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ON EVOLUTION INEQUALITIES OF A MODIFIED NAVIER-STOKES TYPE, I

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INTRODUCTION

It is the purpose of the present paper to develop an existence and regularity theory for a class of evolution inequalities which includes the weak formulation of a type of modified Navier-Stokes equations under certain unilateral boundary conditions.

In the present first part of our paper we are going to prove an existence theorem for a strong solution to a class of abstract evolution inequalities. These studies will be continued in the second part by establishing the existence and uniqueness of a weak solution to the abstract evolution inequality under consideration. Besides we shall prove some regularity results for the strong solution. In both parts, the conditions imposed upon the operators occuring in the evolution inequality reflect the essential features of the boundary value problem we are going to discuss in the third part. This part of our paper will be concerned with the application of the abstract results obtained to the following type of modified Navier-Stokes equations:

$$\frac{\partial u_i}{\partial t} - \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left[\left(\mu_0 + \mu_1 |\nabla u|^{r-2}\right)\frac{\partial u_i}{\partial x_k}\right] +$$

$$+ \sum_{k=1}^{3} u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} = f_i\,, \quad (i = 1, 2, 3)\,; \quad in \quad \Omega \times [0, T]$$

$$\operatorname{div} u = 0$$

(Ω denotes a bounded domain in $\mathbb{R}^3$, $r > 2$, $\mu_j = \text{const} > 0$ $(j = 0, 1)$, $|\nabla u| = [\sum_{i,k=1}^{3} ((\partial u_i)/\partial x_k)^2]^{1/2}$). The boundary conditions we shall consider are as follows:

$$
\left.\begin{aligned}
&u_\tau = 0\,, \quad u \cdot \nu \leqq 0\,, \\
&\left(\mu_0 + \mu_1 |\nabla u|^{r-2}\right) \frac{\partial u}{\partial \nu} \cdot \nu - p \leqq 0\,, \\
&\left(\mu_0 + \mu_1 |\nabla u|^{r-2}\right) \frac{\partial u}{\partial \nu} \cdot u - pu \cdot \nu = 0
\end{aligned}\right\} \quad on \quad \Gamma_1 \times [0, T]\,,
$$

$$
\left.\begin{aligned}
&u_\tau = 0, \quad u \cdot \nu \geqq 0\,, \\
&\left(\mu_0 + \mu_1 |\nabla u|^{r-2}\right) \frac{\partial u}{\partial \nu} \cdot \nu - p \geqq 0\,, \\
&\left(\mu_0 + \mu_1 |\nabla u|^{r-2}\right) \frac{\partial u}{\partial \nu} \cdot u - pu \cdot \nu = 0
\end{aligned}\right\} \quad on\, \Gamma_2 \times [0, T]\,,
$$

$$
u = 0 \quad on \quad \Gamma_3 \times [0, T]
$$

(Γ = boundary of Ω, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, ν = outer unit normal, $u_\tau = u - (u \cdot \nu)\,\nu$ tangential component of u). These boundary conditions will be completed by the usual initial condition. The function u which satisfies the above problem, describes the motion of a viscous, incompressible fluid with big gradient of velocity (cf. [5], [6]) through the "tube" Ω with pressure p and under the external force $f = \{f_1, f_2, f_3\}$. The first two boundary conditions on Γ_1 (Γ_2) express the fact that the fluid runs into Ω along Γ_1 (respectively that it leaves Ω along Γ_2), while the remaining conditions on Γ_1 and Γ_2 are natural boundary conditions. The boundary condition on Γ_3 means that the fluid adheres to this part of the boundary.

It is obvious that we may impose only straightforward estimates from above upon the bilinear operator which represents the convection term. This peculiarity leads to some sharp differences between our theory and the theories known about the Navier-Stokes equations.

The modifications of the Navier-Stokes equations which are based on the concept of the motion of a viscous, incompressible fluid with big gradient of velocity are extensively studied in [5], [6], [7; Chap. 2·5] (under zero boundary conditions). On the other hand, let us refer to the papers [1], [2], [3] where existence theorems for evolution inequalities related to the (usual) Navier-Stokes equations in two dimensions may be found. Another type of unilateral boundary conditions for the (usual) Navier-Stokes equations is discussed in [8].

