Aplikace matematiky

Václav Medek

Boundary of the union of rectangles in the plane

Aplikace matematiky, Vol. 28 (1983), No. 3, 161–172

Persistent URL: `http://dml.cz/dmlcz/104022`

BOUNDARY OF THE UNION OF RECTANGLES IN THE PLANE

VÁCLAV MEDEK

(Received January 12, 1981)

Let n rectangles O_i $(i = 1, 2, \ldots, n)$ in the plane are given, whose sides lie on lines belonging to two directions perpendicular to each other. We choose an orthogonal coordinate system such that its axes also belong to the two given perpendicular directions and that all vertices of the given rectangles O_i have positive coordinates.

Each rectangle O_i is determined by its left lower vertex $D_i[a_i, b_i]$ and by its right upper vertex $H_i[c_i, d_i]$.

We investigate the mutual position of each couple of rectangles O_i, O_j $(i, j = 1, 2, \ldots, n)$. To this purpose we compute the following 8 numbers: $IJ_1 = c_i - a_j$, $IJ_2 = a_i - c_j$, $IJ_3 = d_i - b_j$, $IJ_4 = b_i - d_j$, $IJ_5 = a_i - a_j$, $IJ_6 = b_i - b_j$, $IJ_7 = c_i - c_j$, $IJ_8 = d_i - d_j$. These numbers are not mutually independent as regards their signs. The signs at least two of them depend on the others; in nearly half cases the signs of 4 numbers from $IJ_1 - IJ_8$ depend on the others.

If at least one of the numbers IJ_2, IJ_4 is positive or at least one of the numbers IJ_1, IJ_3 is negative, then the corresponding rectangles O_i, O_j have empty intersection and we shall not investigate such a couple any more. If both numbers IJ_1, IJ_3 are nonnegative and both numbers IJ_2, IJ_4 are nonpositive, we compute the other numbers $IJ_5 - IJ_8$. Since for the classification of the mutual position of rectangles O_i, O_j only the signs of the numbers $IJ_1 - IJ_8$ are interesting, we associate the numbers $1, 0, -1$ with the numbers $IJ_1 - IJ_8$ according to whether these numbers are positive, zero or negative.

We choose an orientation of the plane of rectangles. In this article the clockwise orientation wil be used.

We associate the 4 halflines on the coordinate axes with their origin at the origin of the coordinate system with the numbers 1, 2, 3, 4 as in Fig. 1 (the 4 corresponding directions are associated with the numbers in the same way).

We associate each vertex of each rectangle O_i with an ordered triple of numbers; the first number is its x-coordinate, the second its y-coordinate, while the third number is one of the numbers 1, 2, 3, 4 according to the direction of motion of the vertex of the rectangle along its boundary in the sense of the orientation of the plane (Fig. 2).

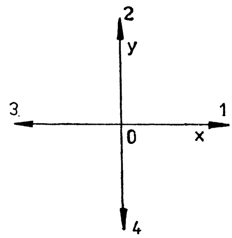


Fig. 1. Labeling of directions.

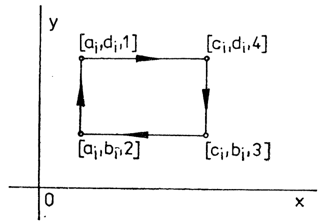


Fig. 2. Numbers associated with the vertices of a rectangle.

We investigate the mutual position of all couples of rectangles O_i, O_j with nonempty intersection and associate each vertex of the boundary of the union $O_i \cup O_j$ with an ordered triple of numbers similarly as was the case with the vertices of a rectangle. In Figs. 3 and 4 two examples of labeling the vertices of the union of two rectangles are shown.

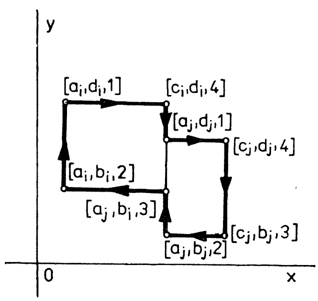


Fig. 3. Numbers associated with the vertices of the union of two rectangles.

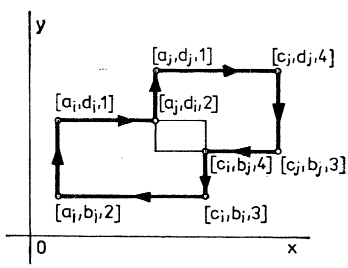


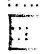
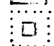
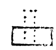
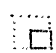
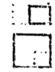
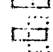
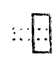
Fig. 4. Numbers associated with the vertices of the union of two rectangles.

In total there are 121 different cases of mutual positions of two rectangles of the type considered with nonempty intersection. One of the possibe cases is that of two identical rectangles while the other 120 cases can be arranged into 60 pairs. The cases belonging to the same pair involve the same rectangles, only reversing their order. All these cases are shown in Tab. 1, where the corresponding ordered triples of numbers associated with the vertices of the boundary of the union of both rectangles are also shown.

