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ON THE CONCRETENESS OF QUANTUM LOGICS

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INTRODUCTION

In mathematical formulations of the foundations of quantum mechanics it is often postulated that the "event structure" of a quantum experiment be a quantum logic, that is, an orthomodular partially ordered set. (See below for definitions.) We emphasise that in this note it is *not* assumed that quantum logics are lattices.

When a quantum logic, L, is isomorphic to a logic of subsets of some given set, it is said to be a *concrete* logic. All Boolean algebras are concrete. So, also, is the lattice of projections in a spin factor. On the other hand, many logics are not concrete. For example, the lattice of all projections in a Hilbert space H is not concrete unless the dimension of H is less than three.

We shall show that for any quantum logic L we can find a concrete logic K and a surjective homomorphism f from K onto L such that f maps the centre of K onto the centre of L. Moreover, we can ensure that each finite set of compatible elements in L is the image of a compatible subset of K. We go on to show that this result is "best possible" – let a logic L be the homomorphic image of a concrete logic under a homomorphism such that, if F is a finite subset of the pre-image of a compatible subset of L then F is compatible. Then we prove that L must also be concrete. We then consider embeddings into concrete logics. We shall show that any concrete logic can be embedded into a concrete logic with preassigned centre and an abundance of two-valued measures. Finally, we prove that an arbitrary logic can be mapped into a concrete logic by a centrally additive mapping which preserves the ordering and complementation.

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1. LOGICS AND CONCRETE LOGICS

We shall need the following definitions.

Definition 1.1. A *logic* is a partially ordered set L with a least element 0 and a greatest element 1, together with an operation $x \to x'$ mapping L to L, such that the following conditions are satisfied. For any a, b in L,

(i) $(a')' = a$,

(ii) $a \leqq b$ implies $b' \leqq a'$,

(iii) $a \vee a'$ exists in L and is 1,

(iv) if $a \leqq b$ then $b \wedge a'$ and $a \vee (b \wedge a')$ exist in L and $a \vee (b \wedge a') = b$.

In many formulations of the foundations of quantum mechanics it is assumed that the "event structure" of a quantum experiment is a logic.

The degree of "quantumness" then corresponds to the degree of nondistributivity of the logic (see e.g. [6], [11]). Standard examples of logics are the lattice of all projections in a Hilbert space and the "classical logics" which are Boolean algebras.

Definition 1.2. Let Q_1 and Q_2 be logics. A mapping $f: Q_1 \to Q_2$ is said to be a *homomorphism* if

(i) $f(0) = 0$,

(ii) $f(a') = f(a)'$ for any $a \in L$,

(iii) $f(a \vee b) = f(a) \vee f(b)$ whenever $a, b \in L$ and $a \leqq b'$.

When $f: L_1 \to L_2$ is bijective and both f and f^{-1} are homomorphisms then f is said to be an *isomorphism*. When f is injective and $f: L_1 \to f(L_1)$ is an isomorphism then f is called an *embedding*, and L_1 is a *sublogic* of L_2.

Let S be a non-empty set and Δ a collection of subsets of S. Partially order Δ by set inclusion and, for each $A \in \Delta$, let A' be the set $S \setminus A$. Then Δ will be a logic when the following three conditions are satisfied

(i) $\emptyset \in \Delta$.

(ii) If $A \in \Delta$ then $A' \in \Delta$.

(iii) If A and B are in Δ and $A \cap B = \emptyset$ then $A \cup B \in \Delta$.

Let us recall that a logic L is said to be *concretely representable*, in short, a *concrete logic*, if there exists a collection of subsets Δ of a non-empty set S, satisfying the above three conditions, such that the logic L is isomorphic to (S, Δ).

We shall now derive a simple characterization of concrete logics which will be useful to us later. We recall that a mapping $m: L \to [0, 1]$ is called a (probability) measure if $m(1) = 1$ and $m(a \vee b) = m(a) + m(b)$ whenever $a, b \in L$ and $a \leqq b'$. Let $\mathscr{M}_2(L)$ be the set of all probability measures on L which take only the values zero and one.