In the first section of the present paper we state the main existence result. Its proof which rests upon a Galerkin type argument combined with the regularization of the functional involved, is carried out in the second section.

Let H be a real Hilbert space with scalar product $(\, , \,)$ and norm $| \; |$. Suppose we are given another real Hilbert space V with scalar product $((\, , \,))$ and norm $\| \; \|$. We further assume that V is compactly and densely imbedded into H. We denote by V^* the dual of V, and by (x^*, x) the dual pairing between $x^* \in V^*$ and $x \in V$. Further, let W be a real, separable, reflexive Banach space that is continuously and densely imbedded into V. Let $||| \; |||$ denote the norm on W, W^* the dual of W, $||| \; |||_*$ the dual norm on W^*, and $(y^*. y)$ the dual pairing between $y^* \in W^*$ and $y \in W$. Identifying H with its dual one obtains continuous and dense imbeddings $H \subset V^* \subset W^*$, and in case $h \in H$ and $x \in V$ (resp. $x \in W$) the dual pairing between h and x coincides with their scalar product in H.

Let $A : W \to W^*$ be a (nonlinear) mapping that satisfies the following conditions:

$$\text{(1.1)} \qquad A \textit{ is monotone and hemi-continuous}^{1)} ;$$

$$\text{(1.2)} \qquad \begin{cases} \textit{there exists a functional } F : W \to \mathbb{R} \textit{ such} \\ \textit{that } A = \operatorname{grad} F, \textit{ where } F(x) \geqq c_1 |||x|||^p + c_2 \\ \forall x \in W,\ c_1 = \text{const} > 0,\ p \geqq 4,\ c_2 = \text{const}; \end{cases}$$

$$\text{(1.3)} \qquad |||Ax|||_* \leqq c_3(|||x|||^{p-1} + 1) \quad \forall x \in W,\ c_3 = \text{const}.$$

Further, let B be a bilinear mapping from $W \times W$ into W^* such that

$$\text{(1.4)} \qquad \begin{cases} |(B(x, y), z)| \leqq c_4 |x| \, |||y||| \, |||z||| , \\ |(B(x, y), z)| \leqq c_4 |||x||| \, |||y||| \, |z| \\ \forall x, y, z \in W,\ c_4 = \text{const}. \end{cases}$$

Let $\varphi : V \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous functional. Let $D(\varphi)$ denote its effective domain, i.e.

$$D(\varphi) = \{x \in V : \varphi(x) < +\infty\} .$$

For our further purposes we suppose that

$$\text{(1.5)} \qquad \varphi(x) \geqq \varphi(0) \quad \forall x \in V .$$

Next, given $u \in L^2(0, T; V)$ $(0 < T < +\infty)$ we set

$$\Phi(u) = \begin{cases} \displaystyle\int_0^T \varphi(u)\, \mathrm{d}t & \textit{if } \ \varphi(u(.)) \in L^1(0, T) , \\ +\infty & \textit{otherwise} . \end{cases}$$

---

[1]) That is, the function $t \mapsto (A(x + ty), z)$ is continuous on the real line for arbitrary but fixed $x, y, z \in W$.

The functional Φ is proper, convex and lower semi-continuous on $L^2(0, T; V)$ (cf. [4; Prop. 2.16]).

The main result of our paper is the following

Theorem. *Let the mapping A satisfy the conditions* (1.1)–(1.3), *while B is assumed to fulfil* (1.4). *Let the functional φ satisfy* (1.5).

Further suppose that

$$f = f_1 + f_2 :$$

$$f_1 \in L^2(0, T; H), \quad f_2, f_2' \in L^{p'}(0, T; W^*) \left(p' = \frac{p-1}{p}\right),^1)$$

$$u_0 \in W \cap D(\varphi) .$$

Then there exists a function $u \in L^\infty(0, T; W) \cap C([0, T]; H)$ such that

$$\Phi(u) < +\infty , \quad u' \in L^2(0, T; H) ; \tag{1.6}$$

$$\begin{cases} \int_0^T (u' + Au + B(u, u), v - u) \, \mathrm{d}t + \Phi(v) - \Phi(u) \geqq \int_0^T (f, v - u) \, \mathrm{d}t \\ \forall v \in L^p(0, T; W) ; \end{cases} \tag{1.7}$$