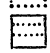
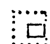
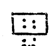
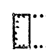
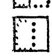
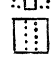
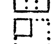
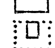
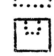
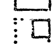
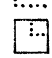
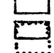
The algorithm for the construction of the boundary of the union O_i $(i = 1, 2, \ldots, n)$ of all given rectangles proceeds as follows:

1. We compile the list of all vertices of the boundaries of the unions of all couples of the given rectangles by Tab. 1. In the list some vertices can occur several times; we retain each vertex in the list only once.

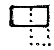
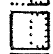
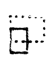
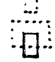
Tab. 1. Numbers associated with the union of two rectangles, the type of their mutual position and numbers associated with the vertices

	I, J_1	I, J_2	I, J_3	I, J_4	I, J_5	I, J_6	I, J_7	I, J_8	Type	
1	1	−1	1	−1	0	0	−1	−1		$[a_2, b_2, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3]$
2	1	−1	1	−1	0	0	1	1		$[a_1, b_1, 2], [a_1, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3]$
3	1	−1	1	−1	1	0	−1	−1		see 1
4	1	−1	1	−1	−1	0	1	1		see 2
5	1	−1	1	−1	0	1	−1	−1		see 1
6	1	−1	1	−1	0	−1	1	1		see 2
7	1	−1	1	−1	1	1	−1	−1		see 1
8	1	−1	1	−1	−1	−1	1	1		see 2
9	1	−1	1	−1	−1	0	0	−1		$[a_1, b_1, 2], [a_1, d_1, 1], [a_2, d_1, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3]$
10	1	−1	1	−1	1	0	0	1		$[a_2, b_2, 2], [a_2, d_2, 1], [a_1, d_2, 2], [a_1, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3]$
11	1	−1	1	−1	−1	0	1	−1		$[a_1, b_1, 2], [a_1, d_1, 1], [a_2, d_1, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_2, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3]$
12	1	−1	1	−1	1	0	−1	1		$[a_2, b_2, 2], [a_2, d_2, 1], [a_1, d_2, 2], [a_1, d_1, 1], [c_1, d_1, 4], [c_1, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3]$
13	1	−1	1	−1	0	0	0	−1		see 1
14	1	−1	1	−1	0	0	0	1		see 2
15	1	−1	1	−1	0	0	1	−1		$[a_1, b_1, 2], [a_1, d_2, 1], [c_2, d_2, 4], [c_2, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3]$
16	1	−1	1	−1	0	0	−1	1		$[a_2, b_2, 2], [a_2, d_1, 1], [c_1, d_1, 4], [c_1, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3]$
17	1	−1	1	−1	1	0	0	−1		see 1
18	1	−1	1	−1	−1	0	0	1		see 2
19	1	−1	1	−1	−1	1	0	−1		$[a_1, b_1, 2], [a_1, d_1, 1], [a_2, d_1, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3], [a_2, b_2, 2], [a_2, b_1, 3]$
20	1	−1	1	−1	1	−1	0	1		$[a_2, b_2, 2], [a_2, d_2, 1], [a_1, d_2, 2], [a_1, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3], [a_1, b_1, 2], [a_1, b_2, 3]$
21	1	−1	1	−1	−1	1	1	−1		$[a_1, b_1, 2], [a_1, d_2, 1], [a_2, d_1, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_2, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3], [c_2, b_1, 4], [c_2, b_2, 3], [a_2, b_2, 2], [a_2, b_1, 3].$

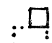
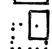
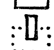
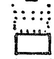
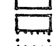
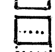
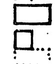
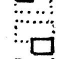
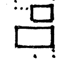
Tab. 1. Continued

