is subjected to a life test.

The procedure obeys generally the following rules: experimenter observes when the first failure ossurs and records the moment, say x_1. Then he removes from the $(n - 1)$ tools which did not fail a random sample of size n_1, and tests the remaining tools till the next failure occurs.

Let the moment of the second failure be x_2. Then a random of n_2 objects are removed from the $(n - n_1 - 2)$ tools still in order.

The procedure is applied till the r-th failure occurs and in this case the rest of items:

$$n_r = n - \sum_{j=1}^{r-1} n_j - r \tag{53}$$

are removed from the test.

It is considered that the life-testing procedure is finished when the r-th tool fails.

In this procedure, the experimenter knows:

1) the size of sample subjected to the life-testing (n);
2) the allowable number of tools which can fail (r);

3) the sizes of samples which are excluded from the test $(n_i;\ i = 1, 2, \ldots, r)$;
4) the moments when the failures occur - that is the experimental data x_i, $i = 1, 2, \ldots, r$.

We are looking for an estimation of the expected life (mean durability) when time to failure obeys the G.R. distribution.

Mean durability estimation reduces in fact to estimate the parameter in the parent density function.

Consider now that the shape parameter in the G. R. variable is known and let $\tilde{x} = (x_1, x_2, \ldots, x_r)$ be the vector of observations.

If we take into account the rules of the test, we have

$$n = r + \sum_{j=1}^{r} n_j . \tag{54}$$

Denote

$$N_i = n - \sum_{j=1}^{i-1} n_j - i + 1 . \tag{55}$$

It is clear that N_i represents the number of tools which the experimenter continues to observe after the $(i - 1)$-st failure and the corresponding exclusion.

The likelihood function is

$$L(\tilde{x}; \theta) = \prod_{i=1}^{r} [N_i f(x_i; \theta, k) (1 - F(x_i; \theta, k))^{n_i}]$$

that is the distribution of all observed failures. In the above formula $f(x_i; \theta, k)$ represents the probability that the next item fail at the moment x_i, $[1 - F(x_i; \theta, k)]^{n_i}$ represents the probability that n_i objects are still in order at the same moment.

The log-likelihood equation

$$\frac{\partial \ln L(\tilde{x}, \hat{\theta})}{\partial \hat{\theta}} = (k + 1)\, r \frac{1}{\hat{\theta}} - 2 \sum_{i-1}^{r} \ln x_i - \frac{1}{2\hat{\theta}} \sum_{i=1}^{r} n_i x_i\, \lambda(x_i) = 0 \tag{56}$$

where $\lambda(x_i) = f(x_i; \theta, k)/(1 - F(x_i; \theta, k))$ is the hazard rate provides the solution

$$\hat{\theta}_{ML} = \frac{2(k + 1)\, r - \sum_{i=1}^{r} n_i x_i\, \lambda(x_i)}{4 \sum_{i=1}^{r} \ln x_i} . \tag{57}$$

Therefore, to obtain a maximum likelihood estimate of $\hat{\theta}$ and implicitly an estimation of the expected-life, we must estimate by means of known methods the hazard rate of the model.

D. *Maximum likelihood estimation in the case when both parameters are unknown*: Let us consider now the case when k is not known. Then the likelihood system is

$$\frac{\mathrm{d} \ln \Gamma(\hat{k} + 1)}{\mathrm{d}\hat{k}} - n \ln \hat{\theta} = 2 \sum_{i=1}^{n} \ln x_i \tag{58}$$

$$n\hat{k} - \hat{\theta} \sum_{i=1}^{n} x_i^2 = -n .$$

If we use the same notation as above, we have

$$\psi(k + 1) = -c + \sum_{j=2}^{\infty} (-1)^i \xi(i) k^{i-1} . \tag{59}$$

Considering the terms up to the fifth degree, we have

$$\psi(k + 1) \cong -0{\cdot}577215664 + 1{\cdot}644934067k - 1{\cdot}202056904k^2 + + 1{\cdot}082323234k^3 - 1{\cdot}036927765k^4 + 1{\cdot}017343062k^5 , \tag{60}$$

$$\ln \theta \cong (\theta - 1) - \tfrac{1}{2}(\theta - 1)^2 + + \tfrac{1}{3}(\theta - 1)^3 - \tfrac{1}{4}(\theta - 1)^4 + \tfrac{1}{5}(\theta - 1)^5 \tag{61}$$

and in this way, the likelihood system can be reduced to an equation of a high order in k or θ.