$$u(0) = u_0 . \tag{1.8}$$

Remark. – Condition (1.1) implies that A maps strongly convergent sequences into weakly convergent sequences. Thus, by Pettis' theorem, the function $t \mapsto A\,v(t)$ is strongly measurable on $[0, T]$ (with respect to W^*) for any $v \in L^p(0, T; W)$ (cf. [9] for details). Setting $(\mathscr{A}(v))(t) = A\,v(t)$ for a. a. $t \in [0, T]$ and any $v \in L^p(0, T; W)$ one easily concludes from hypotheses (1.1), (1.3) that $\mathscr{A}$ is a monotone, hemi-continuous mapping from $L^p(0, T; W)$ into $L^{p'}(0, T; W^*)$.

Further, each of the estimates in (1.4) implies that $B(.,.)$ is a continuous bilinear mapping from $W \times W$ into W^*. Hence the function $t \mapsto B(v(t), v(t))$ is strongly measurable on $[0, T]$ (with respect to W^*) for any $v \in L^p(0, T; W)$ (cf. [9]). Thus, setting $(\mathscr{B}(v))(t) = B(v(t)), v(t))$ for a.a. $t \in [0, T]$ and any $v \in L^p(0, T; W)$ we obtain $\mathscr{B}(v) \in L^{p'}(0, T; W^*)$ (since $2p' \leqq p$).

2. PROOF OF THE THEOREM

We begin by making preliminaries which are necessary for what follows.

Given $\varepsilon < 0$ we set for any $x \in V$

$$\varphi_\varepsilon(x) = \min_{y \in V} \left\{\frac{1}{2\varepsilon} \|y - x\|^2 + \varphi(y)\right\}$$

---

[2]) Throughout the whole paper, the derivatives are to be understood in the sense of vector-valued distributions.

(cf. [4; Prop. 2.11]). Obviously, $\varphi_\varepsilon(x) \leqq \varphi(x)$ for all $x \in V$. The functional φ_ε is convex and Fréchet differentiable on V, and the Fréchet derivative φ'_ε coincides with the Yosida-approximation of $\partial\varphi$:

$$\varphi'_\varepsilon = (\partial\varphi)_\varepsilon = \frac{1}{\varepsilon}(I - J_\varepsilon),$$

$$J_\varepsilon = (I + \varepsilon\,\partial\varphi)^{-1}$$

where I denotes the identity in V, and $\partial\varphi$ the subdifferential mapping of φ (as (multi-valued) mapping of V into itself), i.e.

$$\partial\varphi(x) = \{z \in V : \varphi(y) \geqq \varphi(x) + ((z, y - x))\ \forall y \in V\}$$

(cf. [4; Prop. 2.16]). The mapping $(\partial\varphi)_\varepsilon$ is monotone and Lipschitzian (with Lipschitz constant $1/\varepsilon$). Further, (1.5) implies $\varphi_\varepsilon(x) \geqq \varphi(0)$ for all $x \in V$, and $(\partial\varphi)_\varepsilon(0) = 0$.

We now define a mapping $C_\varepsilon : W \to W^*$ by

$$(C_\varepsilon(x), y) = (((\partial\varphi)_\varepsilon(x), y))\ \forall x, y \in W.$$

It is readily seen that C_ε is monotone and Lipschitzian. As above, defining $(\mathscr{C}_\varepsilon(v))(t) = C_\varepsilon(v(t))$ for a.a. $t \in [0, T]$ and any $v \in L^p(0, T; W)$ one easily verifies that $\mathscr{C}_\varepsilon$ is a monotone and continuous mapping from $L^p(0, T; W)$ into $L^{p'}(0, T; W^*)$.