	I, J_1	I, J_2	I, J_3	I, J_4	I, J_5	I, J_6	I, J_7	I, J_8	Type	
22	1	−1	1	−1	1	−1	−1	1		$[a_2, b_2, 2]$, $[a_2, d_1, 1]$, $[a_1, d_2, 2]$, $[a_1, d_1, 1]$, $[c_1, d_1, 4]$, $[c_1, d_2, 1]$, $[c_2, d_2, 4]$, $[c_2, b_2, 3]$, $[c_1, b_1, 4]$, $[c_1, b_1, 3]$, $[a_1, b_1, 2]$, $[a_1, b_2, 3]$.
23	1	−1	1	−1	0	1	0	−1		see 1
24	1	−1	1	−1	0	−1	0	−1		see 2
25	1	−1	1	−1	0	1	1	−1		$[a_2, b_2, 2]$, $[a_2, d_2, 1]$, $[c_2, d_2, 4]$, $[c_2, d_1, 1]$, $[c_1, d_1, 4]$, $[c_1, b_1, 3]$, $[c_2, b_1, 4]$, $[c_2, b_2, 3]$
26	1	−1	1	−1	0	−1	−1	1		$[a_1, b_1, 2]$, $[a_1, d_1, 1]$, $[c_1, d_1, 4]$, $[c_1, d_2, 1]$, $[c_2, d_2, 4]$, $[c_2, b_2, 3]$, $[c_1, b_2, 4]$, $[c_1, b_1, 3]$
27	1	−1	1	−1	1	1	0	−1		see 1
28	1	−1	1	−1	−1	−1	0	1		see 2
29	1	−1	1	−1	0	−1	−1	0		$[a_1, b_1, 2]$, $[a_1, d_1, 1]$, $[c_2, d_2, 4]$, $[c_2, b_2, 3]$, $[c_1, b_2, 4]$, $[c_1, b_1, 3]$
30	1	−1	1	−1	0	1	1	0		$[a_2, b_2, 2]$, $[a_2, d_2, 1]$, $[c_1, a_1, 4]$, $[c_1, b_1, 3]$, $[c_2, b_1, 4]$, $[c_2, b_2, 3]$
31	1	−1	1	−1	1	−1	−1	0		$[a_2, b_2, 2]$, $[a_2, d_2, 1]$, $[c_2, d_2, 4]$, $[c_2, b_2, 3]$, $[c_1, b_2, 4]$, $[c_1, b_1, 3]$, $[a_1, b_1, 2]$, $[a_1, b_2, 3]$
32	1	−1	1	−1	−1	1	1	0		$[a_1, b_1, 2]$, $[a_1, d_1, 1]$, $[c_1, d_1, 4]$, $[c_1, b_1, 3]$, $[c_2, b_1, 4]$, $[c_2, b_2, 3]$, $[a_2, b_2, 2]$, $[a_2, b_1, 3]$
33	1	−1	1	−1	0	0	−1	0		see 1
34	1	−1	1	−1	0	0	1	0		see 2
35	1	−1	1	−1	1	0	−1	0		see 1
36	1	−1	1	−1	−1	0	1	0		see 2
37	1	−1	1	−1	0	1	−1	0		see 1
38	1	−1	1	−1	0	−1	1	0		see 2
39	1	−1	1	−1	1	1	−1	0		see 1
40	1	−1	1	−1	−1	−1	1	0		see 2
41	1	−1	1	−1	−1	−1	0	0		see 1
42	1	−1	1	−1	1	1	0	0		see 2
43	1	−1	1	−1	0	−1	0	0		see 1
44	1	−1	1	−1	0	1	0	0		see 2

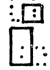
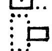
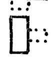
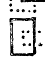
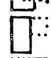
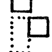
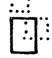
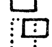
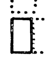
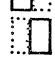
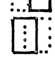
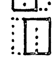
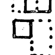
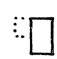
Tab. 1. Continued

	I, J_1	I, J_2	I, J_3	I, J_4	I, J_5	I, J_6	I, J_7	I, J_8	Type	
45	1	−1	1	−1	1	−1	0	0		$[a_2, b_2, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_1, b_1, 3], [a_1, b_1, 2], [a_1, b_2, 3]$
46	1	−1	1	−1	−1	1	0	0		$[a_1, b_1, 2], [a_1, d_1, 1], [c_1, d_1, 4], [c_2, b_2, 3], [a_2, b_2, 2], [a_2, b_1, 3]$
47	1	−1	1	−1	1	0	0	0		see 1
48	1	−1	1	−1	−1	0	0	0		see 2
49	1	−1	1	−1	0	0	0	0		see 2
50	0	−1	0	−1	−1	−1	−1	−1		$[a_1, b_1, 2], [a_1, d_1, 1], [c_1, d_1, 2], [c_1, d_2, 1], [c_2, d_2, 4], [c_2, d_1, 3], [c_1, d_1, 4], [c_1, b_1, 3]$
51	0	−1	0	−1	1	1	1	1		$[a_2, b_2, 2], [a_2, d_2, 1], [c_2, d_2, 2], [c_2, d_1, 1], [c_1, d_1, 4], [c_1, d_2, 3], [c_2, d_2, 4], [c_2, b_2, 3]$
52	1	−1	0	−1	−1	−1	−1	−1		$[a_1, b_1, 2], [a_1, d_1, 1], [a_2, d_1, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3], [c_1, d_1, 4], [c_1, b_1, 3]$
53	1	−1	1	0	1	1	1	1		$[a_2, b_2, 2], [a_2, d_2, 1], [a_1, d_2, 2], [a_1, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3], [c_2, d_2, 4], [c_2, b_2, 3]$
54	0	−1	1	−1	−1	−1	−1	−1		$[a_1, b_1, 2], [a_1, d_1, 1], [c_1, d_1, 2], [c_1, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3], [c_1, b_2, 4], [c_1, b_1, 3]$
55	1	0	1	−1	1	1	1	1		$[a_2, b_2, 2], [a_2, d_2, 1], [c_2, d_2, 2], [c_2, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3], [c_2, b_1, 4], [c_2, b_2, 3]$
56	1	−1	1	−1	−1	−1	−1	−1		$[a_1, b_1, 2], [a_1, d_1, 1], [a_2, d_1, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3], [c_1, b_2, 4], [c_1, b_1, 3]$
57	1	−1	1	−1	1	1	1	1		$[a_2, b_2, 2], [a_2, d_2, 1], [a_1, d_2, 2], [a_1, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3], [c_2, b_1, 4], [c_2, b_2, 3]$
58	1	−1	0	−1	0	−1	−1	−1		$[a_1, b_1, 2], [a_1, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3], [c_1, b_2, 4], [c_1, b_1, 3]$
59	1	−1	1	0	0	1	1	1		$[a_2, b_2, 2], [a_2, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3], [c_2, b_1, 4], [c_2, b_2, 3]$
60	1	−1	1	−1	0	−1	−1	−1		see 58
61	1	−1	1	−1	0	1	1	1		see 59
62	1	−1	0	−1	1	−1	−1	−1		see 31
63	1	−1	1	0	−1	1	1	1		see 32
64	1	−1	1	−1	1	−1	−1	−1		see 31