4. LINEAR ESTIMATION

In this paragraph, we shall construct linear estimation for $\vartheta^{\frac{1}{2}}$ using the so-called "quasi-ranges".

Firstly, let us note that the theoretical median of the G.R. distribution, given by the equation

$$\Gamma_{\theta(m_e)^2}(k + 1) = \tfrac{1}{2}\Gamma(k + 1) , \tag{62}$$

can be written approximately as

$$m_e \cong \left(\frac{1}{2C_0}\right)^{1/(2k+2)} \vartheta^{\frac{1}{2}} \tag{63}$$

where C_0 is a constant which can be read from incomplete Gamma function tables depending on values of k.

Now, let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be order statistics obtained from a random sample $x_1, x_2, \ldots, x_n$ on X.

Define two random variables as follows:

$$X_1 : f_1(x; \theta, k) = \begin{cases} 2 f(x; \theta, k) & \text{for } \quad x \leqq m_e \\ 0 & \text{otherwise}, \end{cases} \tag{64}$$

$$X_2 : f_2(x; \theta, k) = \begin{cases} 0 & \text{for } \quad x < m_e \\ 2 f(x; \theta, k) & \text{otherwise}. \end{cases} \tag{65}$$

The expected values of these variables are

$$E(X_1) = 2 \int_0^{m_e} x f(x; \theta, k) \, \mathrm{d}x \quad \text{and} \quad E(X_2) = 2 \int_{m_e}^{\infty} x f(x; \theta, k) \, \mathrm{d}x . \tag{66}$$

The above mean-values can be estimated by

$$S_1 = \frac{1}{[n/2]} \sum_{i=1}^{[n/2]} x_{(i)} \quad \text{and} \quad S_2 = \frac{1}{[n/2]} \sum_{i=1}^{[n/2]} x_{(n-i+1)} \tag{67}$$

where $[n/2]$ is the integer part of $n/2$.

Therefore we can estimate the parameter from the relation:

$$E(X_2) - E(X_1) = S_2 - S_1 = \frac{1}{[n/2]} \sum_{i=1}^{[n/2]} w_{(i)} \tag{68}$$

where $w_{(i)} = x_{(n-i+1)} - x_{(i)}$ is the i-th quasi-range.

In our case, straightforward computation provides

$$E(X_1) = \frac{(k+1)\, \vartheta^{\frac{1}{2}}}{(2C_0)^{1/(k+1)}} \quad \text{and} \quad E(X_2) = 2 \frac{\Gamma(k+\frac{3}{2})\, \vartheta^{\frac{1}{2}}}{\Gamma(k+1)} - \frac{(k+1)\, \vartheta^{\frac{1}{2}}}{(2C_0)^{1/(k+1)}} . \tag{69}$$

Therefore

$$\hat{\vartheta}^{\frac{1}{2}} = C(n; k) \sum_{i=1}^{[n/2]} w_{(i)} \tag{70}$$

where

$$C(n; k) = \frac{1}{[n/2]} \left[2 \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+1)} - \frac{2(k+1)}{(2C_0)^{1/(k+1)}} \right]^{-1} . \tag{71}$$

Taking $\theta = 1/\lambda^2$ with $\lambda > 0$ and noticing that for $k = 0$ one obtains Rayleigh distribution, we have after some algebra the following estimate for λ:

$$\hat{\lambda} = \frac{1{\cdot}344}{[n/2]} \sum_{i=1}^{[n/2]} w_{(i)} . \tag{72}$$

It should be mentioned that in the case $k = 0$, the median can be expressed exactly as $\lambda \sqrt{\ln 2}$.

Former results form [8] imply that estimators of the type (70) are asymptotically unbiased estimators for the underlying parameters.

We shall prove here

Theorem 1. *The statistic* $h(x) = x\,\Gamma(k+\frac{3}{2})/\Gamma(k+2)$ *where* k *is assumed to be known is an unbiased estimate in Lehmann's sense for* $\vartheta^{\frac{1}{2}}$ *with respect to the loss function*

$$W[\vartheta^{\frac{1}{2}}; h(x)] = \theta\left(\frac{x\,\Gamma(k+\frac{3}{2})}{\Gamma(k+2)} - \vartheta^{\frac{1}{2}}\right)^2. \tag{73}$$

It is also minimax and admissible with respect to the same loss function.

