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RELATIVE CONDITIONAL EXPECTATIONS ON A LOGIC

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It is a well-known fact that by a quantum mechanical experiment the set of all random events is no more a Boolean algebra, but a more general algebraic structure. To describe a quantum mechanical measurement, a generalization of the classical probability theory is needed. In the quantum logic approach, the set of random events is supposed to be a quantum logic.

Conditional expectations play a basic role in the classical probability theory. Some of the most important areas of the theory such as Markov processes and martingales rely heavily on this concept. Although there has been much discussion [16]–[21], conditional expectations have not been satisfactorily generalized to quantum probability.

In this paper, we introduce the notion of a conditional expectation of an observable x on a logic $\mathscr{L}$ with respect to a sublogic $\mathscr{L}_0 \subset \mathscr{L}$ in a state m on $\mathscr{L}$, relative to an element $a \in \mathscr{L}$ such that $m(a) = 1$ and $\mathscr{R}(x) \cup \mathscr{L}_0$ is partially compatible with respect to a. This conditional expectation is an analogue of the conditional expectation of an integrable function f on a probability space $(\Omega, \mathscr{S}, \mu)$ with respect to a sub-σ-field $\mathscr{S}_0$ of $\mathscr{S}$, relativized by a massive set A (i.e. $\mu(A) = 1$); that is, the conditional expectation of f with respect to the σ-field $\mathscr{S}_1$ generated by $\mathscr{S}_0$ and A.

1. BASIC DEFINITIONS

Let $\mathscr{L}$ be a logic (an orthomodular σ-lattice), i.e. a partially ordered set with the first and last elements 0 and 1, respectively, with the orthocomplementation $\perp: \mathscr{L} \to \mathscr{L}$ such that

(i) $(a^\perp)^\perp = a$;

(ii) $a \leqq b$ implies $a^\perp \geqq b^\perp$;

(iii) $a^\perp \vee a = 1$ for all $a \in \mathscr{L}$;

(iv) $\bigvee_{i=1}^{\infty} a_i$ exists in $\mathscr{L}$ for any sequence $\{a_i\}_{i=1}^{\infty}$ in $\mathscr{L}$;

(v) $a \leq b$ implies $b = a \vee (a^\perp \wedge b)$.

Two elements a, b from $\mathscr{L}$ are orthogonal ($a \perp b$) if $a \leq b^\perp$, and they are compatible ($a \leftrightarrow b$) if $a = (a \wedge b) \vee (a \wedge b^\perp)$, $b = (a \wedge b) \vee (a^\perp \wedge b)$. If $\{b_i\}_{i=1}^\infty$ is a sequence of elements of $\mathscr{L}$ and $a \in \mathscr{L}$ is such that $a \leftrightarrow b_i$ for all $i = 1, 2, \ldots$, then $a \leftrightarrow \bigvee_{i=1}^\infty b_i$, and $a \wedge (\bigvee_{i=1}^\infty b_i) = \bigvee_{i=1}^\infty (a \wedge b_i)$ (Cf. [1].)

A set $M \subset \mathscr{L}$ is said to be *compatible* if $a \leftrightarrow b$ for any $a, b \in M$. A subset $\mathscr{L}_0$ of $\mathscr{L}$ is a *sublogic* if (i) $a \in \mathscr{L}_0$ implies $a^\perp \in \mathscr{L}_0$; (ii) $\{a_i\}_{i=1}^\infty \subset \mathscr{L}_0$ implies $\bigvee_{i=1}^\infty a_i \in \mathscr{L}_0$. A sublogic $\mathscr{B}$ of $\mathscr{L}$ is *a Boolean sub-σ-algebra,* if for any there elements a, b, c of $\mathscr{B}$ the distributive law $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds. For any compatible subset M of $\mathscr{L}$ there is a Boolean sub-σ-algebra $\mathscr{B}$ such that $M \subset \mathscr{B} \subset \mathscr{L}$. (Cf. [1].)

A state on $\mathscr{L}$ is the map $m: \mathscr{L} \to [0, 1]$ such that (i) $m(1) = 1$; (ii) $m(\bigvee_{i=1}^\infty a_i) = \sum_{i=1}^\infty m(a_i)$ for any sequence $\{a_i\}_{i=1}^\infty$ of mutually orthogonal elements of $\mathscr{L}$.

