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HOMOGENIZATION OF LINEAR ELASTICITY EQUATIONS

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INTRODUCTION

Homogenization has become an important method in computing problems for composite materials. In the physical sense, homogenization means replacing the periodically heterogeneous material by an "equivalent" homogeneous one. From the mathematical point of view, the equation with highly oscillating periodic coefficients is approximated by a problem with constant coefficients. The mathematical approach to this method, introduced by I. Babuška (1974), is based on considering a sequence of problems with diminishing period. The method was further developed by many authors, e.g. A. Bensoussan, M. Biroli, J. L. Lions, G. Papanicolaou, E. Sanchez Palencia, L. Tartar, etc.

In connection with homogenization we must mention the notion of the operator G-convergence and the functional Γ-convergence, which were studied by many authors, e.g. A. Ambrosetti, E. De Giorgi, T. Franzoni, O. A. Olejnik, C. Sbordone, S. Spagnolo, etc.

For further references see e.g. [3], [5], [7], [10], [18].

In the paper we shall deal with homogenization of problems of linear elasticity. These problems can be formulated either in terms of displacements or in terms of stresses. The former formulations lead to a system of elliptic second-order equations, the homogenization of which has been studied e.g. in [5], [14]. The homogenization of dual formulations has not been studied except for [17], where some results contained in this paper were announced.

In the paper we show (Theorems 1, 2) that with diminishing period ε

– the displacement vector u^ε converges to u^0,
– the stress tensor τ^ε converges to t^0 and
– the local energy $\tau^\varepsilon e(u^\varepsilon)$ converges to $t^0 e(u^0)$,

where u^0, t^0 are the solutions of the unique homogenized problem determined by the so-called homogenized coefficients given by (2.19), (2.32). Formula (2.32) seems to be new.

From the numerical point of view the homogenized solutions u^0, t^0 represent an approximation of the solutions u^ε, τ^ε. This approximation can be improved by means of the correctors U^ε, T^ε (2.33), (2.34). The convergence of $u^\varepsilon - U^\varepsilon$, $\tau^\varepsilon - T^\varepsilon$ is proved in Theorem 3.

The main result is contained in Section 4. The assumptions, the variational formulations of the problems and the convergence theorems are completed by a remark on the G- and Γ-convergences.

The physical formulation is introduced in Section 1. In Section 2 we derive the homogenized problem by means of the so-called multiple-scale method and introduce the correctors. Section 3 deals with the homogenized coefficients – several formulae, properties and examples are introduced. The convergence theorems are proved by means of a simplified version of the so-called local energy method in Section 5.

Throughout the paper we use the convention on the summation over repeated indices and denote partial derivatives by indices after comma:

$$f_{i,j} \text{ means } \frac{\partial f_i}{\partial x_j}, \quad f_{i,y_j} \text{ means } \frac{\partial f_i}{\partial y_j}.$$

Let $Y = [0, \bar{y}_1] \times [0, \bar{y}_2] \times [0, \bar{y}_3]$ $(\bar{y}_i > 0)$ be the unit period. A function $f(x, y)$ is said to be *Y-periodic in y* if

$$f(x, y_1 + k_1\bar{y}_1, y_2 + k_2\bar{y}_2, y_3 + k_3\bar{y}_3) = f(x, y_1, y_2, y_3)$$

for all integers k_1, k_2, k_3. The integral average in y is denoted by

$$\mathcal{M}(f) = \int_Y f(x, y)\, \mathrm{d}y / \operatorname{meas}(Y).$$

Further, we use the usual function spaces of continuously differentiable functions denoted by C_0^∞, C^2, Lebesgue spaces L^p, L^∞ and Sobolev spaces $W^{k,p}$. The subscript *per* denotes Y-periodic functions, *sym* denotes symmetric tensors, see Section 4.

1. SETTING THE PROBLEM

We shall consider simultaneously the first, the second and the mixed boundary value problem of linear elasticity for a body made of a material with periodic structure.

Denote by

Ω – the domain in R^3 representing the volume of the body,
$\partial\Omega$ – its surface (with the normal vector $n = (n_i)$) divided into two parts Γ_u, Γ_τ,
$f = (f_i)$ – the prescribed volume forces in Ω,
$U = (U_i)$ – the prescribed displacement on Γ_u,
$T = (T_i)$ – the prescribed stress-vector on Γ_τ.

Further, we denote by

$u = (u_i)$ – the displacement vector in Ω,

$\mathrm{e} = (\mathrm{e}_{ij})$ – the small strain tensor $[\mathrm{e}_{ij}(u) = (u_{i,j} + u_{j,i})/2]$,

$\tau = (\tau_{ij})$ – the stress tensor $[\tau_{ij} = \tau_{ji}]$.

The relation between the stress and strain tensors is described by the linear Hooke's law

$$\tau_{ij} = a_{ijkl}\,\mathrm{e}_{kl}\,,$$

where the coefficients a_{ijkl} form a matrix of the type $(3 \times 3 \times 3 \times 3)$ satisfying

$$(1.1) \qquad \alpha|\xi|^2 \leqq a_{ijkl}\xi_{ij}\xi_{kl} \leqq A|\xi|^2 \quad \forall \xi \in R^{3\times 3}_{\mathrm{sym}}\,,$$

$$(1.2) \qquad a_{ijkl} = a_{klij} = a_{jikl} = a_{ijlk}\,.$$

In the dual formulation we use the inverse Hooke's law

$$\mathrm{e}_{ij} = b_{ijkl}\tau_{kl}\,,$$

whose coefficients matrix is of the same type and satisfies

$$(1.3) \qquad \frac{1}{A}\,|\eta|^2 \leqq b_{ijkl}\eta_{ij}\eta_{kl} \leqq \frac{1}{\alpha}\,|\eta|^2 \quad \forall \eta \in R^{3\times 3}_{\mathrm{sym}}\,,$$

$$(1.4) \qquad b_{ijkl} = b_{klij} = b_{jikl} = b_{ijlk}\,,$$

$$(1.5) \qquad a_{ijmn}b_{mnkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})^{*)}\,.$$

Particularly, for an isotropic material with Lamé's constants λ, μ we have

$$a_{ijkl} = \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\,\delta_{ij}\delta_{kl}\,,$$

$$b_{ijkl} = \frac{1}{4\mu}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\,\delta_{ij}\delta_{kl}$$

and the constants in the inequalities (1.1), (1.3) are $\alpha = 2\min\{\mu(x);\, x \in \Omega\}$, $A = \max\{2\mu(x) + 3\lambda(x);\, x \in \Omega\}$.

The periodical structure of the material is expressed by the periodic Hooke's law coefficients a_{ijkl}, b_{ijkl}. Since we want to consider a sequence of problems with diminishing period ε, we set

$$(1.6) \qquad a^{\varepsilon}_{ijkl}(x) = a_{ijkl}(x/\varepsilon)\,, \quad b^{\varepsilon}_{ijkl}(x) = b_{ijkl}(x/\varepsilon)\,,$$

where $a_{ijkl}(y)$, $b_{ijkl}(y)$ are Y-periodic functions.

We shall consider even the problem with "non uniformly oscillating" coefficients

$$(1.6^*) \qquad a^{\varepsilon}_{ijkl}(x) = a_{ijkl}(x, x/\varepsilon)\,, \quad b^{\varepsilon}_{ijkl}(x) = b_{ijkl}(x, x/\varepsilon)\,,$$

$$a_{ijkl}(x, y)\,, \quad b_{ijkl}(x, y) \; - \; Y\text{-periodic in } y\,,$$

which are useful in applications.

*) δ_{ij} – Kronecker symbol, $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$.

The solution of the problem with the coefficients a^{ε}_{ijkl}, b^{ε}_{ijkl} will be denoted by u^{ε}, τ^{ε}.

For $\varepsilon > 0$ we have the ε-periodic problem:
Find functions u^{ε}, τ^{ε} satisfying the equations of equilibrium

$$\tau^{\varepsilon}_{ij,j} + f_i = 0\,, \quad \tau^{\varepsilon}_{ij} = \tau^{\varepsilon}_{ji} \quad \text{in} \quad \Omega\,, \tag{1.7}$$

the constitutive equations

$$\tau^{\varepsilon}_{ij} = a^{\varepsilon}_{ijkl}\, e_{kl}(u^{\varepsilon}) \quad \text{or} \quad e_{ij}(u^{\varepsilon}) = b^{\varepsilon}_{ijkl}\tau^{\varepsilon}_{kl} \quad \text{in} \quad \Omega\,, \tag{1.8}$$

and the boundary conditions

$$u^{\varepsilon}_i = U_i \quad \text{on} \quad \Gamma_u\,, \tag{1.9}$$

$$\tau^{\varepsilon}_{ij}n_j = T_i \quad \text{on} \quad \Gamma_\tau\,. \tag{1.10}$$

The aim of the homogenization is to find, to the introduced ε-periodic problem, a homogenized problem (independent of ε) such that its solution (u^0, t^0) is an approximation of $(u^{\varepsilon}, \tau^{\varepsilon})$.