Proof. To prove unbiasedness in Lehmann's sense [12] we shall use a theorem by Goodman [4], [5] which states that if $B = E(\xi/\lambda)/E(\xi/\lambda)^2 = \text{constant}$, where ξ is a random variable with a density function $f(x; \lambda)$, then the statistic $B\xi$ is an unbiased estimate for λ with respect to the loss function

$$W[\lambda; h(x)] = (h(x) - \lambda)^2/\lambda^2. \tag{74}$$

In our case we have immediately:

$$B = \frac{E(X\sqrt{\theta})}{E(X\sqrt{\theta})^2} = \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+2)} = \text{constant} \tag{75}$$

and hence the first part of the theorem is proved.

The risk associated (Wald [20]) with the loss function (73) is given as

$$R_{h(x)}\vartheta^{\frac{1}{2}} = 1 - \frac{\Gamma^2(k+\frac{3}{2})}{\Gamma(k+2)\,\Gamma(k+1)}. \tag{76}$$

To prove that the minimum is reached for $h(x)$ – constructed in this way – we shall proceed in this way: suppose for instance that there exists another estimate $h_1(x)$ for which the risk assumes its minimum. We choose $h_1(x)$ so that

$$E(h_1(x)) = \vartheta^{\frac{1}{2}} + a\vartheta^{\frac{1}{2}} = \varphi\vartheta^{\frac{1}{2}}. \tag{77}$$

The tisk in this case is

$$R_{h_1(x)}\vartheta^{\frac{1}{2}} = \theta[\operatorname{Var}(h_1(x)) + a^2\vartheta^{\frac{1}{2}}]. \tag{78}$$

Apply now the Rao-Cramer inequality

$$R_{h_1(x)}\vartheta^{\frac{1}{2}} \geqq \theta\left[\frac{(1+a'\vartheta^{\frac{1}{2}})^2}{E\left(\dfrac{\partial \ln f}{\partial\vartheta^{\frac{1}{2}}}\right)^2} + a^2\vartheta^{\frac{1}{2}}\right] = \theta a^2\vartheta^{\frac{1}{2}} + \frac{(1+a'\vartheta^{\frac{1}{2}})^2}{4(k+1)}. \tag{79}$$

From this point on the proof consists only in very tedious computations the assumption that $h_1(x)$ attains its minimum leads to a contradiction and hence $h(x)$ is the desired estimate. We omit here these computations.

6. A SPECIAL PROBLEM OF ESTIMATION

In this paragraph, we shall consider the following problem: let X be a generalized Rayleigh variable with a density function $f(x; \theta, k)$ where the parameters θ and K are supposed to be known.

Let $\varphi(x; \omega)$ be the probability that X is observable. We assume that the probability $\varphi(x; \omega)$ is not known and our aim is an estimation of the parameter ω based on a sample of size $r : x_1, x_2, \ldots, x_r$, drawn from the parent population, knowing that we have $(n - r)$ unobserved values.

The problem when the density function of a random variable is assumed to be known except an unknown parameter, was solved by D. E. Lloyd [13]. (His method concerns the χ^2 distribution with unknown scale parameter. The results can be verified easily if we use the square of a generalized Rayleigh variate which includes as a particular case the χ^2 distribution.)

We shall distinguish here the following cases:

a) we draw a sample of size n from the parent population and look for the probability that the sample values be $x_1, x_2, \ldots, x_r$;

b) we draw a sample of size n from the parent population and look for the probability of obtaining r observed values;

c) given r observations from X (the possible total n being unknown) we look for the probability that the selected values be $x_1, x_2, \ldots, x_r$.

In each case, the method of estimation will be that of the maximum likelihood, the likelihood function being in each case the required probability.

Let us note that if we select randomly a value x, the probability that it will be observable is

$$p(\omega) = \int_0^\infty f(x; \theta, k)\, \varphi(x; \omega)\, \mathrm{d}x \,. \tag{80}$$

Straightforward computations provide the following log-likelihood equations:

$$\text{a)}\quad \frac{1}{r} \sum_{i=1}^{r} \frac{\partial}{\partial \omega} \ln \varphi(x_i; \omega) - \left(1 - \frac{r}{n}\right), \tag{81}$$

$$\text{b)}\quad p(\omega) = \frac{r}{n}\,, \quad \text{(an expected result)}\,, \tag{82}$$

$$\text{c)}\quad \frac{1}{r} \sum_{i=1}^{r} \frac{\partial}{\partial \omega} \ln \varphi(x_i; \omega) - \frac{1}{p(\omega)} \cdot \frac{\partial p(\omega)}{\partial \omega} = 0 \tag{83}$$

(see also Lloyd [13]).