Proposition 1.3. *Let L be a logic. The following conditions on L are equivalent.*

1) *L is a concrete logic.*

2) *For each a and b in L either $a \geqq b$ or there exists a two-valued probability measure $m \in \mathscr{M}_2(L)$ such that $m(a) = 0$ and $m(b) = 1$.*

3) *L is isomorphic to a concrete logic K, (corresponding to a pair (S, Δ)), where S has the property: for each $m \in \mathscr{M}_2(K)$ there is a point $p \in S$ such that $m(A) = 1$ if, and only if, $p \in A$.*

4) *There exists an injective homomorphism $f: L \to B$ from L into a Boolean algebra B such that $f(a) \leqq f(b)'$ implies $a \leqq b'$.*

Proof. Let us assume (1) so that L is isomorphic to (S, Δ). When $A, B \in \Delta$ either $A \supset B$ or there exists a point $p \in B \setminus A$. Let m be the evaluation measure at the point. Then $m(A) = 0$ and $m(B) = 1$.

Now assume (2). Let S be the set $\mathscr{M}_2(L)$ and, for each $a \in L$ let $S_a = \{m \in S: m(a) = 1\}$. Let Δ be the collection $\{S_a: a \in L\}$. Then (S, Δ) is a logic of subsets of S and $f: L \to \Delta$, defined by $f(a) = S_a$, is an isomorphism. From the definition of S and Δ, any two-valued measure on L corresponds to a point of S.

Let us assume (3) and let $i: L \to \Delta$ be the isomorphism whose existence is implied by this assumption. Let B be the Boolean algebra of all subsets of S and let $j: \Delta \to B$ be the natural embedding. Let $f = ji$. Let a, b be in L with $f(a) \leqq f(b')$. Then $i(a) \leqq i(b)' = i(b')$. Since i is an isomorphism it follows that $a \leqq b'$.

We now assume 4). Let S be the Stone structure space of the Boolean algebra B and identify B with the clopen subsets of S. Let $\Delta = \{f(a): a \in L\}$. Then (S, Δ) is a logic isomorphic to L.

We see that for a logic to be concrete it is necessary that it possess an abundance of two-valued measures. Since there exist logics without two-valued measures, see [1], [8], or, indeed, without any measures at all, see [5], it is clear that many logics fail to be concrete. In particular, when A is a von Neumann algebra or a JBW-algebra and $L(A)$ is its lattice of projections then $L(A)$ is concrete if, and only if, A is a direct sum of an abelian and a Type I_2 algebra (see [2]).

As non-concrete logics exist in abudance it is surprising, at first sight, that every logic is the homomorphic image of a concrete logic. Before giving our principal results, let us recall the definition of compatibility in a logic.

Definition 1.4. Let F be a finite subset of a logic L. If there exists a Boolean sublogic of L which contains F, then F is said to be *compatible* (in L).

The notion of compatibility is important for quantum mechanics. The rough idea is that compatible events in a quantum mechanical experiment may be considered by "classical" methods. A more detailed exposition can be found in [3, 6, 9, 10].

Definition 1.5. Let L be a logic and let $C(L)$ be the set of all $a \in L$ such that, for every $b \in L$, $\{a, b\}$ is a compatible subset of L. The set $C(L)$ is called the *centre* of L.

Proposition 1.6. (see [3]). *A subset M of a logic L can be enlarged to a Boolean sublogic of L if, and only if, each finite subset of M is compatible. It follows that $C(L)$ is a Boolean subalgebra of L. When f is a homomorphism from a logic L_1 to a logic L_2 then f maps compatible subsets of L_1 to compatible subsets of L_2 and if f is surjective it maps $C(L_1)$ into $C(L_2)$.*

Before turning to our main results in the next section let us observe that when (S, Δ) is a concrete logic and Δ_1 is a finite subset of Δ, then Δ_1 is compatible if, and only if, the intersection of each subfamily of Δ_1 belongs to Δ.

Let us recall two simple constructions we shall make use of in the next section.

The $\{0, 1\}$-pasting of logics. Suppose that we are given a collection $\{L_\alpha \mid \alpha \in I\}$ of logics. Let K stand for the disjoint union of all L_α. Define an equivalence relation on K as follows. If $x \neq 0_\alpha$, $x \neq 1_\alpha$ then $x \sim y$ if and only if $x = y$. If $x = 0_\alpha$ ($x = 1_\alpha$ respectively) then $x \sim y$ if and only if $y = 0_\beta$ ($y = 1_\beta$ respectively) for some $\beta \in I$. It is easy to see that $L = K/\sim$ becomes a logic when endowed with the complementations operation and partial ordering inherited from K.