In Section 2 we show that the homogenized problem consists of the same system of equations as (1.7)–(1.10) but with coefficients a^0_{ijkl}, b^0_{ijkl} and a solution u^0, t^0.

In Section 4 we prove the convergence $u^{\varepsilon} \to u^0$ and $\tau^{\varepsilon} \to t^0$ as $\varepsilon \to 0$.

The convergence theorems remain valid even for more general boundary conditions and right-hand sides, e.g. if in (1.7) the function of volume forces f is replaced by f^{ε} converging to f weakly in $[L^2(\Omega)]^3$.

2. DERIVING THE HOMOGENIZED PROBLEM

In this section we derive the homogenized problem by means of the so-called multiple scale method, for details see e.g. [5].

Besides the "slow" variable $x = (x_1, x_2, x_3)$ we introduce a formal "fast" variable $y = (y_1, y_2, y_3)$ by the relation $y = x/\varepsilon$ and we look for the asymptotic expansion of the solution u^{ε}, τ^{ε} in the form

$$u^{\varepsilon}(x) = u^0(x, x/\varepsilon) + \varepsilon\, u^1(x, x/\varepsilon) + \varepsilon^2\, u^2(x, x/\varepsilon) + \ldots\,, \tag{2.1}$$

$$\tau^{\varepsilon}(x) = \tau^0(x, x/\varepsilon) + \varepsilon\, \tau^1(x, x/\varepsilon) + \varepsilon^2\, \tau^2(x, x/\varepsilon) + \ldots\,, \tag{2.2}$$

where the functions $u^k(x, y)$, $\tau^k(x, y)$ are Y-periodic in y and independent of ε.

We start from the equations (1.7)–(1.10). We consider the variables x, y to be mutually independent and using (1.6), (2.1), (2.2) we obtain the equations with functions independent of ε.

Let us recall that if $f(y)$ is a Y-periodic function then

$$\mathcal{M}(f_{,y_j}) = 0\,. \tag{2.3}$$

A compound function is differentiated as follows:

$$\frac{\partial}{\partial x_i} v\left(x, \frac{x}{\varepsilon}\right) = \left[\frac{\partial v}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial v}{\partial y_i}\right](x, y)\Big|_{y=x/\varepsilon} \equiv v_{,i} + \frac{1}{\varepsilon} v_{,y_i}.$$

With this in mind, inserting (2.1), (2.2), into (1.7)–(1.10) and comparing the coefficients at the corresponding powers of ε we obtain the following system (we introduce only the first equations):

(2.4) $$\tau^0_{ij,y_j} = 0 \quad \text{in} \quad \Omega \times Y,$$

(2.5) $$0 = a_{ijkl}\, e^y_{kl}(u^0) \quad \text{or} \quad e^y_{ij}(u^0) = 0 \quad \text{in} \quad \Omega \times Y,$$

(2.6) $$\tau^1_{ij,y_j} + \tau^0_{ij,j} + f_i = 0 \quad \text{in} \quad \Omega \times Y,$$

(2.7a) $$\tau^0_{ij} = a_{ijkl}[e^y_{kl}(u^1) + e_{kl}(u^0)] \quad \text{in} \quad \Omega \times Y,$$

or

(2.7b) $$e^y_{ij}(u^1) + e_{ij}(u^0) = b_{ijkl}\tau^0_{kl} \quad \text{in} \quad \Omega \times Y,$$

(2.8) $$u^0_i = U_i \quad \text{on} \quad \Gamma_u \times Y,$$

(2.9) $$\tau^0_{ij} n_j = T_i \quad \text{on} \quad \Gamma_\tau \times Y,$$

where derivatives are taken in the weak sense and

$$e_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad e^y_{ij}(u) = \tfrac{1}{2}(u_{i,y_j} + u_{j,y_i}).$$

The first method. The only Y-periodic solutions of (2.5) are functions independent of y, therefore

(2.10) $$u^0 = u^0(x).$$

Inserting (2.7a) into (2.4) and using (1.2) and (2.10) we obtain

(2.11) $$-\left(a_{ijkl}u^1_{k,y_l}\right)_{,y_j} = a_{ijkl,y_j}u^0_{k,l}.$$

By the separation of variables we can find the solution u^1 in the form

(2.12) $$u^1_g(x, y) = -\chi^{kl}_g(y)\, u^0_{k,l}(x) + \tilde{u}_g(x),$$

where $\tilde{u}$ is independent of y and $\chi^{kl} \in W_{\text{per}} \equiv [W^{1,2}_{\text{per}}]^3$ is the Y-periodic solution of

(2.13) $$(\mathscr{A}^y\chi^{kl})_i \equiv -\left(a_{ijgh}\chi^{kl}_{g,y_h}\right)_{,y_j} = -a_{ijkl,y_j}.$$

Existence and uniqueness of the functions χ^{kl}. Due to (1.1), (1.2) the operator $\mathscr{A}^y$ is a selfadjoint elliptic differential operator. The only Y-periodic solutions of the homogeneous problem $\mathscr{A}^y u = 0$ are constants. Therefore, the problem

(2.14) $$\mathscr{A}^y u = f \quad \text{in} \quad Y,$$

$$u - Y\text{-periodic in } y$$

admits a unique solution (up to an additive constant) if and only if $\mathcal{M}(f) = 0$. But by virtue of (2.3) $\mathcal{M}[-a_{ijkl,y_j}] = 0$, thus the functions χ^{kl} exist and are unique if we add the condition

$$\mathcal{M}(\chi^{kl}) = 0 \,. \tag{2.15}$$

Moreover, due to (1.2) we have

$$\chi^{kl} = \chi^{lk} \,. \tag{2.16}$$

Inserting (2.12) into (2.7a), using symmetry of a_{ikjl} and χ^{kl} we can write

$$\tau^0_{ij} = a_{ijgh}(\delta_{gk}\delta_{hl} - \chi^{kl}_{g,y_h})\, u^0_{k,1} = a_{ijgh}(\delta_{gk}\delta_{hl} - \chi^{kl}_{g,y_h})\, e_{kl}(u^0) \,. \tag{2.17}$$

Using the notation

$$t^0_{ij}(x) = \mathcal{M}[\tau^0_{ij}(x, y)] \,, \tag{2.18}$$

$$a^0_{ijkl} = \mathcal{M}[a_{ijgh}(\delta_{gk}\delta_{hl} - \chi^{kl}_{g,y_h})] \,, \tag{2.19}$$

integrating (2.17) with respect to y we can write

$$t^0_{ij} = a^0_{ijkl} u^0_{k,l} = a^0_{ijkl}\, e_{kl}(u^0) \,. \tag{2.20}$$

Finally, integrating (2.6), (2.8), (2.9) and using (2.3), we obtain the homogenized problem

$$t^0_{ij,j} + f_i = 0 \quad \text{in} \quad \Omega \,. \tag{2.21}$$

$$t^0_{ij} = a^0_{ijkl}\, e_{kl}(u^0) \quad \text{in} \quad \Omega \,, \tag{2.22a}$$

$$u^0_i = U_i \quad \text{on} \quad \Gamma_u \,, \tag{2.23}$$

$$t^0_{ij} n_j = T_i \quad \text{on} \quad \Gamma_\tau \,. \tag{2.24}$$

Similarly, if b^0_{ijkl} denotes the inverse matrix to a^0_{ijkl}, the equation

$$e_{kl}(u^0) = b^0_{ijkl} t^0_{kl} \tag{2.22b}$$

is equivalent to (2.22a).