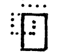
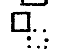
Tab. 1. Continued

	I, J_1	I, J_2	I, J_3	I, J_4	I, J_5	I, J_6	I, J_7	I, J_8	Type	
65	1	−1	1	−1	−1	1	1	1		see 32
66	0	−1	1	−1	−1	0	−1	−1		see 9
67	1	0	1	−1	1	0	1	1		see 10
68	1	−1	1	−1	−1	0	−1	−1		see 9
69	1	−1	1	−1	1	0	1	1		see 10
70	0	−1	1	−1	−1	1	−1	−1		see 19
71	1	0	1	−1	1	−1	1	1		see 20
72	1	−1	1	−1	−1	1	−1	−1		see 19
73	1	−1	1	−1	1	−1	1	1		see 20
74	1	−1	0	−1	−1	−1	0	−1		$[a_1, b_1, 2]$, $[a_1, d_1, 1]$, $[a_2, d_1, 2]$, $[a_2, d_2, 1]$, $[c_2, d_2, 4]$, $[c_1, b_1, 3]$
75	1	−1	1	0	1	1	0	1		$[a_2, b_2, 2]$, $[a_2, d_2, 1]$, $[a_1, d_2, 2]$, $[a_1, d_1, 1]$, $[c_1, d_1, 4]$, $[c_2, b_2, 3]$
76	1	−1	1	−1	−1	−1	0	−1		see 74
77	1	−1	1	−1	1	1	0	1		see 75
78	1	−1	0	−1	−1	−1	1	−1		see 11
79	1	−1	1	0	1	1	−1	1		see 12
80	1	−1	1	−1	−1	−1	1	−1		see 11
81	1	−1	1	−1	1	1	−1	1		see 12
82	1	−1	0	−1	0	−1	0	−1		$[a_1, b_1, 2]$, $[a_2, d_2, 1]$, $[c_2, d_2, 4]$, $[c_1, b_1, 3]$
83	1	−1	1	0	0	1	0	1		$[a_2, b_2, 2]$, $[a_1, d_1, 1]$, $[c_1, d_1, 4]$, $[c_2, b_2, 3]$
84	1	−1	1	−1	0	−1	0	−1		see 82
85	1	−1	1	−1	0	1	0	1		see 83
86	1	−1	0	−1	0	−1	1	−1		see 15
87	1	−1	1	0	0	1	−1	1		see 16
88	1	−1	1	−1	0	−1	1	−1		see 15
89	1	−1	1	−1	0	1	−1	1		see 16
90	1	−1	0	−1	1	−1	0	−1		see 45
91	1	−1	1	0	−1	1	0	1		see 46

Tab. 1. Continued

	I, J_1	I, J_2	I, J_3	I, J_4	I, J_5	I, J_6	I, J_7	I, J_8	Type	
92	1	−1	1	−1	1	−1	0	−1		see 45
93	1	−1	1	−1	−1	1	0	1		see 46
94	1	−1	0	−1	1	−1	1	−1		$[a_2, b_2, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_2, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3], [a_1, b_1, 2], [a_1, b_2, 3]$
95	1	−1	1	0	−1	1	−1	1		$[a_1, b_1, 2], [a_1, d_1, 1], [c_1, d_1, 4], [c_1, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3], [a_2, b_2, 2], [a_2, b_1, 3]$
96	1	−1	1	−1	1	−1	1	−1		see 94
97	1	−1	1	−1	−1	1	−1	1		see 95
98	1	0	1	−1	1	0	1	−1		$[a_2, b_2, 2], [a_2, d_2, 1], [c_2, d_2, 4], [c_2, d_1, 1], [c_1, d_1, 4], [c_1, b_1, 3]$
99	0	−1	1	−1	−1	0	−1	1		$[a_1, b_1, 2], [a_1, d_1, 1], [c_1, d_1, 4], [c_1, d_2, 1], [c_2, d_2, 4], [c_2, b_2, 3]$
100	1	−1	1	−1	1	0	1	−1		see 98
101	1	−1	1	−1	−1	0	−1	1		see 99
102	1	0	1	−1	1	1	1	−1		see 25
103	0	−1	1	−1	−1	−1	−1	1		see 26
104	1	−1	1	−1	1	1	1	−1		see 25
105	1	−1	1	−1	−1	−1	−1	1		see 26
106	0	−1	1	−1	−1	−1	−1	0		see 29
107	1	0	1	−1	1	1	1	0		see 30
108	1	−1	1	−1	−1	−1	−1	0		see 29
109	1	−1	1	−1	1	1	1	0		see 30
110	0	−1	1	−1	−1	0	−1	0		$[a_1, b_1, 2], [a_1, d_1, 1], [c_2, d_2, 4], [c_2, b_2, 3]$
111	1	0	1	−1	1	0	1	0		$[a_2, b_2, 2], [a_2, d_2, 1], [c_1, d_1, 4], [c_1, b_1, 3]$
112	1	−1	1	−1	−1	0	−1	0		see 110
113	1	−1	1	−1	1	0	1	0		see 111
114	0	−1	1	−1	−1	1	−1	0		$[a_1, b_1, 2], [a_1, d_1, 1], [c_2, d_2, 4], [c_2, b_2, 3], [a_2, b_2, 2], [a_2, b_1, 3]$
115	1	0	1	−1	1	−1	1	0		$[a_2, b_2, 2], [a_2, d_2, 1], [c_1. d_1. 4], [c_1, b_1, 3], [a_1, b_1, 2], [a_1, b_2, 3]$

