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A REMARK ON JORDAN ELIMINATION

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A simplified version of the Jordan elimination algorithm and the modified Jordan elimination algorithm [1], suitable for hand as well as machine computation, is presented.

INTRODUCTION

In [1], *Jordan elimination* is defined as follows: consider the system

(1) $$y_i + a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \quad i = 1, \ldots, n$$

of m linear forms in n variables $x_1, \ldots, x_n$. Such a system can be represented by the table

(2) $$\begin{array}{rcl} & & x_1 x_2 \quad \ldots x_s \quad \ldots x_n \\ y_1 & = & a_{11} a_{12} \ldots a_{1s} \ldots a_{1n} \\ y_2 & = & a_{21} a_{22} \ldots a_{2s} \ldots a_{2n} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ y_r & = & a_{r1} a_{r2} \ldots a_{rs} \ldots a_{rn} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ y_m & = & a_{m1} a_{m2} \ldots a_{ms} \ldots a_{mq} \end{array}$$

To perform one step of Jordan elimination with the *pivot element* a_{rs}, r-th *pivot row* and s-th *pivot column* means to find the coefficients b_{ij} $i = 1, \ldots, m; j = 1, \ldots, n$ of the system

(3) $$\begin{array}{rcl} & & x_1 x_2 \quad \ldots y_r \quad \ldots x_m \\ y_1 & = & b_{11} b_{12} \ldots b_{1r} \ldots b_{1m} \\ y_2 & = & b_{21} b_{22} \ldots b_{2r} \ldots b_{2n} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ x_s & = & b_{r1} b_{r2} \ldots b_{rs} \ldots b_{rn} \\ \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ y_m & = & b_{m1} b_{m2} \ldots b_{ms} \ldots b_{mn} \end{array}$$

in which x_r and y_s have exchanged places. It is easy to prove that the following relations hold:

Let $z = a_{rs}$. Then

(4a) for $i \neq r, j \neq s$

$$b_{ij} = a_{ij} - \frac{u_i v_j}{z}$$

where $u_i = a_{is}$, $v_j = a r_j$;

(4b) for $j \neq s$

$$b_{rj} = -a_{rj}/z\ ;$$

(4c) for $i \neq r$

$$b_{is} = a_{is}/z\ ;$$

(4d)

$$b_{rs} = 1/z\ .$$

The *modified Jordan elimination step*, used e.g. in linear programming, is defined as the transition from the system

(5)

$$\begin{array}{ll} & -x_1 \dots \quad -x_s \dots \quad -x_n \\ y_1 = & c_{11} \quad \dots c_{1s} \quad \dots c_{1n} \\ \dots\dots & \dots\dots\dots\dots\dots\dots \\ y_r = & c_{r1} \quad \dots c_{rs} \quad \dots c_{\cdot n} \\ \dots\dots & \dots\dots\dots\dots\dots\dots \\ y_m = & c_{m1} \quad c_{ms} \quad \dots c_{mn} \end{array}$$

to the system

(6)

$$\begin{array}{ll} & -x_1 \dots \quad -y_r \dots \quad -x_n \\ y_1 = & d_{11} \quad \dots d_{1s} \quad \dots d_{1n} \\ \dots\dots & \dots\dots\dots\dots\dots\dots \\ x_s = & d_{r1} \quad \dots d_{rs} \quad \dots d_{rn} \\ \dots\dots & \dots\dots\dots\dots\dots\dots \\ y_m = & d_{m1} \quad \dots d_{ms} \quad \dots d_{mn} \end{array}$$

Again, the following formulae are easily proved: Let $z = c_{rs}$. Then

(7a) for $i \neq r, j \neq s$

$$d_{ij} = c_{ij} - \frac{u_i v_j}{z}$$

where $u_i = c_{is}$, $v_j = c_{rj}$;

(7b) for $j \neq s$

$$d_{rj} = c_{rj}/z;$$

(7c) for $i \neq r$

$$d_{is} = -c_{is}/z \; ;$$

$$d_{rs} = 1/z \, .$$

It is easy to see the only difference between Jordan and modified Jordan elimination is that in the former the sign changes in the pivot row, while in the latter it is in the pivot column.

ANOTHER ALGORITHM

Two objections can be made to these algorithms:

a) there are "too many cases" to be considered in both machine and hand computation.

b) for hand computation, the basic scheme involves four quantities a_{ij}, a_{is}, a_{rj}, a_{rs} located in the four corners of a rectangle. Since all possible rectangles must be considered, it is only too possible to make a mistake.

Consider the formulae (4a)–(4d). We see that u_r and v_s are not used in the computation. Is it possible to define them in such a way that the formula

$$b_{ij} = a_{ij} - \frac{u_i v_j}{z}$$

might then be applicable to any i and j?

Let us try to find the necessary value for u_r. To have

$$b_{rj} = a_{rj} - \frac{u_r v_j}{z}$$

for $j \neq s$, we must have

$$u_r = \frac{a_{rj} - b_{rj}}{v_j} \, . \, z = z + 1 .$$

Similarly $v_s = z - 1$.

It remains to be seen whether these values yield the correct result if both $i = r$ and $j = s$. We have

$$b_{rs} = a_{rs} - \frac{u_r v_s}{z} = z - \frac{(z+1)(z-1)}{z} = \frac{1}{z} \, .$$

Thus it is possible to write the following formulae for the Jordan elimination: Let $z = a_{rs}$; then for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$

(8a) $$b_{ij} = a_{ij} - \frac{u_i v_j}{z}$$

where

$$u_i = a_{is} + \delta_{ir}, \tag{8b}$$

$$v_j = a_{rj}\delta_{js}, \tag{8c}$$

and

$$\delta_{ij} = 0 \quad \text{for} \quad i \neq j, \qquad 1 \quad \text{for} \quad i = j. \tag{8d}$$

For the modified Jordan elimination the formulae are quite similar, only the signs to go with the delta – symbols are reversed.

CONCLUSION

A modified formula for the Jordan elimination has been presented. It should make for simpler programs for machine computation and better organization of hand computation.

References

[1] *С. И. Зуховицкий, Л. И. Авдеева:* Линейное и выпуклое програмирование. Наука, Москва 1967.

Souhrn

POZNÁMKA K JORDANOVĚ ELIMINACI

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V článku se navrhuje zjednodušení vzorců pro Jordanovu eliminaci, které umožňuje používat jediný vzorec pro výpočet všech prvků nové matice. To dovoluje a) zjednodušení programů pro strojový výpočet, b) lepší organizaci ručních výpočtů.

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