Proposition 1.7. *Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of logics and let each L_α be concrete. Then the $\{0, 1\}$-pasting L of $\{L_\alpha \mid \alpha \in I\}$ is concrete as well and if card $I \geqq 2$ and the logics L_α are non-trivial then $C(L) = \{0, 1\}$.*

Proof. The first part follows from Proposition 1.3, the second is trivial.

The product of logics. Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of logics. Take the cartesian product $\prod_{\alpha \in I} L_\alpha$ of the sets L and endow it canonically with the operation $\,'\,$, and the partial ordering $\leqq$. Thus, if $k, h \in \prod_{\alpha \in I} L_\alpha$ and $k = \{k_\alpha \mid \alpha \in I\}$, $h = \{h_\alpha \mid \alpha \in I\}$ then $k \leqq h$ ($k = h'$, resp.) if and only if $k_\alpha \leqq h_\alpha$ ($k_\alpha = h'_\alpha$, resp.) for any $\alpha \in I$. Denote by P the triple $(\prod_{\alpha \in I} L_\alpha\,, ', \leqq)$. It is easily seen that P is a logic.

Proposition 1.8. *Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of logics and let any L_α be concrete. Then the product P of $\{L_\alpha \mid \alpha \in I\}$ is concrete as well and if $C(L_\alpha) = \{0, 1\}$ for all L_α then $C(P)$ is Boolean isomorphic to the Boolean algebra of all subsets of I.*

Proof. According to Proposition 1.2, we have to show that if $k \nleqq h$ for some $k, h \in P$ then there is a measure $m \in \mathcal{M}_2(P)$ such that $m(k) = 1$ and $m(h) = 0$. Write $k_\alpha = k(\alpha)$ and $h_\alpha = h(\alpha)$. Since $k \nleqq h$, there exists an index $\alpha \in I$ such that $k_\alpha \nleqq h_\alpha$. We therefore have a measure $m_\alpha \in \mathcal{M}_2(L_\alpha)$ such that $m_\alpha(k_\alpha) = 1$ and $m_\alpha(h_\alpha) = 0$. Let $\pi_\alpha: P \to L_\alpha$ denote the projection onto the factor L_α. Then π_α is a homomorphism and $m = m_\alpha \pi_\alpha \in \mathcal{M}_2(P)$. Since $m(k) = 1$, $m(h) = 0$, we see that P is concrete.

Secondly, if $C(L_\alpha) = \{0, 1\}$ for any $\alpha \in I$ then $k = \{k_\alpha \mid \alpha \in I\}$ belongs to $C(P)$ if and only if each k_α is either 0_α or 1_α. Therefore the central elements of P are in a one-to-one correspondence with the subsets of I and this completes the proof.

Our aim is to show that every logic is a homomorphic image of a concrete one. Moreover, we shall find such a homomorphism which respects the compatibility relation as much as possible. Let us call a homomorphism $f: L_1 \to L_2$ *strong* if

(1) $f(C(L_1)) = C(L_2)$,

(2) when $B = \{b_1, b_2, \ldots, b_m\}$ is a compatible subset of L_2 then there is a compatible subset $A = \{a_1, a_2, \ldots, a_m\}$ of L_1 such that $f(a_i) = b_i$ for each $i \in I$ and moreover, if $B \cup \{b\}$ is compatible in L_2 then there exists an element $a \in L$, such that $A \cup \{a\}$ is compatible in L_1 and $f(a) = b$.

Before proving our first theorem we must recall the notion of the product of a logic over a Boolean algebra. Let L be a logic and let B be a Boolean algebra with Stone structure space I. Then we may identify B with the clopen subsets of I.

The Cartesian product L^I becomes a product logic when the complementation and partial ordering are defined "coordinatewise". Give L the discrete topology and let H be the set of all functions $k: I \to L$ which are continuous. Since I is compact we find that for each $k \in H$, there is a finite sequence of pairwise disjoint clopen subsets of I, $(I_1, I_2, \ldots, I_p)$ such that k is constant on I_j for $j = 1, 2, \ldots, p$. It is straightforward to verify that H is a sublogic of L^I. We call H the B-product of L. When L is concrete then the B-product will also be concrete .