Finally, setting for $\varepsilon > 0$ and $u \in L^2(0, T; V)$

$$\Phi_\varepsilon(u) = \min_{v \in L^2(0,T;V)} \left\{\frac{1}{2\varepsilon}\|v - u\|^2_{L^2(0,T;V)} + \Phi(v)\right\}$$

we have

$$\Phi_\varepsilon(u) = \frac{1}{2\varepsilon}\|u - \mathscr{J}_\varepsilon(u)\|^2_{L^2(0,T;V)} + \Phi(\mathscr{J}_\varepsilon(u))$$

$$= \int_0^T \varphi_\varepsilon(u)\,dt$$

for all $u \in L^2(0, T; V)$, where $\mathscr{J}_\varepsilon = (I + \varepsilon\,\partial\Phi)^{-1}$ (I denotes the identity in $L^2(0, T; V)$) and

$$\partial\Phi(u) = \{w \in L^2(0, T; V) : \Phi(v) \geqq \Phi(u) + \int_0^T ((w, v - u))\,dt\ \forall v \in L^2(0, T; V)\}$$

(cf. [4; Prop. 2.11, Prop. 2.16]).

1° *Approximate solutions.* Let $\{w_1, w_2, \ldots, w_n, \ldots\}$ be a system of elements in W having the following properties:

a) *the elements* $\{w_1, w_2, \ldots, w_n\}$ *are linearly independent for each* n;

b) $\overline{\bigcup_{n=1}^{\infty} W_n} = W$ *where* $W_n = \operatorname{span}\{w_1, \ldots, w_n\}$.

Without any loss of generality we may assume that $u_0 \in W_{n_0}$ for a (fixed) natural number n_0.

We now consider the following initial value problem for real functions g_{ni} ($i = 1, \ldots, n$; $n \geqq n_0$):

$$(1.9)\qquad \begin{aligned}(u_n'(t), w_i) + (Au_n(t), w_i) + (B(u_n(t), u_n(t)), w_i) + (C_\varepsilon(u_n(t)), w_i) = \\ = (f(t), w_i)\,, \quad (i = 1, \ldots, n)\end{aligned}$$

$$(1.10)\qquad u_n(0) = u_0$$

where

$$u_n(t) = u_{\varepsilon,n}(t) = \sum_{i=1}^{n} g_{ni}(t)\, w_i\,.$$

From the theory of ordinary differential equations we obtain for each $n \geqq n_0$ a number $t_n \in (0, T]$ and absolutely continuous functions g_{ni} on $[0, t_n]$ ($i = 1, \ldots, n$) that fulfil (1.9) for a.a. $t \in [0, t_n]$, and the initial condition (1.10).

2° *A – priori – estimates.* Multiplying (1.9) by $g_{ni}'(t)$, summing over $i = 1, \ldots, n$ and integrating over the interval $[0, t]$ ($0 < t \leqq t_n$) one obtains

$$\int_0^t |u_n'(s)|^2\, ds + F(u_n(t)) - F(u_0) +$$

$$+ \int_0^t (B(u_n(s), u_n(s)), u_n'(s))\, ds + \varphi_\varepsilon(u_n(t)) - \varphi_\varepsilon(u_0) = \int_0^t (f(s), u_n'(s))\, ds$$

(here we have used the fact that A maps strongly convergent sequences into weakly convergent sequences, and that $A = \operatorname{grad} F$; cf. (1.1), (1.2)). Thus, by (1.2) and (1.5),

$$(1.11)\qquad \begin{aligned}\int_0^t |u_n'(s)|^2\, ds + c_1 |||u_n(t)|||^p \leqq \text{const} - \int_0^t (B(u_n(s), u_n(s)), u_n'(s))\, ds + \\ + \int_0^t (f(s), u_n'(s))\, ds\end{aligned}$$

for all $t \in (0, t_n]$ where the constant depends neither on t_n nor on n and ε.

In order to evaluate the first integral on the right hand side of (1.11) we use the second inequality in (1.4):

$$- \int_0^t (B(u_n(s), u_n(s)), u_n'(s))\, ds \leqq \frac{1}{4} \int_0^t |u_n'(s)|^2\, ds + \text{const}\,(1 + \int_0^t |||u_n(s)|||^p\, ds)\,.$$

Further, by our hypothesis on f (note that $f_2 \in C([0, T]; W^*)$):

$$\int_0^t (f(s), u_n'(s))\, ds = \int_0^t (f_1(s), u_n'(s))\, ds +$$

$$+ (f_2(t), u_n(t)) - (f_2(0), u_0) - \int_0^t (f_2'(s), u_n(s))\, ds \leqq$$

$$\leqq \frac{1}{4}\int_0^t |u_n'(s)|^2 \, ds + \frac{1}{2} c_1 |||u_n(t)|||^p + \text{const}\,(1 + \int_0^t |||u_n(s)|||^p \, ds)\,.$$