The second method. The equation (2.5) again implies (2.10). It follows from (2.4) and the symmetry of the tensor τ^0 that the functions τ^0_{ij} can be found in the form

$$\tau^0_{ij}(x, y) = t^0_{ij}(x) + (\mathrm{ROT}^y\, \Theta(x, y))_{ij} \,, \tag{2.25}$$

where t^0_{ij} is given by (2.18), Θ_{ij} are Y-periodic (the so-called stress functions) and the operator $ROT^y = rot^y\, rot^y$ is defined by

$$(\mathrm{ROT}^y\, \Theta)_{ij} = \Theta_{kl,y_m y_n} \varepsilon_{ikm} \varepsilon_{jln} \,. \; {}^{*)} \tag{2.26}$$

Consider the equation (2.7b). Necessary condition for the existence of a solution u^1

*) Levi-Civita's antisymmetric tensor has the only non-zero components $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$.

are the conditions of compatibility that can be written in the form

$$\mathrm{ROT}^y(b_{ijkl}\tau^0_{kl} - \mathrm{e}_{ij}(u^0)) = 0 \, .$$

Using (2.25) we obtain

$$\mathrm{ROT}^y(b_{ijkl}(\mathrm{ROT}^y\, \Theta(x, y))_{kl}) = -\mathrm{ROT}^y(b_{ijkl}\, t^0_{kl}(x)) \, . \tag{2.27}$$

By the separation of variables the solution Θ can be found in the form

$$\Theta_{gh}(x, y) = -\vartheta^{kl}_{gh}(y)\, t^0_{kl}(x) + \tilde{\Theta}_{gh}(x) \, , \tag{2.28}$$

where $\tilde{\Theta}$ is independent of y and ϑ^{kl} satisfy

$$(\mathscr{B}^y \vartheta^{kl})_{ij} \equiv (\mathrm{ROT}^y(b_{ijgh}(\mathrm{ROT}^y \vartheta^{kl})_{gh})) = \mathrm{ROT}^y(b_{ijkl}) \, , \tag{2.29}$$

$$\vartheta^{kl}_{gh} = \vartheta^{kl}_{hg} \, .$$

Existence and uniqueness of the solution of (2.29). Introduce the following Sobolev spaces of Y-periodic functions:

$$Q = [W^{2,2}_{\mathrm{per}}]^{3\times 3}_{\mathrm{sym}} \, ,$$

$$Q_0 = \{\varphi \in Q;\ \mathrm{ROT}^y \varphi = 0\} \, ,$$

Q_1 – the orthogonal complement of Q_0 in Q.

Due to (1.3), (1.4) the operator $\mathscr{B}^y = \mathrm{ROT}^y(b\ \mathrm{ROT}^y)$ is a selfadjoint elliptic differential operator on Q_1. The only Y-periodic solutions of $\mathscr{B}^y\varphi = 0$ are functions from Q_0. But the right-hand side of (2.29) is orthogonal to the functions from Q_0 because

$$\mathscr{M}[\mathrm{ROT}^y(b_{ijkl})\, \varphi_{ij}] = \mathscr{M}[b_{ijkl}\, \mathrm{ROT}^y(\varphi_{ij})] = 0 \quad \forall \varphi \in Q_0 \, .$$

So, the functions ϑ^{kl} exist and are uniquely determined by the condition $\vartheta^{kl} \in Q_1$.

Using (2.28) we rewrite (2.25)

$$\tau^0_{gh} = t^0_{kl}(\delta_{gk}\delta_{hl} - (\mathrm{ROT}^y\, \vartheta^{kl})_{gh}) \, . \tag{2.30}$$

Inserting τ^0 from (2.30) and integrating with respect to y we obtain from (2.7b)

$$\mathscr{M}[b_{ijgh}(\delta_{gk}\delta_{hl} - (\mathrm{ROT}^y\, \vartheta^{kl})_{gh})]\, t^0_{kl} = \mathrm{e}_{ij}(u^0) \, . \tag{2.31}$$

The last equation along with (2.21), (2.23), (2.24) yields an equivalent form of the homogenized problem (2.21)–(2.24). Comparing (2.22b) and (2.31) we can see that

$$b^0_{ijkl} = \mathscr{M}[b_{ijgh}(\delta_{gk}\delta_{hl} - (\mathrm{ROT}^y\, \vartheta^{kl})_{gh})] \tag{2.32}$$

expresses the homogenized coefficients b^0_{ijkl} directly in terms of b_{ijkl}.

Remarks. 1. Take notice of the homogenized equations (2.21)–(2.24) having the same form as equations (1.7)–(1.10), only with periodic coefficients a^ε_{ijkl}, b^ε_{ijkl} being replaced by the homogenized ones.

2. So far we have given no proof of convergence of the solutions $(u^\varepsilon, \tau^\varepsilon)$ to the

homogenized solution (u^0, t^0). We justify the asymptotic expansions in Section 4.

3. In the case of coefficients of the form (1.6*) the derivation remains valid, only the auxiliary functions χ^{kl}, ϑ^{kl} and the homogenized coefficients are, in addition, dependent on the variable x.

Correctors. For fixed $\varepsilon > 0$ the functions u^0, t^0 represent an approximation of u^ε, τ^ε. This approximation can be improved (without solving the periodic problem) by means of correcting functions called correctors. Define an improved approximation U^ε by taking the first two terms $u^0 + \varepsilon u^1$ of the expansion (2.1):

$$U_i^\varepsilon(x) = u_i^0(x) - \varepsilon\chi_i^{kl}(x/\varepsilon)\, u_{k,l}^0(x)\,. \tag{2.33}$$

Similarly, we define the corrector of the stress tensor τ^ε by taking the first term of the expansion (2.2). Using (2.17) we define

$$T_{ij}^\varepsilon(x) = a_{ijgh}(x/\varepsilon)\left[\delta_{gk}\delta_{hl} - \chi_{g,y_h}^{kl}(x/\varepsilon)\right] u_{k,l}^0(x)\,. \tag{2.34}$$

According to (2.30) we have an equivalent formula

$$T_{ij}^\varepsilon(x) = t_{kl}^0(x)\left[\delta_{ik}\delta_{jl} - (\mathrm{ROT}^y\, \vartheta^{kl}(x/\varepsilon))_{ij}\right].$$

3. COEFFICIENTS OF THE HOMOGENIZED PROBLEM

In this section we introduce various formulae, some properties, estimates and examples of the homogenized coefficients.

Formulae for a_{ijkl}^0. In Section 2 we have derived the following formula (2.19)

$$a_{ijkl}^0 = \mathcal{M}\left[a_{ijgh}(y)\left[\delta_{gk}\delta_{hl} - \chi_{g,y_h}^{kl}(y)\right]\right], \;*) \tag{3.1}$$

where the functions $\chi^{kl} \in W_{\mathrm{per}}$ are solutions of the equation (2.13), rewritten in the weak form

$$\mathcal{M}\left[a_{ijgh}\left[\delta_{gk}\delta_{hl} - \chi_{g,y_h}^{kl}\right]\varphi_{i,y_j}\right] = 0 \quad \forall \varphi \in W_{\mathrm{per}}\,. \tag{3.2}$$

The solution ϑ^{kl} exists and is unique up to additive constants which do not influence the value of a_{ijkl}^0. So we can choose e.g. (2.15).

Choosing $\chi^{ij} \in W_{\mathrm{per}}$ for the test function in (3.2) and subtracting the equation from (3.1) we obtain a symmetric formula

$$a_{ijkl}^0 = \mathcal{M}\left[a_{pqgh}\left[\delta_{pi}\delta_{qj} - \chi_{p,y_q}^{ij}\right]\left[\delta_{gk}\delta_{hl} - \chi_{g,y_h}^{kl}\right]\right]. \tag{3.3}$$

Formulae for b_{ijkl}^0. The coefficients of the inverse Hooke's law are defined either by the matrix inverse to the matrix (a_{ijkl}^0) or by (2.32):

$$b_{ijkl}^0 = \mathcal{M}\left[b_{ijgh}\left[\delta_{gk}\delta_{hl} - (\mathrm{ROT}^y\, \vartheta^{kl})_{gh}\right]\right], \tag{3.4}$$

*) Due to symmetry (1.2), χ_{g,y_h}^{kl} can be replaced by $e_{gh}^y(\chi^{kl}) \equiv (\chi_{g,y_h}^{kl} + \chi_{h,y_g}^{kl})/2$ in the formulae (3.1)–(3.3).

where the functions $\vartheta^{kl} \in Q_1$ are solutions of the equation (2.29) rewritten in the weak form

(3.5) $$\mathscr{M}[b_{ijgh}[\delta_{gk}\delta_{hl} - (\mathrm{ROT}^y\, \vartheta^{kl})_{gh}]\,(\mathrm{ROT}^y\, \psi)_{ij}] = \forall \psi \in Q_1 \,.$$

The solution ϑ^{kl} exists and is unique in Q_1 as we have proved in Section 2. Again subtracting (3.5) with $\psi = \vartheta^{ij}$ from (3.4) we obtain a symmetric formula

(3.6) $$b^0_{ijkl} = \mathscr{M}[b_{pqgh}[\delta_{ip}\delta_{jq} - (\mathrm{ROT}^y\, \vartheta^{ij})_{pq}]\,[\delta_{gk}\delta_{hl} - (\mathrm{ROT}^y\, \vartheta^{kl})_{gh}]] \,.$$

Properties. 1. *The homogenized coefficients satisfy the same symmetry conditions as* (1.2), (1.4), *i.e.*

(3.7) $$a^0_{ijkl} = a^0_{klij} = a^0_{jikl} = a^0_{ijlk} \,,$$

(3.8) $$b^0_{ijkl} = b^0_{klij} = b^0_{jikl} = b^0_{ijlk} \,.$$

Proof. The identity $a_{ijkl} = a_{jikl}$ and formula (3.1) imply $a^0_{ijkl} = a^0_{jikl}$. The identity $a^0_{ijkl} = a^0_{klij}$ follows from $a_{ijkl} = a_{klij}$ and (3.3). Composing the preceding identities we obtain the third identity $a^0_{ijkl} = a^0_{ijlk}$. The symmetry of b^0_{ijkl} follows from (1.4), (3.4), (3.6) in the same way.