Lemma 2.1. *Let L be a logic and let B be a Boolean subalgebra of $C(L)$. Let M be the B-product of L. Then there exists a strong morphism H from M onto L.*

Let I be the Stone structure space of B, Δ the collection of all clopen subsets of I and $t: \Delta \to B$ the Stone isomorphism.

Let $f \in M$. Then the range of f is a finite subset of L, say, $\{k_1, k_2, \ldots, k_p\}$. Let I_j be the clopen set $f^{-1}(k_j)$, for $j = 1, 2, \ldots, p$. Let

$$H(f) = \bigvee_{j=1}^{p} t(I_j) \wedge k_j .$$

A straightforward verification shows that H is a logic morphism from M onto L. For each $x \in L$ let $\bar{x}$ be the function on I which takes the constant value x. Since $x \to \bar{x}$ is an embedding of L into M and since $H(\bar{x}) = x$, for all $x \in L$, H is a strong morphism.

Theorem 2.2. *Let L be an arbitrary logic. Then L is the strong homomorphic image of some concrete logic.*

If L is a Boolean algebra there is nothing to prove.

Suppose that L is not Boolean. Let $\{C_\alpha \mid \alpha \in S\}$ be the collection of all maximal Bollean subalgebras of L. Then *card* $S \geqq 2$ and since each Boolean sub-algebra of L can be embedded in a maximal Boolean subalgebra (Zorn's lemma), we see that each finite compatible family $A \subset L$ belongs to some C_α. Further, if $A \cup \{a\}$ is compatible then $A \cup \{a\} \subset C_\beta$ for some $\beta \in I$. Let C be the $\{0, 1\}$-pasting of $\{C_\alpha \mid \alpha \in I\}$ and let K be the $C(L)$-product of C. Obviously, K is concrete and we have $C(K) = C(L)$.

By Lemma 2.1, we have a strong morphism $H: M \to L$, where M is the $C(L)$-product of L. We also have a natural (strong) morphism $D: K \to M$; each h in K is a function from the Stone space ,S, of $C(L)$ into C. So, for each $x \in S$, $h(x) \in C_\alpha$ for some α. Hence we may regard h as taking its values in L. So Dh is the same function as h but now regarded as Lvalued. We finally take $F: K \to L$ to be the composition HD. Then F is a strong homomorphism from a concrete logic onto L and the proof of the Theorem is complete.

The notion of a strong homomorphism can be further strengthened by requiring a correspondence between compatible sets. A homomorphism f from L_1 onto L_2 is called *very strong* if the following condition is satisfied: if $f(a_i) = b_i$ for all $i \leqq n$ and if $b_1, b_2, \ldots, b_n$ is a compatible set in L_2 then $\{a_1, a_2, \ldots, a_n\}$ is a compatible set in L_1.

Obviously, a very strong homomorphism is automatically strong. It turns out that Theorem 2.2 cannot be generalised to very strong homomorphisms.

Theorem 2.3. *Let f be a very strong homomorphism of a logic K onto a logic L. If K is concrete then L is also concrete.*

Proof. We write $K = (S, \Delta)$ and denote by $\exp S$ the Boolean algebra of all subsets of S. Let $i: \Delta \to \exp S$ be the canonical embedding. Let us consider the set $I = \{A \in \Delta \mid f(A) = 0\}$. Then $I \subset C(K)$ and I is the base of an ideal in the Boolean algebra $\exp S$. Take the ideal $J = \{B \in \exp S \mid B \subset A \text{ for some } A \in I\}$ and put $F = \exp S/J$. Let $g: \exp S \to F$ be the natural quotient mapping. We define a mapping $h: L \to F$ by setting $h(f(A)) = g\, i(A)$. We shall show that h is a mapping satisfying the fourth condition in Proposition 1.3.

Let us first verify that h is well defined. Suppose that $f(C) = f(D)$. Since C, D are then compatible in K, we see that $(C \cap D') \cup (D \cap C') \in \Delta$ and moreover, $f((C \cap D') \cup (D \cap C')) = f(C \cap D') \cup f(D \cap C') = 0$. It follows that $(C \cap D') \cup (D \cap C') \in J$ and therefore $g\, i(C) = g\, i(D)$. Thus the mapping h is well defined.