Now, the problem reduces in fact to choosing the form of $\varphi(x; \omega)$ a continuous function such that $0 \leqq \varphi(x; \omega) \leqq 1$ for every $x > 0$, $\omega > 0$ being unknown.

In our case we choose

$$(84) \qquad \varphi(x; \omega) = \exp\{-\omega x^2\}, \quad x > 0, \quad \omega > 0.$$

The conditions requested are obviously satisfied.

Therefore

$$(85) \qquad \text{a)} \quad -\frac{1}{r}\sum_{i=1}^{r} x_2^i + \left(\frac{n}{r} - 1\right)\left(k + \tfrac{1}{2}\right)\frac{\theta^{k+1}}{(\theta + \omega)^{k+\frac{1}{2}} - \theta^{k+1}} \cdot \frac{1}{\theta + \omega} = 0,$$

$$(86) \qquad \text{b)} \quad -\frac{1}{r}\sum_{i=1}^{r} x_i^2 + \frac{k + \frac{1}{2}}{\theta + \omega} = 0.$$

Notice that $(1/r)\sum_{i=1}^{r} x_i^2$ is the second noncentral sample moment calculated with r observations. Denote $r^{-1}\sum_{i=1}^{r} x_i^2 = M_{2(r)}$.

From (c) we obtain

$$(87) \qquad \hat{\omega} = \left(k + \tfrac{1}{2}\right) M_{2(r)}^{-1} - \theta.$$

Taking into account the invariance property of the likelihood estimation, we get

$$(88) \qquad E\left[\left(\widehat{\frac{1}{\omega + \theta}}\right)\right] = \frac{k + 1}{k + \frac{1}{2}}\,\vartheta.$$

For the simple Rayleigh model $(k = 0)$ we obtain from a)

$$(89) \qquad M_{2(r)}u^3 - \theta M_{2(r)}u^2 + \tfrac{1}{2}\theta\left(1 - \frac{n}{r}\right) = 0$$

where we have denoted $\theta + \omega = u^2$. Direct calculations yield the desired estimate for ω.

Remark. The converse problem – which means in our case that the scale θ in $f(x; \theta, k)$ is considered unknown and we wish to estimate θ by means of $x_1, x_2, \ldots$ $\ldots, x_r$ and $\varphi(x; \omega) = e^{-\omega x^2}$, $\omega > 0$ known (the shape parameter K is also assumed to be known – leads to the following equations for a) and c):

$$(90) \qquad \frac{k + 1}{\theta} - M_{2(r)} + \left(\frac{n}{r} - 1\right)\frac{(\theta + \omega)^{k+1/2} - \theta^{k+1}}{\theta^k} \cdot \frac{(k + 1)\,\omega + \frac{1}{2}\theta}{\theta + \omega} = 0$$

and

$$(91) \qquad \frac{k + 1}{\theta} - M_{2(r)} - \frac{(k + 1)\,\omega + \frac{1}{2}\theta}{\theta(\theta + \omega)} = 0.$$

For $k = 0$ we obtain respectively

$$\frac{1}{u^2 - \omega} - M_{2(r)} + \frac{\left(1 - \frac{n}{r}\right)[\omega + \frac{1}{2}(u^2 - \omega)]}{u^3 - u^2 + \omega u} = 0 \tag{92}$$

and

$$\vartheta - M_{2(r)} - \frac{\omega + \frac{1}{2}\theta}{\theta(\theta + \omega)} = 0 . \tag{93}$$

(We have put $\theta + \omega = u^2$.)

The last equation yields easily

$$\hat{\theta} = [2M_{2(r)}]^{-1} - \omega . \tag{94}$$

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Souhrn

PROCEDURY STATISTICKÉ INDUKCE PRO ZOBECNĚNOU RAYLEIGHOVU PROMĚNNOU (I)

V. GH. VODĂ

V práci se studuje určitá jednorozměrná náhodná proměnná, zahrnující některé důležité speciální případy jako Rayleighovu, Maxwellovu proměnnou a některé další. Tato část je věnována různým problémům odhadů.

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