Tab. 1. Continued

	I, J_1	I, J_2	I, J_3	I, J_4	I, J_5	I, J_6	I, J_7	I, J_8	Type	
116	1	−1	1	−1	−1	1	−1	0		see 114
117	1	−1	1	−1	1	−1	1	0		see 115
118	0	−1	1	0	−1	1	−1	1		$[a_1, b_1, 2]$, $[a_1, d_1, 1]$, $[c_1, d_1, 4]$, $[c_1, b_1, 1]$, $[c_2, d_2, 4]$, $[c_2, b_2, 3]$, $[a_2, b_2, 2]$, $[a_2, d_2, 3]$
119	1	0	0	−1	1	−1	1	−1		$[a_2, b_2, 2]$, $[a_2, d_2, 1]$, $[c_2, d_2, 4]$, $[c_2, b_2, 1]$, $[c_1, d_1, 4]$, $[c_1, b_1, 3]$, $[a_1, b_1, 2]$, $[a_1, d_1, 3]$
120	0	−1	1	−1	−1	1	−1	1		$[a_1, b_1, 2]$, $[a_1, d_1, 1]$, $[c_1, d_1, 4]$, $[a_2, d_2, 1]$, $[c_2, d_2, 4]$, $[c_2, b_2, 3]$, $[a_2, b_2, 2]$, $[c_1, b_1, 3]$
121	1	0	1	−1	1	−1	1	−1		$[a_2, b_2, 2]$, $[a_2, d_2, 1]$, $[c_2, d_2, 4]$, $[a_1, d_1, 1]$, $[c_1, d_1, 4]$, $[c_1, b_1, 3]$, $[a_1, b_1, 2]$, $[c_2, b_2, 3]$

2. From all the vertices constructed in the way described above we choose those with the smallest x-coordinate and from them the vertex A with the smallest y coordinate. Let the ordered triple of numbers associated with the point A be $[a_i, b_i, 2]$. As the point A moves in the direction 2, we find in the list of vertices all vertices different from A, whose x-coordinate is a_i, and choose from all of them the vertex with the smallest y-coordinate. Let it be the point B and let the ordered triple of numbers associated with it be $[a_i, b_j, k]$; then the number k determines the direction of the subsequent motion of the point B along the boundary of the union $\bigcup O_i$. If $k = 1$, the next vertex of the boundary of the union $\bigcup O_i$ is the vertex (different from B) with the y-coordinate b_j and with the smallest x-coordinate. If $k = 2$, the next vertex of the boundary of the union $\bigcup O_i$ is the vertex (different from B) with the x-coordinate a_i and with the smallest y-coordinate. If $k = 3$, the next vertex of the boundary of the union $\bigcup O_i$ is the vertex (different from B) with the y-coordinate b_j, whose x-coordinate is the greatest of the x-coordinates of all vertices with x-coordinates less than a_i. If $k = 4$, the next vertex of the boundary of the union $\bigcup O_i$ is the vertex (different from B) with the x-coordinate a_i and whose y-coordinate is the greatest of the y-coordinates of all vertices with y-coordinates less than b_j.

3. We retain the point A in the list and we delete from it all other vertices obtained by the above procedure. If after a finite number of steps we again obtain the point A, we also delete the point A.

4. If no other vertices remain in the list, then the whole boundary of the union $\bigcup O_i$ is constructed.

5. If some vertices remain in the list, we choose an arbitrary one of them, e.g. $M[c, d, k]$, and we determine whether it is an interior point of any rectangle O_i

or not, i.e. whether the following inequalities are valid simultaneously:

$$(1) \qquad c - a_i > 0 \,, \quad c - c_i < 0 \,, \quad d - b_i > 0 \,, \quad d - d_i < 0$$

for some $i \in \{1, 2, \ldots, n\}$. If there is an i such that all inequalities (1) are fulfilled, we also delete the point M from the list. If in the list there is a vertex V such that it is an interior point of no rectangle O_i, it is again a vertex of the boundary of the union $\bigcup O_i$ and we handle it similarly as the point A. This situation occurs if the union $\bigcup O_i$ consists of several connected components, and/or it is not simply connected.

6. We repeat this procedure until no point remains in the list.

For a large number of rectangles it is possible to do first some segmentation.