Let us show that h is a homomorphism. Obviously, $h(0) = 0$. If $f(A) \in L$ then $(h\, f(A))' = (g\, i(A))' = g\, i(A') = h\, f(A')$. Finally, if $f(A) \leqq f(B)'$ then $f(B) \leqq f(A')$. and therefore $f(B) = f(B \cap A')$. This implies that $h(f(B)) = h(f(B \cap A'))$ and therefore $h(f(A)) = g\, i(A) \leqq g\, i(B \cap A')' = h(f(B)')$. So h is a homomorphism.

It remains to prove that if $h(f(A)) \leqq h(f(B)')$, then $f(A) \leqq f(B)'$. Since F is a Boolean algebra, our assumption says that $h(f(A)) \wedge h(f(B)) = 0$ and hence $g\, i(A) \wedge g\, i(B) = 0$. It follows that $A \cap B \in J$ and therefore there is a set D, $D \in I$ such that $A \cap B \subset D$. Since $D \in C(K)$ and $f(D) = 0$, we have the equations $f(A \cap D') = f(A), f(B \cap D') = f(B)$ and moreover, $(A \cap D') \cap (B \cap D') = \emptyset$. Therefore $f(A \cap D') \leqq f(B \cap D')'$ which implies that $f(A) \leqq f(B)'$. The proof is complete.

The last result may be restated in the following algebraic form. Let us call a subset J of a logic L an *ideal* if J fulfils the following conditions

1) if $a \in J$ and $b \leqq a$ then $b \in J$,
2) if $a, b \in J$ and $a \leqq b'$ then $a \vee b \in J$,
3) if $a \in J$ then $a' \notin J$.

Let us call an ideal *central* if it consist of central elements.

Theorem 2.4. *If J is a central ideal in a logic then the quotient L/J is again a logic. Moreover, if L is concrete then L/J is also concrete.*

Proof. The first part of Theorem 2.4 is easy, the second part follows from Theorem 2.3.

Corollary 2.5. *Each central ideal of a concrete logic is a subset of the collection of zero sets of a two-valued measure on the logic.*

The following simple example shows that not every ideal of a concrete logic can be embedded in the zero sets of such a measure. Take S to be $\{1, 2, 3, 4, 5, 6\}$ and let Δ consist of all subsets of S with an even number of elements. Put $I = \{A \in \Delta \mid \text{card } A = 2 \text{ and } 1 \in A\}$. Then I is an ideal in (S, Δ) and if $C = \{3, 4\}$ then there is no ideal in (S, Δ) containing either $I \cup \{C\}$ or $I \cup \{C'\}$.

3. EMBEDDINGS AND PSEUDO-EMBEDDINGS INTO CONCRETE LOGICS

In the first part of this section we shall consider embeddings of concrete logics into concrete logics with preassigned centres. Let us start with the following observation.

Proposition 3.1. *Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of concrete logics. Put $P = \prod_{\alpha \in I} L_\alpha$ and consider the set $S = \{p \in P \mid \text{for each } \alpha \in I, p(\alpha) \text{ is } 0 \text{ or } 1\}$. The set S is a Boolean sublogic of P and if $\bar{m} \in \mathcal{M}_2(S)$ then there is a measure $m \in \mathcal{M}_2(P)$ such that $m \mid S = \bar{m}$. (Moreover, if $p \in P$ is an element such that $\bar{m}(s) = 1$ for each $s \in S$, $s \geqq p$ then the measure $m \in \mathcal{M}_2(P)$ can be chosen so that $m(p) = 1$).*

Proof. For each $\alpha \in I$ choose a measure $m_\alpha \in \mathcal{M}_2(L_\alpha)$. For each element p in P, let $\tilde{p}$ be the element of S satisfying $\tilde{p}(\alpha) = 1$ if and only if $m_\alpha(p(\alpha)) = 1$. Let us now define a function $m: P \to \{0, 1\}$ by setting $m(p) = \bar{m}(\tilde{p})$. One easily checks that $m \in \mathcal{M}_2(P)$ and m extends $\bar{m}$. (The rest of Prop. 3.1 follows from a suitable choice of m_α's).