2. *The homogenized coefficients satisfy the ellipticity conditions* (1.1), (1.3) *with the same constants* α, Λ, *i.e.*

(3.9) $$\alpha|\xi|^2 \leqq a^0_{ijkl}\xi_{ij}\xi_{kl} \leqq \Lambda|\xi|^2 \quad \forall \xi \in R^{3\times 3}_{\mathrm{sym}} \,,$$

(3.10) $$\frac{1}{\Lambda}\,|\eta|^2 \leqq b^0_{ijkl}\eta_{ij}\eta_{kl} \leqq \frac{1}{\alpha}\,|\eta^2| \quad \forall \eta \in R^{3\times 3}_{\mathrm{sym}} \,.$$

Proof. Inserting (3.3) into the desired inequality we obtain

$$a^0_{ijkl}\xi_{ij}\xi_{kl} = \mathscr{M}[a_{pqgh}[\delta_{pi}\delta_{qj} - \chi^{ij}_{p,y_q}]\,\xi_{ij}[\delta_{gk}\delta_{hl} - \chi^{kl}_{g,y_h}]\,\xi_{kl}] \geqq$$
$$\geqq (1.1) \geqq \alpha\,\mathscr{M}[[\delta_{pi}\delta_{qj} - \chi^{ij}_{p,y_q}]\,\xi_{ij}[\delta_{pk}\delta_{ql} - \chi^{kl}_{p,y_q}]\,\xi_{kl}] = (2.3) =$$
$$= \alpha[\xi_{pq}\xi_{pq} + \mathscr{M}[\sum_{p,q}[\sum_{i,j}\chi^{ij}_{p,y_q}\xi_{ij}]^2]] \geqq \alpha|\xi|^2 \,.$$

Similarly, using (3.6), (1.3), (2.3) we obtain

$$b^0_{ijkl}\eta_{ij}\eta_{kl} \geqq \frac{1}{\Lambda}\,|\eta|^2 \,.$$

The other inequalities follow from the properties of the inverse symmetric positive matrices.

3. *Denote* $a^M_{ijkl} = \mathscr{M}[a_{ijkl}]$, $b^M_{ijkl} = \mathscr{M}[b_{ijkl}]$. *Further, let* (a^m_{ijkl}) *be the inverse matrix to* (b^M_{ijkl}) *and* (b^m_{ijkl}) *the inverse matrix to* (a^M_{ijkl}). *Then we have the estimates*

(3.11) $$a^m_{ijkl}\xi_{ij}\xi_{kl} \leqq a^0_{ijkl}\xi_{ij}\xi_{kl} \leqq a^M_{ijkl}\xi_{ij}\xi_{kl} \quad \forall \xi \in R^{3\times 3}_{\mathrm{sym}} \,,$$

(3.12) $$b^m_{ijkl}\eta_{ij}\eta_{kl} \leqq b^0_{ijkl}\eta_{ij}\eta_{kl} \geqq b^M_{ijkl}\eta_{ij}\eta_{kl} \quad \forall \eta \in R^{3\times 3}_{\mathrm{sym}} \,.$$

Proof. The inequality

$$a^0_{ijkl}\xi_{ij}\xi_{kl} \leqq a^M_{ijkl}\xi_{ij}\xi_{kl}$$

can be proved as follows. From (3.1) we have

$$a^0_{ijkl}\xi_{ij}\xi_{kl} = a^M_{ijkl}\xi_{ij}\xi_{kl} - \mathscr{M}[a_{ijgh}\chi^{kl}_{g,y_h}]\,\xi_{ij}\xi_{kl}\,,$$

but due to (1.1), (1.2) the equation (3.2) with $\varphi_g = \chi^{kl}_g$ yields

$$\mathscr{M}[a_{ijgh}\chi^{kl}_{g,y_h}]\,\xi_{ij}\xi_{kl} = (1.2) = \mathscr{M}[a_{ghpq}\delta_{pi}\delta_{qj}\chi^{kl}_{g,y_h}]\,\xi_{ij}\xi_{kl} =$$

$$= (3.2) = \mathscr{M}[a_{ghpq}\chi^{ij}_{p,y_q}\chi^{kl}_{g,y_h}]\,\xi_{ij}\xi_{kl} \geqq \alpha\,\mathscr{M}[\sum_{p,q}[\sum_{i,j}\chi^{ij}_{p,y_q}\xi_{ij}]^2] \geqq 0\,.$$

Analogously, using (3.4), (1.3), (1.4) and the equation (3.5) we can prove the inequality

$$b^0_{ijkl}\eta_{ij}\eta_{kl} \leqq b^M_{ijkl}\eta_{ij}\eta_{kl}\,.$$

The remaining inequalities follow from the properties of the inverse symmetric positive matrices.

Remarks. 1. From the symmetries (1.2), (1.4) we have also the symmetries

$$\chi^{kl} = \chi^{lk}\,, \quad \vartheta^{gh} = \vartheta^{hg}\,.$$

2. The above introduced properties hold in the same form even for the coefficients of the form (1.6*).

Remarks on computation of the homogenized coefficients. The homogenized coefficients are given by the formula (3.1) that contains the unknown functions χ^{kl}. These auxiliary functions are the Y-periodic solution of the so-called "cell problem" (3.2) which represents an elliptic system of second-order differential equations on the unit period Y called the unit cell.

We illustrate the problem on an example of a two-component material. Let the first component occupy the volume Y_1 in the unit cell Y, the second one $Y_2 = Y - \overline{Y}_1$. Then the problem has piece-wise constant coefficients

$$a_{ijkl}(y) = \left\langle \begin{array}{ll} {}^1a_{ijkl} & \text{if} \quad y \in Y_1\,, \\ {}^2a_{ijkl} & \text{if} \quad y \in Y_2\,. \end{array} \right.$$

We look for the functions

$${}^m\chi^{kl} \in [C^2(Y_m)]^3 \cap [C^1(\overline{Y}_m)]^3 \quad m = 1, 2$$

satisfying the equation

$$-{}^m a_{ijgh}\,{}^m\chi_{g,y_hy_j}(y) = 0 \quad \text{in} \quad Y_m\,,$$

the continuity conditions on $\partial Y_1 \cap \partial Y_2$ and the transmission conditions

$$[{}^1a_{ijkl} - {}^1a_{ijgh}\,{}^1\chi^{kl}_{g,y_h}]\,n^\Gamma_j = [{}^2a_{ijkl} - {}^2a_{ijgh}\,{}^2\chi^{kl}_{g,y_h}]\,n^\Gamma_j$$

on $\partial Y_1 \cap \partial Y_2$ at the points where the normal vector n^Γ exists. Similar continuity and transmission conditions are required on the opposite sides of the unit cell Y.

If the coefficients depend on two variables only, then the functions χ^{kl} depend on two variables as well, and the cell problem is reduced to a two-dimensional one.

Numerically the problem can be solved by the finite element method, see e.g. [8], [9] where some numerical examples of two-dimensional homogenization are introduced.

One-dimensional homogenization. If the coefficients depend on one variable only then the homogenized coefficients can be computed explicitly. Let e.g. $a_{ijkl} = a_{ijkl}(y_1)$ then also $\chi^{kl} = \chi^{kl}(y_1)$, (3.2) reduces to a system of ordinary differential equations, the solution χ^{kl} can be expressed analytically and we obtain the following formula:

$$(3.13) \quad a^0_{ijkl} = \mathcal{M}[a_{ijkl}] - \mathcal{M}[a_{ijg1} g_{ge} a_{e1kl}] + G_{fg} \mathcal{M}[a_{ijh1} g_{hf}] \mathcal{M}[g_{ge} a_{e1kl}] ,$$

where (g_{hf}) is the inverse matrix to the matrix $(a_{i1g1})^3_{i,g=1}$ and (G_{fg}) is the inverse matrix to $(\mathcal{M}[g_{gi}])$.