Fig. 5 shows a simple example. 12 rectangles $O_1, \ldots, O_{12}$ are given by their vertices

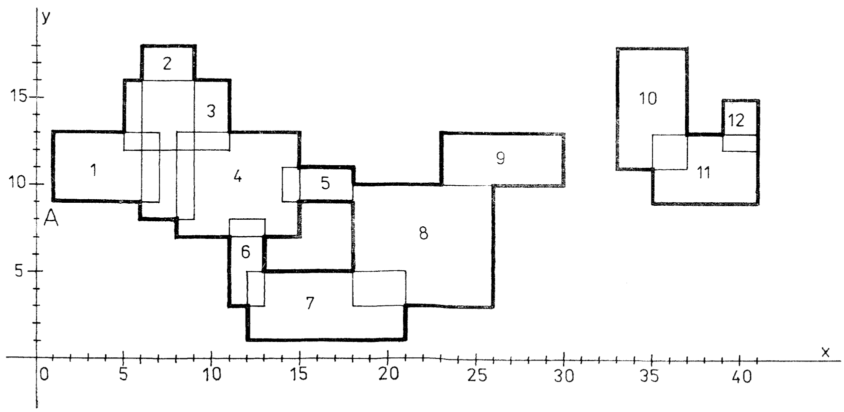


Fig. 5. Boundary of the union of 12 rectangles.

$D_1, \ldots, D_{12}$, $H_1, \ldots, H_{12}$: $D_1[1, 9]$, $H_1[7, 13]$, $D_2[6, 8]$, $H_2[9, 18]$, $D_3[5, 12]$, $H_3[11, 16]$, $D_4[8, 7]$, $H_4[15, 13]$, $D_5[14, 9]$, $H_5[18, 11]$, $D_6[11, 3]$, $H_6[13, 8]$, $D_7[12, 1]$, $H_7[21, 5]$, $D_8[18, 3]$, $H_8[26, 10]$, $D_9[23, 10]$, $H_9[30, 13]$, $D_{10}[33, 11]$, $H_{10}[37, 18]$, $D_{11}[35, 9]$, $H_{11}[41, 13]$, $D_{12}[39, 12]$, $H_{12}[41, 15]$.

The numbers $IJ_1, \ldots, IJ_8$ are as follows (we denote the ordered set of numbers $IJ_1, \ldots, IJ_8$ by IJ) : $1, 2 = [1, -8, 5, -9, -5, 1, -2, -5]$; $1, 3 = [8, -15, 1, -7, -4, -3, -4, -3]$; $1, 4_1 = -1 \Rightarrow O_1 \cap O_4 = \emptyset$; $1, 5_1 = -7 \Rightarrow O_1 \cap O_5 = \emptyset$; $1, 6_1 = -4 \Rightarrow O_1 \cap O_6 = \emptyset$; $1, 7_1 = -5 \Rightarrow O_1 \cap O_7 = \emptyset$; $1, 8_1 = -11 \Rightarrow O_1 \cap O_8 = \emptyset$; $1, 9_1 = -16 \Rightarrow O_1 \cap O_9 = \emptyset$; $1, 10_1 = -26 \Rightarrow O_1 \cap O_{10} = \emptyset$; $1, 11_1 = -28 \Rightarrow O_1 \cap O_{11} = \emptyset$; $1, 12_1 = -32 \Rightarrow O_1 \cap O_{12} = \emptyset$; $2, 3 = [4, -5, 6, -8, 1, -4, -2, 2]$; $2, 4 = [1, -5, 6, -8, -2, 1, -6, 5]$; $2, 5_1 = -5 \Rightarrow O_2 \cap O_5 = \emptyset$; $2, 6_1 = -2 \Rightarrow O_2 \cap O_6 = \emptyset$; $2, 7_1 = -3 \Rightarrow O_2 \cap O_7 = \emptyset$; $2, 8_1 = -9 \Rightarrow O_2 \cap O_8 = \emptyset$; $2, 9_1 = -14 \Rightarrow O_2 \cap O_9 = \emptyset$; $2, 10_1 = -24 \Rightarrow O_2 \cap O_{10} = \emptyset$; $2, 11_1 = -26 \Rightarrow O_2 \cap O_{11} = \emptyset$; $2, 12_1 = -30 \Rightarrow O_2 \cap O_{12} = \emptyset$; $3, 4 = [3, -10, 9, -1,$