Theorem 3.2. *Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of concrete logics and let B be a Boolean algebra. Then there is a concrete logic L such that*

1) for each $\alpha \in I$ there exists an embedding $e_\alpha: L_\alpha \to L$ (and therefore L_α is a sublogic of L for every $\alpha \in I$),

2) $C(L) = B$,

3) *if* $\overline{m} \in \mathscr{M}_2(C(L))$ *and* $m_\alpha \in \mathscr{M}_2(L_\alpha)$ *for all* $\alpha \in I$ *then there exists a measure* $m \in \mathscr{M}_2(L)$ *such that* $m \mid C(L) = \overline{m}$ *and* $m(e_\alpha(x)) = m_\alpha(x)$ *for each* $\alpha \in I$, $x \in L_\alpha$ *(and therefore m is a common extension of* $\overline{m}$ *and all* m_α*'s).*

Moreover, the above construction has the following uniform property. Let $\{L_\alpha \mid \alpha \in I\}$ *be a collection of concrete logics and* $\{B_\beta \mid \beta \in J\}$ *a collection of Boolean algebras. Then there are concrete logics* L_β, $\beta \in J$ *such that*

(1a) *for any* $\beta \in J$, *the logic* L_β *fulfils the above properties* 1), 2), 3) *with respect to* $\{L_\alpha \mid \alpha \in I\}$ *and the Boolean algebra* B_β,

(2a) *there exists a concrete logic L such that, for any* L_β, *there is an embedding* $f_\beta: L_\beta \to L$,

(3a) *each* B_β *is a Boolean subalgebra of* $C(L)$.

(4a) *if* $m_1 \in \mathscr{M}_2(C(L))$ *and* $m_\alpha \in \mathscr{M}_2(L_\alpha)$ *for all* $\alpha \in I$ *then there is a measure* $m \in \mathscr{M}_2(L)$ *such that* $m \mid C(L) = m_1$ *and* $m(f_\beta e_\alpha)(x) = m_\alpha(x)$ *for any* $\alpha \in I$, $\beta \in J$, $x \in L_\alpha$.

Proof. Let L_0 be a four point Boolean algebra and let M be the $\{0, 1\}$-pasting of $\{L_\alpha \mid \alpha \in I\} \cup \{L_0\}$. Let L be the B-product of M. Then $C(L) = B$ and the property 1) is obvious, the property 3) follows from Prop. 3.1. If $\{B_\beta \mid \beta \in J\}$ is a collection of Boolean algebras then there is a set I such that each B_β can be represented by a collection of subsets of I. We then take $L = \prod_{\alpha \in I} M_\alpha$, here each M_α equals M, and the rest of Theorem 3.2 follows from the construction of B_β-products and Prop. 3.1.

Let us now state our final result. We have seen that a sublogic of a concrete logic has to be concrete (Prop. 1.2). We shall examine here the best possible way of "embedding" an arbitrary logic into a concrete one. Let us call a mapping $f: L_1 \to L_2$ a *pseudo-embedding* if f satisfies the following properties

(1) $f(0) = 0$,

(2) $f(a') = f(a)'$ for every $a \in L$,

(3) $f(a) \leqq f(b)$ if and only if $a \leqq b$,

(4) $f(a \vee b) = f(a) \vee f(b)$ whenever $a \in L$, $b \in C(L)$.

Theorem 3.3. *Let L be a logic. Then there exists a concrete logic K and a pseudo-embedding* $f: L \to K$. *Moreover, if* $g: L \to M$ *is a pseudo-embedding into a concrete logic then there exists a unique homomorphism* $h: K \to M$ *such that* $g = hf$.

Before we can prove this theorem we must establish several technical lemmas generalizing results of [7] and [12]. In the following L is a fixed but arbitary logic. We shall call a subset I of L *absorbing* when the following conditions are satisfied

(i) If $a \in I$ and $b \leqq a$ then $b \in I$.

(ii) If $a \in I$ and $b \in I \cap C(L)$ then $a \vee b \in I$.

(iii) If $a \in I$ then $a' \notin I$.

Lemma 3.4. *Let c and d be elements of L. Either $c \geqq d$ or there is an absorbing set I such that $\{c, d'\} \subset I$.*

Proof. For each $x \in L$, let $I_x = \{y \in L: y \leqq x\}$. Let $I = \{x \in L: x \leqq m \vee n \vee k$ for some $m \in I_c \cap C(L)$, $n \in I_{d'} \cap C(L)$ and $k \in I_c \cup I_{d'}\}$.