Similarly, it is possible to solve (3.5) and derive an explicit formula for the coefficients b^0_{ijkl}.

Examples. For the sake of brevity denote

$$(3.14) \qquad c_{ijkl}(\mu, \lambda) = \mu\, (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\, \delta_{ij}\delta_{kl} .$$

1. *A two-component material.* Consider a composite material composed of p_A . . 100% component A and of p_B . 100% component B ($p_A + p_B = 1$). Let both components be isotropic with Lamé's constants μ_A, λ_A and μ_B, λ_B. The coefficients of Hooke's law form an isotropic tensor

$$a_{ijkl}(y) = c_{ijkl}(\mu(y), \lambda(y)) .$$

The inequalities (3.11), (3.12) yield estimates of the homogenized coefficients. For example in (3.11) the upper bound is

$$a^0_{ijkl}\xi_{ij}\xi_{kl} \leqq c_{ijkl}(\mu_M, \lambda_M)\, \xi_{ij}\xi_{kl} \quad \forall \xi \in R^{3\times 3}_{\mathrm{sym}} ,$$

where $\mu^M = p_A\mu_A + p_B\mu_B$, $\lambda^M = p_A\lambda_A + p_B\lambda_B$.

Let e.g. $p_A = 0{\cdot}1$, $p_B = 0{\cdot}9$, $\mu_A = 11$, $\mu_B = 1$, $\lambda_A = 22$, $\lambda_B = 2$. Then in (3.11) we obtain

$$a^m_{ijkl} = c_{ijkl}(\mu^m, \lambda^m) , \quad a^M_{ijkl} = c_{ijkl}(\mu^M, \lambda^M) ,$$

where $\mu^m = 1{\cdot}1$, $\lambda^m = 2{\cdot}2$, $\mu^M = 2$, $\lambda^M = 4$.

2. *The layered material.* Let the components of the material in the previous example be arranged in layers orthogonal to y_1 of thicknesses $\varepsilon . p_A$, $\varepsilon . p_B$. Then using formula (3.13) we have the homogenized coefficients. We introduce them

along with the coefficients from the estimate (3.11) in the following table:

Coefficient	a^0_{ijkl}	a^m_{ijkl}	a^M_{ijkl}
1111	4·4	4·4	8·0
2222,2222	7·1		
1122,1133	2·2	2·2	4·0
2233	3·1		
1212,1313	1·1	1·1	2·0
2323	2·0		

4. THE HOMOGENIZATION RESULT

In this section we introduce assumptions, variational formulations and convergence theorems. The section is completed by a remark on G- and Γ-convergences.

Assumptions. Let the domain Ω in R^3 have a Lipschitz boundary which is divided into two parts Γ_u, Γ_τ (Γ_u, Γ_τ are disjoint, either empty or open in $\partial\Omega$, and the surface measure of $\partial\Omega - (\Gamma_u \cup \Gamma_\tau)$ is zero).

Let the prescribed body forces f, displacement U and stress T satisfy

$$f \in [L^2(\Omega)]^3\,, \quad U \in [W^{1,2}(\Omega)]^3\,, \quad T \in [L^2(\Gamma_\tau)]^3\,. \tag{4.1}$$

Let the coefficients of Hooke's law be given by (1.6) or (1.6*), where the functions a_{ijkl}, b_{ijkl} satisfy (1.1)–(1.4) and

$$a_{ijkl}\,, \quad b_{ijkl} \in L^\infty_{\text{per}} \tag{4.2}$$

or

$$a_{ijkl}\,, \quad b_{ijkl} \in C^2(\overline{\Omega}, L^\infty_{\text{per}})\,. \tag{4.2*}$$

Introduce the Sobolev spaces

$$W = [W^{1,2}(\Omega)]^3\,, \quad W^0 = [W^{1,2}_0(\Omega)]^3\,, \quad W' = [W^{-1,2}(\Omega)]^3\,,$$

$$W_{\text{per}} = [W^{1,2}_{\text{per}}]^3\,,$$

$$V^0 = \{u \in W;\ u = 0 \text{ on } \Gamma_u \text{ in the sense of traces}\}\,,$$

a geometrically admissible displacement field

$$V^U = \{u \in W;\ u - U \in V^0\}\,,$$

further spaces of symmetric tensors

$$H = [L^2(\Omega)]^{3\times 3}_{\text{sym}}\,,$$

$$\sum{}^0 = \{\tau \in H;\ \int_\Omega \tau_{ij}\, e_{ij}(v)\, dx = 0 \quad \forall v \in V^0\}\,,$$

and a statistically admissible stress field

$$\Sigma^{Tf} = \left\{ \tau \in H; \int_\Omega \tau_{ij}\, e_{ij}(v)\, dx = \int_\Omega f_i v_i\, dx + \int_{\Gamma_\tau} T_i v_i\, dS \quad \forall v \in V^0 \right\}.$$

In the case of the first boundary value problem, Γ_u is empty and we put $U = 0$, assume the conditions of total equilibrium

$$(4.3) \qquad \int_\Omega F_i\, dx + \int_{\Gamma_\tau} T_i\, dS = 0\,, \quad \int_\Omega (x \times F)\, dx + \int_{\Gamma_\tau} (x \times T)\, dS = 0$$

and look for the solutions u^ε, u^0 in Q_p, where Q_p is the orthogonal complement of the set $\{v \in W;\ v = a + b \times x\}$ in the space W.

In the case of the second boundary value problem $\Gamma_\tau = \emptyset$ and the integral over Γ_τ vanishes.

Variational formulations (see [13]). The ε-periodic problem (1.7)–(1.10) can be mathematically formulated by means of the basic variational principles.

Formulations in terms of displacements (problem P^ε): 1. (*principle of virtual displacements*)

Find $u^\varepsilon \in V^U$ *such that*

$$(4.4) \qquad \mathscr{A}^\varepsilon(u^\varepsilon, \tilde{u}) \equiv \int_\Omega a^\varepsilon_{ijkl} u^\varepsilon_{k,l} \tilde{u}_{i,j}\, dx = \int_\Omega f_i \tilde{u}_i\, dx + \int_{\Gamma_\tau} T_i \tilde{u}_i\, dS \quad \forall \tilde{u} \in V^0\,.$$

2. (*principle of minimum potential energy*)

Find $u^\varepsilon \in V^U$ *minimizing the functional*

$$(4.5) \qquad \Phi^\varepsilon(u) = \tfrac{1}{2}\, \mathscr{A}^\varepsilon(u, u) - \int_\Omega f_i u_i\, dx - \int_{\Gamma_\tau} T_i u_i\, dS$$

on the set V^U.

Recall that in the case of the first problem we suppose (4.3) and replace the set V^U by Q_p in the formulations.

Formulation in terms of stresses (problem Q^ε): 3. (*principle of virtual stresses*).

Find $\tau^\varepsilon \in \Sigma^{Tf}$ *such that*

$$(4.6) \qquad \mathscr{B}^\varepsilon(\tau^\varepsilon, \tilde{\tau}) \equiv \int_\Omega b^\varepsilon_{ijkl} \tau^\varepsilon_{kl} \tilde{\tau}_{ij}\, dx = \int_\Omega e_{ij}(U)\, \tilde{\tau}_{ij}\, dx \quad \forall \tilde{\tau} \in \Sigma^0\,.$$

4. (*principle of minimum complementary energy*).

Find $\tau^\varepsilon \in \Sigma^{Tf}$ *minimizing the functional*

$$(4.7) \qquad \mathscr{S}^\varepsilon(\tau) = \tfrac{1}{2}\, \mathscr{B}^\varepsilon(\tau, \tau) - \int_\Omega e_{ij}(U)\, \tau_{ij}\, dx$$

on the set Σ^{Tf}.

Both the formulations 1, 2 of the problem P^ε and both the formulations 3, 4 of the problem Q^ε are equivalent. Under the above introduced assumptions the bilinear forms $\mathscr{A}^\varepsilon$, $\mathscr{B}^\varepsilon$ and functionals Φ^ε, $\mathscr{S}^\varepsilon$ are coercive. The solutions u^ε of P^ε and τ^ε of Q^ε exist, are unique and are connected by the relation

$$(4.8) \qquad \tau^\varepsilon_{ij} = a^\varepsilon_{ijkl}\, \mathrm{e}_{kl}(u^\varepsilon) \left[= a^\varepsilon_{ijkl} u^\varepsilon_{k,l}\right].$$

Moreover, they satisfy the estimates

$$(4.9) \qquad \|u^\varepsilon\|_W \leqq \text{const}. \qquad \|\tau^\varepsilon\|_H \leqq \text{const}.$$

with constants independent of ε.