Inserting these estimates into (1.11) we get

$$\int_0^t |u_n'(s)|^2 \, ds + |||u_n(t)|||^p \leqq \text{const}\,(1 + \int_0^t |||u_n(s)|||^p \, ds)$$

for all $t \in (0, t_n]$. Thus

$$\int_0^t |u_n'(s)|^2 \, ds + |||u_n(t)||| \leqq \text{const}$$

for all $t \in (0, t_n]$ where the constant does not depend on t_n, n and ε. Hence the solution $g_{\varepsilon i}$ $(i = 1, \ldots, n)$ to (1.9), (1.10) must exist on the whole interval $[0, T]$ and it holds

$$\text{(1.12)} \qquad \begin{aligned} &|||u_n(t)||| \leqq k_1 \ \ \forall t \in [0, T], \ \forall n \geqq n_0, \ \forall \varepsilon > 0\,, \\ &\|u_n'\|_{L^2(0,T;H)} \leqq k_1 \ \ \forall n \geqq n_0, \ \forall \varepsilon > 0\,. \end{aligned}$$

3° *Passage to limit* $n \to \infty$ ($\varepsilon > 0$ *fixed*). From (1.12) we may conclude (by passing to a subsequence if necessary) that

$$\text{(1.13)} \qquad u_n \to u_\varepsilon \quad \textit{weakly* in} \quad L^\infty(0, T; W)\,,$$

$$\text{(1.14)} \qquad (\mathscr{A}(u_n) + \mathscr{C}_\varepsilon(u_n)) \to \chi_\varepsilon \quad \textit{weakly in} \quad L^{p'}(0, T; W^*)\,,$$

$$\text{(1.15)} \qquad u_n' \to u_n' \quad \textit{weakly in} \quad L^2(0, T; H)$$

as $n \to \infty$. Consequently,

$$\text{(1.12}_1\text{)} \qquad \begin{aligned} &|||u_\varepsilon(t)||| \leqq k_1 \quad \textit{for a.a.} \quad t \in [0, T], \ \forall \varepsilon > 0\,, \\ &\|u_n'\|_{L^2(0,T;H)} \leqq k_1 \ \ \forall \varepsilon > 0\,. \end{aligned}$$

Further, by a well-known compactness theorem (cf. [7; Chap. 1, 5.2]), (1.13) and (1.15) imply (passing to a subsequence if necessary)

$$\text{(1.16)} \qquad u_n \to u_\varepsilon \quad \textit{strongly in} \quad L^2(0, T; H)$$

as $n \to \infty$. On the other hand, using integration by parts, from (1.13) and (1.15) one easily concludes that

$$\text{(1.17)} \qquad u_n(t) \to u_\varepsilon(t) \quad \textit{weakly in } H\,, \quad \forall t \in [0, T]$$

as $n \to \infty$. Hence

$$\text{(1.18)} \qquad u_\varepsilon(0) = u_0 \ \ \forall \varepsilon > 0\,.$$

Next, we assert that

$$\text{(1.19)} \qquad \mathscr{B}(u_n) \to \mathscr{B}(u_\varepsilon) \quad \textit{weakly in} \quad L^{p'}(0, T; W^*)$$

as $n \to \infty$. Indeed, let $v \in L^p(0, T; W)$ be arbitrarily given. Our hypothesis (1.4) implies that

$$w \mapsto \int_0^t (B(u_\varepsilon, w), v)\, dt$$

is a linear, continuous functional on $L^p(0, T; W)$. Consequently,

$$\int_0^T (B(u_\varepsilon, u_n - u_\varepsilon), v)\, dt \to 0 \quad as \quad n \to \infty ,$$

and (1.19) is easily seen when combining the first inequality in (1.4) with (1.16) and the latter convergence property.