By construction, c and d' are in I, so it remains to show that either I is absorbing or $c \geqq d'$.

First, we observe that if $b \leqq a$ and $a \in I$ then, from the definition of I, $b \in I$.

Let b be any element of $I \cap C(L)$. We shall show that there exist $u \in I_c \cap C(L)$ and $v \in I_d \cap C(L)$ such that $b = u \vee v$.

Since b is in I, $b \leqq p \vee q \vee s$ where $p \in I_c \cap C(L)$, $q \in I_{d'} \cap C(L)$ and $s \in I_c \cup I_{d'}$. Since b, p, q are in the centre $\{b, p, q, s\}$ is a compatible subset of L and so generates a Boolean sublogic.

So $b = (b \wedge p) \vee (b \wedge q) \vee (b \wedge s)$. It follows that there is no loss of generality in supposing that

$$b = p \vee q \vee s .$$

Let $t = b \wedge (p \vee q)'$. Then t is in the centre.

Also $t = s \wedge (p \vee q)'$.

So t is in $I_c \cap C(L)$ or $I_{d'} \cap C(L)$.

If t is in I_c we put $u = p \vee t$ and $v = q$. Similarly, if t is in $I_{d'}$ we put $u = p$ and $v = q \vee t$.

Let a be any element of I and let b be as above. Then $a \leqq m \vee n \vee k$ for some $m \in I_c \cap C(L)$, $n \in I_{d'} \cap C(L)$ and $k \in I_c \cup I_{d'}$. Then $a \vee b \leqq (m \vee u) \vee (n \vee v) \vee k$.

Hence $a \vee b \in I$.

We now assume that there exists an a such that $a \in I$ and $a' \in I$. We also assume that it is false that $c \geqq d$.

We may suppose, without loss of generality, that $a \leqq m \vee k$ and $a' \leqq n \vee h$ where $m \in I_c \cap C(L)$, $n \in I_{d'} \cap C(L)$, $k \in I_d$, $h \in I_c$, $m \leqq k'$ and $n \leqq h'$.

Since m and n are central, m, n and a are compatible. So we have

$$a = (m \wedge a) \vee (m' \wedge a)$$

and

$$a' = (n \wedge a') \vee (n' \wedge a') .$$

It follows that

$$1 = a \vee a' = (m \wedge a) \vee (m' \wedge a) \vee (n \wedge a') \vee (n' \wedge a') .$$

We observe that $m \wedge a$, $m' \wedge a$, $n \wedge a'$, $n' \wedge a'$ are mutually orthogonal.

Since $a' \leqq n \vee h$ we have

$$a' \wedge n' \leqq h \wedge n' \leqq c .$$

So $a' \wedge n' \in I_{c'}$. Similarly, $a \wedge m' \in I_{d'}$.

Then $$(a \wedge m') \vee n \leqq d'$$

so $$n' \wedge (a' \vee m) \geqq d$$

so $$(a, \wedge n') \vee (n' \wedge m) \geqq d$$

so $$(a' \wedge n') \vee m \geqq d$$

so $$c \geqq d\,.$$

This contradiction shows that either $c \geqq d$ or I is absorbing and $\{c, d'\} \subset I$.

Lemma 3.5. *Let I be an absorbing subset of L. Let e and e' be elements of the logic L neither of which is in I. Then there is an absorbing set J containing $I \cup \{e\}$.*

Proof. As before, let $I_e = \{x \in L\colon\ x \leqq e\}$. Let J be the set of all x in L such that

$$x \leqq m \vee n \vee k$$

for some $m \in I \cap C(L)$, some $n \in I_e \cap C(L)$ and some $k \in I \cup I_{e'}$. Then, arguing as in the preceding lemma, we find that J is absorbing.

Lemma 3.6. *Let c and d be elements of L. Then either $c \geqq d$ or there exists a function $s\colon L \to \{0, 1\}$ with the following properties*

(1) $s(c) = 0$ *and* $s(d) = 1$.
(2) $s(a) + s(a') = 1$ *for each* $a \in L$.
(3) *If* $s(b) = 0$ *and* $a \leqq b$ *then* $s(a) = 0$.
(4) *If* $b \in C(L)$, $a \in L$ *and* $a \leqq b'$ *then* $s(a \vee b) = s(a) + s(b)$.