The homogenized problem. Similarly, the homogenized problem (2.21)–(2.24) with the homogenized coefficients can be mathematically formulated by two equivalent formulations in terms of displacements (problem P^0):

1. *Find* $u^0 \in V^U$ *such that*

$$(4.10) \qquad \mathscr{A}^0(u^0, \tilde{u}) \equiv \int_\Omega a^0_{ijkl} u^0_{k,l} \tilde{u}_{i,j}\, \mathrm{d}x = \int_\Omega f_i \tilde{u}_i\, \mathrm{d}x + \int_{\Gamma_\tau} T_i \tilde{u}_i\, \mathrm{d}S \quad \forall \tilde{u} \in V^\circ.$$

2. *Find* $u^0 \in V^U$ *minimizing the functional*

$$(4.11) \qquad \Phi^0(u) = \tfrac{1}{2}\, \mathscr{A}^0(u, u) - \int_\Omega f_i u_i\, \mathrm{d}x - \int_{\Gamma_\tau} T_i u_i\, \mathrm{d}S$$

on the set V^U;

and two equivalent formulations in terms of stresses (problem Q^0):

3. *Find* $t^0 \in \Sigma^{Tf}$ *such that*

$$(4.12) \qquad \mathscr{B}^0(t^0, \tilde{\tau}) \equiv \int_\Omega b^0_{ijkl} t^0_{kl} \tilde{\tau}_{ij}\, \mathrm{d}x = \int_\Omega \mathrm{e}_{ij}(U)\, \tilde{\tau}_{ij}\, \mathrm{d}x \quad \forall \tilde{\tau} \in \Sigma^0.$$

4. *Find* $t^0 \in \Sigma^{Tf}$ *minimizing the functional*

$$(4.13) \qquad \mathscr{S}^0(\tau) = \tfrac{1}{2}\, \mathscr{B}^0(\tau, \tau) - \int_\Omega \mathrm{e}_{ij}(U)\, \tau_{ij}\, \mathrm{d}x$$

on the set Σ^{Tf}.

Also the bilinear forms $\mathscr{A}^0$, $\mathscr{B}^0$ and the functionals Φ^0, $\mathscr{S}^0$ are coercive (as we have proved in Section 3), the solutions u^0, t^0 exist, are unique and are connected by the relation

$$(4.14) \qquad t^0_{ij} = a^0_{ijkl}\, \mathrm{e}_{kl}(u^0) \left[= a^0_{ijkl} u^0_{k,l}\right].$$

Convergence theorems. Suppose the assumptions introduced in this section are satisfied.

Theorem 1. *The displacement vector and the stress tensor converge*:

$$(4.15) \qquad u^\varepsilon \to u^0 \quad \textit{weakly in } W,$$

$$(4.16) \qquad \tau^\varepsilon \to t^0 \quad \textit{weakly in } H.$$

Theorem 2. *The functions of local energy and of complementary energy converge as follows:*

(4.17) $$a^{\varepsilon}_{ijkl}u^{\varepsilon}_{k,l}u^{\varepsilon}_{i,j} \to a^{0}_{ijkl}u^{0}_{k,l}u^{0}_{i,j} \quad \textit{star weakly in } L^1(\Omega)\,,$$

(4.18) $$b^{\varepsilon}_{ijkl}\tau^{\varepsilon}_{kl}\tau^{\varepsilon}_{ij} \to b^{0}_{ijkl}t^{0}_{kl}t^{0}_{ij} \quad \textit{star weakly in } L^1(\Omega)\,. \; {}^{*)}$$

Further, the functionals of potential energy and of complementary energy converge:

(4.19) $$\Phi^{\varepsilon}(u^{\varepsilon}) \to \Phi^{0}(u^{0})\,,$$

(4.20) $$\mathscr{S}^{\varepsilon}(\tau^{\varepsilon}) \to \mathscr{S}^{0}(t^{0})\,.$$

Let U^{ε}, T^{ε} be the corrected solutions defined by (2.33), (2.34).

Theorem 3. *In addition, let* $u^0 \in [C^2(\bar{\Omega})]^3$. *Then we have*

(4.21) $$u^{\varepsilon} - U^{\varepsilon} \to 0 \quad \textit{strongly in } W\,,$$

(4.22) $$\tau^{\varepsilon} - T^{\varepsilon} \to 0 \quad \textit{strongly in } H\,.$$

The proofs are sketched in Section 5. The assumption $u^0 \in [C^2(\bar{\Omega})]^3$ is natural, it assures $U^{\varepsilon} \in W$.

Remark. The variational functionals Φ^{ε}, Φ^{0} can be written as

$$\Phi^{\varepsilon}(u) = \tfrac{1}{2}\int_{\Omega} g\left(\frac{x}{\varepsilon}, \mathrm{D}u\right)\mathrm{d}x - L(u)\,,$$

$$\Phi^{0}(u) = \tfrac{1}{2}\int_{\Omega} g^{0}(\mathrm{D}u)\,\mathrm{d}x - L(u)\,,$$

where

$$L(u) = \int_{\Omega} f_i u_i\,\mathrm{d}x + \int_{\Gamma_\tau} T_i u_i\,\mathrm{d}S\,, \quad \mathrm{D}u = (u_{i,j})$$

and

(4.23) $$g(y, \xi) = a_{ijkl}(y)\,\xi_{ij}\xi_{kl}\,,$$
$$g^{0}(\xi) = a^{0}_{ijkl}\xi_{ij}\xi_{kl}\,.$$

The homogenized function g^0 can be given as the minimum of the functional

(4.24) $$g^{0}(\xi) = \min\{\mathscr{M}[g(y, \xi - \mathrm{D}^{y}v)];\ v \in W_{\mathrm{per}}\}\,,$$

where $\mathrm{D}^{y}v = (v_{i,y_j})$. We obtain, of course, the same function g^0. Indeed, the Euler equation for the minimizing function v in (4.24)

(4.25) $$-(a_{ijkl}(\xi_{kl} - v_{k,y_l}))_{,y_j} = 0$$

has a solution $v \in W_{\mathrm{per}}$,

(4.26) $$v_{g} = \chi^{kl}_{g}(y)\,\xi_{kl}\,,$$

*) precisely in the topology $\sigma(\mathscr{M}(\Omega), C^0_0(\Omega))$ where $\mathscr{M}(\Omega)$ is the space of bounded measures on Ω, $L^1(\Omega)$ is a subspace of $\mathscr{M}(\Omega)$.

where χ^{kl} is given by (3.2). Inserting (4.26) into (4.24), and using (3.3) we obtain (4.23).

The variational functionals $\mathscr{S}^\varepsilon$, $\mathscr{S}^0$ can be expressed similarly:

$$\mathscr{S}^\varepsilon(\tau) = \tfrac{1}{2}\int_\Omega h\left(\frac{x}{\varepsilon}, \tau\right) \mathrm{d}x - \int_\Omega e_{ij}(U)\,\tau_{ij}\,\mathrm{d}x\,,$$

$$\mathscr{S}(\tau) = \tfrac{1}{2}\int_\Omega h^0(\tau)\,\mathrm{d}x - \int_\Omega \mathrm{e}_{ij}(U)\,\tau_{ij}\,\mathrm{d}x\,,$$

where

(4.27) $$h(y, \eta) = b_{ijkl}(y)\,\eta_{ij}\eta_{kl}\,,\qquad h^0(\eta) = b^0_{ijkl}\eta_{ij}\eta_{kl}\,.$$

The homogenized function h^0 can be also expressed as the minimum of the functional

(4.28) $$h^0(\eta) = \min\{\mathscr{M}[h(y, \eta - \sigma)];\ \sigma \in \Sigma^0_{\mathrm{per}}\}\,,$$

where

$$\Sigma^0_{\mathrm{per}} = \{\sigma \in [L^2_{\mathrm{per}}]^{3\times 3}_{\mathrm{sym}},\ \mathscr{M}[\sigma_{ij}\,\mathrm{e}^y_{ij}(v)] = 0 \quad \forall v \in W_{\mathrm{per}},\ \mathscr{M}[\sigma_{ij}] = 0\,.$$

In (4.28) we have, of course, an equivalent expression of the function h^0. Indeed, let σ be the minimizing function. Then according to the Lagrange multipliers theorem there exist a function $v \in W_{\mathrm{per}}$ and constants c_{ij} satisfying

(4.29) $$b_{ijkl}(y)\,[\eta_{kl} + \sigma_{kl}(y)] = \mathrm{e}_{ij}(v) + c_{ij}\,,$$

(4.30) $$\sigma_{ij,y_j} = 0\,,$$

(4.31) $$\mathscr{M}[\sigma_{ij}] = 0\,.$$

Let ϑ^{kl} be given by (2.29). Then the function

(4.32) $$\sigma_{gh}(y) = (\mathrm{ROT}^y\,\vartheta^{kl}(y))_{gh}\,\eta_{kl}$$

is the minimizing function because it satisfies (4.30), (4.31). Further, in (4.29) $c_{ij} = \mathscr{M}[b_{ijkl}(\eta_{kl} + \sigma_{kl})]$ and the function $v \in W_{\mathrm{per}}$ exists because (2.29) yields the compatibility conditions. Finally, inserting (4.32) into (4.28) and using (3.6) we obtain (4.27).