Let $\psi \in C_c^\infty((0, T))$ (= space of all real infinitely differentiable functions on $\mathbb{R}$ having their support in $(0, T)$) be arbitrary, let i_0 be an arbitrary natural number $\geqq n_0$, and let a_i ($i = 1, \ldots, i_0$) be arbitrary reals. Multiplying (1.9) (for $n \geqq i_0$) by $\psi(t)\, a_i$, summing over $i = 1, \ldots, i_0$ and integrating over $[0, T]$ yields

$$\int_0^T (u_n'(t) + Au_n(t) + B(u_n(t), u_n(t)) + C_\varepsilon(u_n(t)) - f(t) ,$$
$$\psi(t) \sum_{i=1}^{i_0} a_i w_i)\, dt = 0 .$$

Observing (1.13)–(1.15) and (1.19) we conclude from this identity after $n \to \infty$ that

$$\int_0^T [u_\varepsilon'(t) + \chi_\varepsilon(t) + B(u_\varepsilon(t), u_\varepsilon(t)) - f(t)]\, \psi(t)\, dt = 0 ,$$

i.e.

$$u_\varepsilon' + \chi_\varepsilon + \mathscr{B}(u_\varepsilon) = f . \tag{1.20}$$

We show that $\chi_\varepsilon = (\mathscr{A} + \mathscr{C}_\varepsilon)(u_\varepsilon) = \mathscr{A}(u_\varepsilon) + \mathscr{C}_\varepsilon(u_\varepsilon)$. To this end, let $v \in$ $\in L^p(0, T; W)$ be arbitrary. We have

$$0 \leqq \int_0^T ((A + C_\varepsilon)(v) - (A + C_\varepsilon)(u_n), v - u_n)\, dt =$$
$$= \int_0^T ((A + C_\varepsilon)(v), v - u_n)\, dt - \int_0^T ((A + C_\varepsilon)(u_n), v)\, dt$$
$$+ \int_0^T (f, u_n)\, dt - \tfrac{1}{2}|u_n(T)|^2 + \tfrac{1}{2}|u_n(0)|^2 - \int_0^T (B(u_n, u_n), u_n)\, dt .$$

Taking the lim sup on the right hand side of this inequality and using (1.20) one finds

$$0 \leqq \int_0^T ((A + C_\varepsilon)(v), v - u_\varepsilon)\, dt - \int_0^T (\chi_\varepsilon, v)\, dt +$$

$$+ \int_0^T (f, u_\varepsilon)\,dt - \tfrac{1}{2}|u_\varepsilon(T)|^2 + \tfrac{1}{2}|u_\varepsilon(0)|^2 - \int_0^T (B(u_\varepsilon, u_\varepsilon), u_\varepsilon)\,dt =$$

$$= \int_0^T ((A + C_\varepsilon)(v) - \chi_\varepsilon, v - u_\varepsilon)\,dt$$

(here we have used the fact that $u_n(T) \to u_\varepsilon(T)$ weakly in H as $n \to \infty$; cf. (1.17)). Thus, by a standard argument from the theory of monotone operators, $\chi_\varepsilon = (\mathscr{A} + \mathscr{C}_\varepsilon)(u_\varepsilon)$. Hence

$$(1.20_1) \qquad u_\varepsilon' + \mathscr{A}(u_\varepsilon) + \mathscr{B}(u_\varepsilon) + \mathscr{C}_\varepsilon(u_\varepsilon) = f.$$

4° *Passage to limit* $\varepsilon \to 0$. As above, by passing to a subsequence if necessary, we may conclude from (1.12_1) that

$$(1.21) \qquad u_\varepsilon \to u \quad \textit{weakly* in} \quad L^\infty(0, T; W),$$

$$(1.22) \qquad u_\varepsilon \to u' \quad \textit{weakly in} \quad L^2(0, T; H)$$

as $\varepsilon \to 0$. By the same argument as above, $u(0) = u_0$ (cf. (1.17), (1.18)).