Proof. Let A be the family of all absorbing subsets $I \subset L$ such that $\{c, d'\} \subset I$. By Lemma 3.4 this family is not empty. Partially order A by set inclusion. By Zorn's lemma we see that A has a maximal element M. Let e be any element of L. If $\{e, e'\} \cap \cap M = \emptyset$ then, by Lemma 3.5 M cannot be maximal. So, for every e, $\{e, e'\} \cap M \neq 0$ So, for every e, either $e \in M$ or $e' \in M$ but not both.

Let $s(x)$ be 1 if $x \notin M$ and let $s(x)$ be 0 if $x \in M$. A straightforward verification shows that s has the four properties required.

Let us now prove Theorem 3.3. Let S be the set of all mappings $s\colon L \to \{0, 1\}$ with the properties (2), (3), (4) listed in the statement of Lemma 3.6. Let us put $\tilde{\Delta} = \{S_k \mid k \in L\}$, where $S_k = \{s \in S \mid s(k) = 1\}$. Take the least concrete logic $K = (S, \Delta)$ such that $\tilde{\Delta} \subset \Delta$ and define $f(k) = S_k$. We shall show that f is a pseudo-embedding with the universal property.

With the help of Lemma 3.6, we can simply verify that the axioms (1), (2), (3) in the definition of pseudo-embedding are fulfilled. Let us consider two elements a and b in L where b is in $C(L)$. We must show that any mapping $s\colon L \to \{0, 1\}$ with the properties (2), (3), (4) satisfies the following requirement: If $s(a \vee b) = 1$ then either $s(a) = 1$ or $s(b) = 1$. Indeed, since $a \vee b = (a \wedge b') \vee b$ and $a \wedge b' \leqq a$, we have $s(a \vee b) = s(a \wedge b') + s(b) = 1$. Therefore either $s(b) = 1$ or $s(a \wedge b') = 1 \leqq \leqq s(a)$. This implies that $f(a \vee b) = f(a) \cup f(b)$.

Finally, suppose that $g: L \to M$ is a pseudo-embedding into a concrete logic M. We have to show that the mapping $\tilde{h}(S_k) = g(k)$, $k \in L$ admits a unique extension to a homomorphism $h: K \to M$. The uniqueness is obvious since the generation of Δ from $\tilde{\Delta}$ involves only the formation of complements and disjoint unions. The existence of the extension can be established as follows. Let B_1, B_2 be the Boolean algebras generated by $\tilde{\Delta}$, $\tilde{h}(\tilde{\Delta})$, respectively.

We shall see that there is an extension of $\tilde{h}$ to a Boolean homomorphism H from B_1 to B_2. We then define h to be the restriction of H to K. The existence and uniqueness of H will follow from Theorem 12-2 [10] if whenever $\{S_{k_j}: j = 1, 2, \ldots, n\}$ has empty intersection then so, also, does $\{\tilde{h}(S_{k_j}): j = 1, 2, \ldots, n\}$. Suppose H does not exist. Then for some $k_1, \ldots, k_n$, we have $\bigcap_{j=1}^{n} S_{k_j} = \emptyset$ and $\bigcap_{j=1}^{n} \tilde{h}(S_{k_j}) \neq \emptyset$.

Let p be any point in $\bigcap_{j=1}^{n} \tilde{h}(S_{k_j})$ and let m be in $\mathscr{M}_2(B_2)$ and such that $m(\{p\}) = 1$. Then $mg \in \bigcap_{i=1}^{\infty} S_{k_i}$, a contradiction. Finally, we must show that h, which is the restriction of H to K, maps K into M.

Let $\tilde{K}$ be $H^{-1}[M]$. Then $\tilde{K}$ contains $\tilde{h}^{-1}[M]$, that is, $\tilde{K}$ contains a set of generators for K. Hence $\tilde{K}$ contains K, as required.

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Souhrn

KONKRETNOST KVANTOVÝCH LOGIK

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Logika se nazývá konkretní, jestliže je representovatelná soustavou podmnožin nějaké množiny. V článku se zkoumá, kdy je obecná logika homomorfním obrazem konkretní a kdy lze obecnou logiku vnořovat do konkretní logiky.

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