$-3, 5, -4, 3]$; $3, 5_1 = -3 \Rightarrow O_3 \cap O_5 = \emptyset$; $3, 6_4 = 4 \Rightarrow O_3 \cap O_6 = \emptyset$; $3, 7_1 = -1 \Rightarrow O_3 \cap O_7 = \emptyset$; $3, 8_1 = -7 \Rightarrow O_3 \cap O_8 = \emptyset$; $3, 9_1 = -12 \Rightarrow O_3 \cap O_9 = \emptyset$; $3, 10_1 = -22 \Rightarrow O_3 \cap O_{10} = \emptyset$; $3, 11_1 = -24 \Rightarrow O_3 \cap O_{11} = \emptyset$; $3, 12_1 = -28 \Rightarrow O_3 \cap O_{12} = \emptyset$; $4, 5 = [1, -10, 4, -4, -6, -2, -3, 2]$; $4, 6 = [4, -5, 10, -1, -3, 4, 2, 5]$; $4, 7_4 = 2 \Rightarrow O_4 \cap O_7 = \emptyset$; $4, 8_1 = -3 \Rightarrow O_4 \cap O_8 = \emptyset$; $4, 9_1 = -8 \Rightarrow O_4 \cap O_9 = \emptyset$; $4, 10_1 = -8 \Rightarrow O_4 \cap O_{10} = \emptyset$; $4, 11_1 = -20 \Rightarrow O_4 \cap O_{11} = \emptyset$; $4, 12_1 = -24 \Rightarrow O_4 \cap O_{12} = \emptyset$; $5, 6_2 = 1 \Rightarrow O_5 \cap O_6 = \emptyset$; $5, 7_4 = 4 \Rightarrow O_5 \cap O_7 = \emptyset$; $5, 8 = [0, 12, 8, -1, -4, 6, -8, 1]$; $5, 9_1 = -5 \Rightarrow O_5 \cap O_9 = \emptyset$; $5, 10_1 = -15 \Rightarrow O_5 \cap O_{10} = \emptyset$; $5, 11_1 = -17 \Rightarrow O_5 \cap O_{11} = \emptyset$; $5, 12_1 = -21 \Rightarrow O_5 \cap O_{12} = \emptyset$; $6, 7 = [1, -10, 7, -2, -1, 2, -8, 3]$; $6, 8_1 = -5 \Rightarrow O_6 \cap O_8 = \emptyset$; $6, 9_1 = -10 \Rightarrow O_6 \cap O_9 = \emptyset$; $6, 10_1 = -20 \Rightarrow O_6 \cap O_{10} = \emptyset$; $6, 11_1 = -22 \Rightarrow O_6 \cap O_{11} = \emptyset$; $6, 12_1 = -26 \Rightarrow O_6 \cap O_{12} = \emptyset$; $7, 8 = [3, -14, 2, -9, -6, -2, -5, -5]$; $7, 9_1 = -2 \Rightarrow O_7 \cap O_9 = \emptyset$; $7, 10_1 = -12 \Rightarrow O_7 \cap O_{10} = \emptyset$; $7, 11_1 = -14 \Rightarrow O_7 \cap O_{11} = \emptyset$; $7, 12_1 = -18 \Rightarrow O_7 \cap O_{12} = \emptyset$; $8, 9 = [3, -12, 0, -10, -15, -7, -4, -3]$; $8, 10_1 = -7 \Rightarrow O_8 \cap O_{10} = \emptyset$; $8, 11_1 = -9 \Rightarrow O_8 \cap O_{11} = \emptyset$; $8, 12_1 = -13 \Rightarrow O_8 \cap O_{12} = \emptyset$; $9, 10_1 = -3 \Rightarrow O_9 \cap O_{10} = \emptyset$; $9, 11_1 = -5 \Rightarrow O_9 \cap O_{11} = \emptyset$; $9, 12_1 = -9 \Rightarrow O_9 \cap O_{12} = \emptyset$; $10, 11 = [2, -8, 9, -2, -2, 2, -4, 5]$; $10, 12_1 = -2 \Rightarrow O_{10} \cap O_{12} = \emptyset$; $11, 12 = [2, -6, 1, -6, -4, -3, 0, -2]$.

It follows that only 13 couples of rectangles have nonempty intersections and the ordered 8-tuples of numbers 1, 0, −1 are associated with them as follows:

(2)
$$\begin{aligned}
1,\ 2 &\to 1, -1, 1, -1, -1, 1, -1, -1\\
1,\ 3 &\to 1, -1, 1, -1, -1, -1, -1, -1,\\
2,\ 3 &\to 1, -1, 1, -1, 1, -1, -1, 1\\
2,\ 4 &\to 1, -1, 1, -1, -1, 1, -1, 1\\
3,\ 4 &\to 1, -1, 1, -1, -1, 1, -1, 1\\
4,\ 5 &\to 1, -1, 1, -1, -1, -1, -1, 1\\
4,\ 6 &\to 1, -1, 1, -1, -1, 1, 1, 1\\
5,\ 8 &\to 0, -1, 1, -1, -1, 1, -1, 1\\
6,\ 7 &\to 1, -1, 1, -1, -1, 1, -1, 1\\
7,\ 8 &\to 1, -1, 1, -1, -1, -1, -1, -1\\
8,\ 9 &\to 1, -1, 0, -1, -1, -1, -1, -1\\
10, 11 &\to 1, -1, 1, -1, -1, 1, -1, 1\\
11, 12 &\to 1, -1, 1, -1, -1, -1, 0, -1
\end{aligned}$$

By (1) and Tab. 1 it is possible to compile the list of vertices:

[1, 9, 2], [1, 13, 1], [6, 13, 2], [6, 18, 1], [9, 18, 4], [9, 8, 3], [6, 8, 2], [6, 9, 3], [5, 13, 2], [5, 16, 1], [11, 16, 4], [11, 12, 3], [7, 12, 4], [7, 9, 3], [5, 12, 2], [6, 16, 2], [9, 16, 1], [9, 12, 4], [6, 12, 3], [9, 13, 1], [15, 13, 4], [15, 7, 3], [8, 7, 2], [8, 8, 3], [11, 13, 1], [8, 12, 3], [8, 13, 1], [15, 11, 1], [18, 11, 4], [18, 9, 3], [15, 9, 4], [13, 7, 4], [13, 3, 3], [11, 3, 2], [11, 7, 3], [14, 9, 2], [14, 11, 1], [18, 10, 1], [26, 10, 4]

[26, 3, 3], [18, 3, 2], [11, 8, 1]. [13, 8, 4], [13, 5, 1], [21, 5, 4], [21, 1, 3], [12, 1, 2], [12, 3, 3], [12, 5, 1], [18, 5, 2], [21, 3, 4], [23, 10, 2], [23, 13, 1], [30, 13, 4] [30, 10, 3], [33, 11, 2], [33, 18, 1], [37, 18, 4], [37, 13, 1], [41, 13, 4], [41, 9, 3], [35, 9, 2], [35, 11, 3], [35, 13, 1], [39, 13, 2], [39, 15, 1], [41, 15, 4].

From all the vertices with the smallest x-coordinate the point $A[1, 9]$ has the smallest y-coordinate. From this vertex we start the motion along the boundary of the union of the given 12 rectangles.

We first find the following vertices of the boundary of the union (according to the procedure described above): [1, 9], [1,13], [5, 13], [5, 16], [6, 16], [6, 18], [9, 18], [9, 16], [11, 16], [11, 13], [15, 13], [15, 11], [18, 11], [18, 10], [23, 10], [23, 13], [30, 13], [30, 10], [26, 10], [26, 3], [21, 3], [21, 1], [12, 1], [12, 3], [11, 3], [11, 7], [8, 7], [8,8], [6, 8], [6,9].

When we delete these vertices from the list, we get a new list of vertices: [6, 13, 2], [9, 8, 3], [11, 12, 3], [7, 12, 4], [7, 9, 3], [5, 12, 2], [9, 12, 4], [6, 12, 3], [9, 13, 1], [15, 7, 3], [8, 12, 3], [8, 13, 1], [18, 9, 3], [15, 9, 4], [13, 7, 4], [13, 3, 3], [14, 9, 2], [14, 11, 1], [18, 3, 2], [11, 8, 1], [13, 8, 4], [13, 5, 1], [21, 5, 4], [12, 5, 1], [18, 5, 2], [33, 11, 2], [33, 18, 1], [37, 18, 4], [37, 13, 1], [41, 13, 4], [41, 9, 3], [35, 9, 2], [35, 11, 3], [35, 13, 1], [39, 13, 2], [39, 15, 1], [41, 15, 4].

We test whether these vertices are interior points of any rectangle $O_1, \ldots, O_{12}$. The first vertex in the list, which is not an interior point of any rectangle O_i, is the vertex [15, 7]. According to the procedure described above we get the following vertices of the boundary of the hole: [15, 7], [13, 7], [13, 5], [18, 5], [18, 9], [15, 9].

After this procedure, the vertices remaining in the list are [8, 12, 3], [8, 13, 1], [13, 3, 3], [14, 9, 2], [14, 11, 1], [18, 3, 2], [11, 8, 1], [13, 8, 4], [21, 5, 4], [12, 5, 1], [33, 11, 2], [33, 18, 1], [37, 18, 4], [37, 13, 1], [41, 13, 4], [41, 9, 3], [35, 9, 2], [35, 11, 3], [35, 13, 1], [39, 13, 2], [39, 15, 1], [41, 15, 4].

We test these vertices again whether they are interior points of any rectangle O_i. The first vertex in the list, which is an interior point of no rectangle O_i, is the vertex [33, 11]. We get according to the above mentioned procedure the following vertices of the boundary of the union: [33, 11], [33, 18], [37, 18], [37, 13], [39, 13], [39, 15], [41, 15], [41, 9], [35, 9], [35, 11].

Since we have spent all vertices from the list in this way, the problem is solved.

The above mentioned algorithm can be modified in such a way that after compiling the list of the vertices of the union, we test all these vertices according to whether they are interior points of any one of the given rectangles O_i. We omit from the list all vertices which are interior points of some rectangles.

References

[1] *A. Rosenfeld* and *A. C. Kak:* Digital Picture Processing, Academic Press, New York—London, 1976,

[2] *A. Rosenfeld:* Digital Topology, Amer. Math. Monthly, 86, 1979, 621—630.

Súhrn

HRANICA ZJEDNOTENIA OBDĽŽNIKOV V ROVINE

VÁCLAV MEDEK

Pri rysovaní masiek integrovaných obvodov sa vyskytuje úloha nájsť hranicu množiny obdĺžnikov, ktorých strany patria do dvoch navzájom kolmých smerov. V práci je uvedený algoritmus na riešenie tejto úlohy.

Author's address: Prof. RNDr. *Václav Medek*, Stavebná fakulta SVŠT, Radlinského 11, 813 68 Bratislava.