In (4.24) and (4.28) we have functional forms of the cell problem. Thus, we completed four formulations of the periodic problem (4.4)–(4.7) and the homogenized problem (4.10)–(4.13) by the corresponding four formulations of the cell problem (3.2), (4.24), (3.5), (4.28).

Homogenization and G- and Γ-convergences. The operator convergence, introduced by S. Spagnolo (1968), the so-called G-convergence and the functional convergence the so-called Γ-convergence generalize the notion of homogenization in the following sense: the sequence of the operators in the ε-periodic problems G-converges to the operator in the homogenized problem and the sequence of functionals in the ε-periodic problems Γ-converges to the functional in the homogenized problem.

Let $A^\varepsilon, A^0 : W \to W'$ be the differential operators associated with the bilinear forms $\mathscr{A}^\varepsilon, \mathscr{A}^0$ and denote by w the weak topology in W and by s the strong topology in W'. Then A^ε G-converges to A^0 in the sense of definition in [7], [16], [18], A^ε $G(s, w)$-converges to A^0 in the sense of definition in [1]. Moreover, A^ε strongly G-converges to A^0 in the sense of definition in [18] that requires besides the convergence of the solution u^ε even the convergence of the generalized gradients $\tau^\varepsilon_{ij} = a^\varepsilon_{ijkl} u^\varepsilon_{k,l}$.

Similarly, choosing convenient topological spaces we obtain G-convergence of the operators B^ε, B^0 associated with the bilinear forms $\mathscr{B}^\varepsilon, \mathscr{B}^0$.

Finally, the variational functionals

$$\Phi^\varepsilon \; \Gamma\text{-converges to } \Phi^0 \,,^{*)}$$

$$\mathscr{S}^\varepsilon \; \Gamma\text{-converges to } \mathscr{S}^0$$

in the sense of definitions in [1], [10].

This topological approach suggests the way of mastering the nonlinear problems.

5. PROOFS OF THE CONVERGENCE THEOREMS

Denote the weak and the strong convergence in $L^2(\Omega)$ by $\rightharpoonup$ and $\to$, respectively. In the proofs we shall make use of the following properties:

– if f is a Y-periodic function then

$$f^\varepsilon(x) = f(x/\varepsilon) \rightharpoonup \mathscr{M}[f(y)] \,. \tag{5.1}$$

– we have

$$\frac{\partial}{\partial x_p} f\left(x, \frac{x}{\varepsilon}\right) = \left[f_{,p} + \frac{1}{\varepsilon} f_{,y_p}\right]\left(x, \frac{x}{\varepsilon}\right). \tag{5.2}$$

– if $u^\varepsilon \rightharpoonup u$ and $v^\varepsilon \to v$ such that either u^ε or v^ε are uniformly bounded in L^∞, then

$$u^\varepsilon v^\varepsilon \rightharpoonup uv \,. \tag{5.3}$$

P r o o f of Theorem 1. We prove the theorem by the local energy method which was introduced by L. Tartar, see [5], [7], [16]. We present here a simplified version. The crucial point of the proof – the identity (5.7) (Lemma 1) – follows from the so-called "adjoint N-condition" which is equivalent and in the case of selfadjoint operator coincides with N-condition introduced by O. A. Olejnik in [18].

Due to (4.9) we can extract a subsequence $\varepsilon_n \to 0$ such that

(5.4) $\quad u^{\varepsilon_n}$ converges weakly in W to a function u, which implies $u_k^{\varepsilon_n} \to u_k$

and $u_{k,l}^{\varepsilon_n} \rightharpoonup u_{k,l}$,

---

*) Let F^k, F^0 be functionals on a topological space (X, τ). F^k *Γ-converges to* F^0 if $\forall u \in X$:

1) $\forall u_k \xrightarrow{\tau} u$ holds $F^0(u) \leqq \liminf F^k(u_k)$,

2) there exists $v_k \xrightarrow{\tau} u$ such that $F^0(u) = \lim F^k(v_k)$.

(5.5) τ^{ε_n} converges weakly in H to a tensor t i.e.

$$\tau_{ij}^{\varepsilon_n} = (4.8) = a_{ijkl}^{\varepsilon_n} u_{k,l}^{\varepsilon_n} \rightharpoonup t_{ij} .$$

Consider the variational formulation (4.4). Using (5.5) we can pass to the limit

$$\int_\Omega t_{ij} \tilde{u}_{i,j} \, \mathrm{d}x = \int_\Omega f_i \tilde{u}_i \, \mathrm{d}x + \int_{\Gamma_\tau} T_i \tilde{u}_i \, \mathrm{d}S \quad \forall \tilde{u} \in V^0 . \tag{5.6}$$

In Lemma 1 we shall prove the identity

$$t_{ij} = \overset{0}{a}_{ijkl} u_{k,l} \quad \text{a.e.} \quad \text{in } \Omega . \tag{5.7}$$

Since $u \in V^U$, (5.6) and (5.7), the function u is a solution of the homogenized problem (4.10). The uniqueness of the solution and (4.14) yield $u = u^0$, $t = t^0$ and not only the extracted subsequence but the whole sequences u^ε and τ^ε converge (4.15), (4.16) which was to be proved.

Lemma 1. *Under the assumptions of Theorem* 1, (5.7) *holds.*

First of all, we introduce the definition of N-condition. *We say that the sequence of operators A^ε (with coefficients a_{ijkl}^ε) satisfies the adjoint N-condition with respect to the operator A^0 if there exists a sequence of functions $N^{ij\varepsilon} \in W$ such that*:

$$\mathrm{N}^{ij\varepsilon} \to 0 \ \textit{weakly in } W, \tag{5.8}$$

$$a_{ijkl}^\varepsilon - a_{ghkl}^\varepsilon N_{g,h}^{ij\varepsilon} \rightharpoonup \overset{0}{a}_{ijkl} \ \textit{weakly in } L^2(\Omega) , \tag{5.9}$$

$$-(a_{ijkl}^\varepsilon - a_{ghkl}^\varepsilon N_{g,h}^{ij\varepsilon} - \overset{0}{a}_{ijkl})_{,l} \to 0 \ \textit{strongly in dual } W' . \tag{5.10}$$

Proof of Lemma 1. Let $\varphi \in C_0^\infty(\Omega)$ and let $\chi^{ij} \in W_{\mathrm{per}}$ be the solution of (3.2) and put

$$N_g^{ij\varepsilon}(x) = \varepsilon \, \chi_g^{ij}(x/\varepsilon) . \tag{5.11}$$

It is easy to verify that the functions $N^{ij\varepsilon} \in W$ defined by (5.11) satisfy (5.8)–(5.10). Indeed, (5.8) is obvious, (5.9) follows from the symmetry (1.2), formula (3.1) and (5.1). In (5.10) even equality is true. It follows from the equation (2.13) using symmetry (1.2) just replacing variable y by x/ε.