Further, equation (1.20_1) implies that, for any $\varepsilon > 0$,

$$(1.23) \qquad \begin{aligned} &\int_0^T (u_\varepsilon' + Au_\varepsilon + B(u_\varepsilon, u_\varepsilon), v - u_\varepsilon)\,dt + \Phi(v) - \Phi_\varepsilon(u_\varepsilon) \geqq \\ &\geqq \int_0^T (f, v - u_\varepsilon)\,dt \ \forall v \in L^p(0, T; W). \end{aligned}$$

Set $v = 0$ in (1.23). Then

$$\text{const} \geqq \Phi_\varepsilon(u_\varepsilon) = \frac{1}{2\varepsilon}\|u_\varepsilon - \mathscr{J}_\varepsilon(u_\varepsilon)\|^2_{L^2(0,T;V)} + \Phi(\mathscr{J}_\varepsilon(u_\varepsilon))$$

where the constant does not depend on ε. Hence

$$(1.24) \qquad \mathscr{J}_\varepsilon(u_\varepsilon) \to u \quad \textit{weakly in} \quad L^2(0, T; V)$$

as $\varepsilon \to 0$, and therefore $\Phi(u) < +\infty$.

Finally, (1.23) implies

$$\int_0^T (Au_\varepsilon, u_\varepsilon - v)\,dt \leqq \int_0^T (u_\varepsilon', v)\,dt - \tfrac{1}{2}|u_\varepsilon(T)|^2 + \tfrac{1}{2}|u_\varepsilon(0)|^2 +$$

$$+ \int_0^T (B(u_\varepsilon, u_\varepsilon), v - u_\varepsilon)\,dt + \Phi(v) - \Phi(\mathscr{J}_\varepsilon(u_\varepsilon)) + \int_0^T (f, u_\varepsilon - v)\,dt$$

for any $v \in L^p(0, T; W)$. Observing that $\mathscr{B}(u_\varepsilon) \to \mathscr{B}(u)$ weakly in $L^{p'}(0, T; W^*)$ as $\varepsilon \to 0$ (cf. (1.19)), and that

$$\left|\int_0^T (B(u_\varepsilon, u_\varepsilon), u - u_\varepsilon)\,dt\right| \leqq$$

$$\leqq c_4 \int_0^T |||u_\varepsilon|||^2 \, |u - u_\varepsilon| \, dt \leqq c_4 k_1^2 \int_0^T |u - u_\varepsilon| \, dt \to 0$$

as $\varepsilon \to 0$, we find

$$\lim \int_0^T (B(u_\varepsilon, u_\varepsilon), v - u_\varepsilon) \, dt = \int_0^T (B(u, u), v - u) \, dt$$

($v \in L^p(0, T; W)$ arbitrary). Thus, by (1.21), (1.22) and (1.24),

$$\limsup \int_0^T (Au_\varepsilon, u_\varepsilon - u) \, dt \leqq 0 \, .$$

The operator $\mathscr{A}$ being monotone and hemi-continuous, it holds

$$\int_0^T (Au, u - v) \, dt \leqq \liminf \int_0^T (Au_\varepsilon, u_\varepsilon - v) \, dt \leqq$$

$$\leqq \int_0^T (u' + B(u, u), v - u) \, dt + \Phi(v) - \Phi(u) + \int_0^T (f, u - v) \, dt$$

for all $v \in L^p(0, T; W)$.

The proof of Theorem is complete.

Remark. — It is easy to see that the assertion of the theorem holds if condition (1.2) is replaced by the following ones:

(i) $(Ax, x) \geqq c_0 |||x|||^p \; \forall x \in W, \; c_0 = \text{const} > 0, \, p > 3$;

(ii) *there exists a functional* $F : W \to \mathbb{R}$ *such that* $A = \operatorname{grad} F$

cf. also Part II of our paper). Indeed, the estimate $\|u_n\|_{L^p(0,T;W)} \leqq \text{const}$ for all $n \geqq n_0$ and all $\varepsilon > 0$ is readily verified when multipliying (1.9) by $g_{n_i}(t)$, summing over $i = 1, \ldots, n$ and integrating over the interval $[0, t]$. In virtue of this estimate we get (1.12).

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Souhrn

O EVOLUČNÍCH NEROVNOSTECH MODIFIKOVANÉHO NAVIEROVA-STOKESOVA TYPU, I

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V článku je dokázána existenční věta pro silné řešení abstraktní nerovnosti, při čemž vlastnosti uvažovaných operátorů jsou motivovány modifikovanými Navierovými-Stokesovými rovnicemi při jistých jednostranných okrajových podmínkách. Metoda důkazu spočívá v úvaze Galerkinova typu kombinované s regularisací funkcionálu.

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