In the case of coefficients of the form (1.6*), (4.2*) the functions $\chi^{ij} = \chi^{ij}(x, y)$ are from the space $C^2(\bar{\Omega}, W_{\mathrm{per}})$ and (5.8)–(5.10) is satisfied with the functions $N^{ij\varepsilon} \in W$ defined by

$$N_g^{ij\varepsilon}(x) = \varepsilon \, \chi_g^{ij}(x, x/\varepsilon) . \tag{5.11*}$$

Consider the extracted subsequence $\varepsilon_n \to 0$, writing briefly only $\varepsilon \to 0$. The proof is divided into two steps:

1. Choose $\tilde{u} = N^{ij\varepsilon}\varphi \in W^0$ in (4.4):

$$\int_\Omega a_{ghkl}^\varepsilon u_{k,l}^\varepsilon (N_g^{ij\varepsilon}\varphi)_{,h} \, \mathrm{d}x = \int_\Omega f_g N_g^{ij\varepsilon} \varphi \, \mathrm{d}x .$$

Passing to the limit due to (5.8) we obtain

$$\int_\Omega a^\varepsilon_{ghkl} u^\varepsilon_{k,l} N^{ij\varepsilon}_{g,h} \varphi \, dx \to 0 . \tag{5.12}$$

2. The condition (5.10) multiplied by $u^\varepsilon_k \varphi \in W^0$ yields

$$\int_\Omega (a^\varepsilon_{ijkl} - a^\varepsilon_{ghkl} N^{ij}_{g,h} - a^0_{ijkl}) (u^\varepsilon_k \varphi)_{,l} \, dx = \int_\Omega (a^\varepsilon_{ijkl} u^\varepsilon_{k,l} - a^0_{ijkl} u^\varepsilon_{k,l}) \varphi \, dx -$$

$$- \int_\Omega a^\varepsilon_{ghkl} N^{ij\varepsilon}_{g,h} u^\varepsilon_{k,l} \varphi \, dx + \int_\Omega (a^\varepsilon_{ijkl} - a^\varepsilon_{ghkl} N^{ij\varepsilon}_{g,h} - a^0_{ijkl}) u^\varepsilon_k \varphi_{,l} \, dx \to 0 .$$

Due to (5.12) and (5.9) the second and the third integral tends to zero and using (5.4), (5.5) we obtain the identity

$$\int_\Omega (t_{ij} - a^0_{ijkl} u_{k,l}) \varphi \, dx = 0 ,$$

which holds for arbitrary $\varphi \in C^\infty_0(\Omega)$, and (5.7) follows.

Proof of Theorem 2. Let $\varphi \in C^\infty_0(\Omega)$. Consider the integral $I^\varepsilon = \int_\Omega a^\varepsilon_{ijkl} u^\varepsilon_{k,l} u^\varepsilon_{i,j} \varphi \, dx$. Since $u^\varepsilon_i \varphi \in V^0$, using $u^\varepsilon_{i,j}\varphi = (u^\varepsilon_i \varphi)_{,j} - u^\varepsilon_i \varphi_{,j}$ and (4.4) we have

$$I^\varepsilon = \int_\Omega f_i u^\varepsilon_i \varphi \, dx - \int_\Omega \tau^\varepsilon_{ij} u^\varepsilon_i \varphi_{,j} \, dx .$$

The integral $I^0 = \int_\Omega a^0_{ijkl} u^0_{k,l} u^0_{i,j} \varphi \, dx$ can be transformed in the same way. But

$$I^\varepsilon - I^0 = \int_\Omega f_i (u^\varepsilon_i - u^0_i) \varphi \, dx - \int_\Omega (\tau^\varepsilon_{ij} u^\varepsilon_i - t^0_{ij} u^0_i) \varphi_{,j} \, dx$$

tends to zero due to the convergences (4.15), (4.16). Since $C^\infty_0(\Omega)$ is dense in $C^0_0(\Omega)$- the dual of $\mathcal{M}(\Omega)$, the convergence (4.17) follows.

The convergence (4.18) can be transformed to the previous case by virtue of (4.8), (4.14).

Consider the functional Φ^ε. Using (4.4) with $\tilde{u} = u^\varepsilon - U \in V^0$ and (4.10) with $\tilde{u} = u^0 - U \in V^0$, we can write

$$\Phi^\varepsilon(u^\varepsilon) - \Phi^0(u^0) = \tfrac{1}{2} \left[\int_\Omega (\tau^\varepsilon_{ij} - t^0_{ij}) U_{i,j} \, dx - \int_\Omega f_i (u^\varepsilon_i - u^0_i) \, dx - \int_{\Gamma_\tau} T_i (u^\varepsilon_i - u^0_i) dS \right],$$

but (4.15), (4.16) yield the desired convergence (4.19).

The convergence (4.20) can be proved similarly.

Proof of Theorem 3. The theorem is proved by the method introduced in [18] by O. A. Olejnik. The method was developed from the multiple-scale methods, see e.g. [5], [7]. The crucial point is the convergence

$$A^\varepsilon(u^\varepsilon - U^\varepsilon) \to 0 \quad \text{strongly in} \quad W' , \tag{5.13}$$

(where $(A^\varepsilon u)_i = -(a^\varepsilon_{ijkl} u_{k,l})_{,j}$) which follows from the so-called N-condition as we shall prove in Lemma 2.

Convergence $u^\varepsilon - U^\varepsilon \to 0$. Suppose (5.13) is true. Since u^ε is bounded (4.9) we have

$$\mathscr{A}^\varepsilon(u^\varepsilon - U^\varepsilon, u^\varepsilon - U^\varepsilon) \to 0\,, \tag{5.14}$$

which with (1.1) and $u^\varepsilon - U^\varepsilon \to 0$ on Γ_u implies the desired convergence (4.21).

Convergence $\tau^\varepsilon - T^\varepsilon \to 0$. Thanks to (1.3) it suffices to prove

$$\mathscr{B}^\varepsilon(\tau^\varepsilon - T^\varepsilon, \tau^\varepsilon - T^\varepsilon) \to 0\,. \tag{5.15}$$

But (2.34), (1.8) yield

$$\tau^\varepsilon_{ij} - T^\varepsilon_{ij} = a^\varepsilon_{ijgh}(u^\varepsilon_{g,h} - U^\varepsilon_{g,h})$$

and using (1.5) we can transform the convergence (5.15) into (5.14) which has been proved above.

Lemma 2. *Under the assumptions of Theorem* 3 (5.13) *holds.*

We begin with the definition of N-condition introduced in [18]. *We say that the sequence of operators A^ε satisfies N-condition with respect to the operator A^0 if there exists a sequence of functions $N^{kl\varepsilon} \in W$ such that*:

$$N^{kl\varepsilon} \to 0 \quad \textit{weakly in } W\,, \tag{5.16}$$

$$a^\varepsilon_{ijkl} - a^\varepsilon_{ijgh} N^{kl\varepsilon}_{g,h} \to a^0_{ijkl} \quad \textit{weakly in } L^2(\Omega)\,, \tag{5.17}$$

$$-(a^\varepsilon_{ijkl} - a^\varepsilon_{ijgh} N^{kl\varepsilon}_{g,h} - a^0_{ijkl})_{,j} \to 0 \quad \textit{strongly in } W'\,. \tag{5.18}$$

Let us remark that (5.8)–(5.10) and (5.16)–(5.18) coincide because the operator is selfadjoint. In [18] the equivalence of the N-condition and strong G-convergence is proved.

Proof of Lemma 2. The functions $N^{kl\varepsilon} \in W$ defined by (5.11), (5.11*) satisfy (5.16)–(5.18) because they satisfy (5.8)–(5.10).

Consider the expression $A^\varepsilon(u^\varepsilon - U^\varepsilon)$. Due to $A^\varepsilon u^\varepsilon = f = A^0 u^0$ and $U^\varepsilon_g = u^0_g - N^{kl\varepsilon}_g u^0_{k,l}$ we have

$$A^\varepsilon(u^\varepsilon - U^\varepsilon)_i = (a^\varepsilon_{ijkl} - a^\varepsilon_{ijgh} N^{kl\varepsilon}_{g,h} - a^0_{ijkl})_{,j} u^0_{k,l} + \\ + (a^\varepsilon_{ijkl} - a^\varepsilon_{ijgh} N^{kl\varepsilon}_{g,h} - a^0_{ijkl})\, u^0_{k,lj} - (a^\varepsilon_{ijgh} N^{kl\varepsilon}_g u^0_{k,lh})_{,j}\,.$$

Since $u^0 \in [C^2(\bar{\Omega})]^3$, (5.18) yiedls the strong convergence in W' of the first term, (5.17) the weak convergence in $L^2(\Omega)$ (which implies the strong convergence in W') of the second term, (5.16) the strong convergence in W' of the third term and hence (5.13) follows.

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Souhrn

HOMOGENIZACE ROVNIC LINEÁRNÍ PRUŽNOSTI

JAN FRANCŮ

Práce se zabývá homogenizací (t.j. aproximací materiálu s periodickou strukturou materiálem homogenním) rovnic lineární pružnosti. Zkoumáme obě formulace v posunutích i v napětích a výsledky jsou srovnány. Homogenizované rovnice jsou odvozeny metodou „multiple scale". Uvádíme různé vzorce, vlastnosti homogenizovaných koefficientů a korektory. Konvergence vektoru posunutí, tenzoru napětí a funkce lokální energie je dokázána zjednodušenou metodou lokální energie.

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