An observable on $\mathscr{L}$ is a σ-homomorphism from the Borel subsets $\mathscr{B}(\mathscr{R})$ of the real line $\mathscr{R}$ to $\mathscr{L}$; i.e. a map $x: \mathscr{B}(\mathscr{R}) \to \mathscr{L}$ such that (i) $x(\mathscr{R}) = 1$; (ii) $x(E^c) = x(E)^\perp$ for any $E \in \mathscr{B}(\mathscr{R})$ and (iii) $x(\bigcup_{i=1}^\infty E_i) = \bigvee_{i=1}^\infty x(E_i)$ for any sequence $\{E_i\}_{i=1}^\infty$ of $\mathscr{B}(\mathscr{R})$.

If x is an observable and $f: \mathscr{R} \to \mathscr{R}$ is a Borel measurable function, then the map $f(x) = x \circ f^{-1}: \mathscr{B}(\mathscr{R}) \to \mathscr{L}$ is also an observable. It is called *the function f* of the observable x. *The range* of an observable x, $\mathscr{R}(x) = \{x(E); E \in \mathscr{B}(\mathscr{R})\}$, is a Boolean sub-$\sigma$-algebra of $\mathscr{L}$. A Boolean sub-σ-algebra of $\mathscr{L}$ is the range of an observable if and only if it is countably generated; and $\mathscr{R}(y) \subset \mathscr{R}(x)$ implies that the observable y is a function of x, i.e. there is a Borel function $f: \mathscr{R} \to \mathscr{R}$ such that $y = f(x)$. (Cf. [1].)

If x is an observable and m is a state on $\mathscr{L}$, then the map $m_x: \mathscr{B}(\mathscr{R}) \to [0, 1]$ where

$$m_x(E) = m(x(E)),$$

is *a probability measure* on $\mathscr{B}(\mathscr{R})$. It is called the probability distribution of the observable x in the state m. *The expectation* of x in the state m is

$$m(x) = \int_{\mathscr{R}} \lambda m_x(\mathrm{d}\lambda) \tag{1}$$

if the integral exists. The observable x on $\mathscr{L}$ is called *integrable* in the state m if $m(x)$ exists and is finite. If f is any Borel function on $\mathscr{R}$, then

$$m(f(x)) = \int_{\mathscr{R}} f(\lambda)\, m_x(\mathrm{d}\lambda), \tag{2}$$

if the integral exists. The observable x is called *square integrable* in the state m, if

(3) $$m(x^2) = \int_{\mathscr{R}} \lambda^2 m_x(\mathrm{d}\lambda)$$

exists and is finite.

Let $a \in \mathscr{L}$, $a \neq 0$. The set $\mathscr{L}_{[0,a]} = \{b \in \mathscr{L}: b \leqq a\}$ is a logic with the partial ordering inherited from $\mathscr{L}$, with the greatest element a and with the relative orthocomplementation $b' = b^{\perp} \wedge a$, $b \in \mathscr{L}_{[0,a]}$. If x is an observable on $\mathscr{L}$ such that $x \leftrightarrow a$ (i.e. $x(E) \leftrightarrow a$ for any $E \in \mathscr{B}(\mathscr{R})$), then the map $x \wedge a: \mathscr{B}(\mathscr{R}) \to \mathscr{L}_{[0,a]}$,

$$E \mapsto x(E) \wedge a$$

is an observable on the logic $\mathscr{L}_{[0,a]}$. If m is a state on $\mathscr{L}$ such that $m(a) = 1$, then the restriction of m to $\mathscr{L}_{[0,a]}$ is a state on $\mathscr{L}_{[0,a]}$.

Let $a \in \mathscr{L}$, $a \neq 0$ and let $M \subset \mathscr{L}$ be any subset. We say that M is *partially compatible with respect to a* (M is p.c. (a)) if (i) $M \leftrightarrow a$ (i.e. $b \leftrightarrow a$ for all $b \in M$) and (ii) the set $M \wedge a = \{b \wedge a: b \in M\}$ is compatible. It can be easily seen that the set $M \wedge a$ is compatible in $\mathscr{L}$ if and only if it is compatible in the logic $\mathscr{L}_{[0,a]}$.

Let $F = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of $\mathscr{L}$. Let us put $D = \{0, 1\}$, $d = (d_1, d_2, \ldots, d_n) \in D^n$, $a^0 = a^{\perp}$, $a^1 = a$ $(a \in \mathscr{L})$. The element

(4) $$\operatorname{com}(F) = \bigvee_{d \in D^n} a_1^{d_1} \wedge a_2^{d_2} \wedge \ldots \wedge a_n^{d_n}$$

is called *the commutator* of the set F. It was shown ([2], [3]) that F is p.c. (com (F)).

The logic $\mathscr{L}$ is called *separable* if any subset of mutually orthogonal elements is at most countable. If $\{a_\alpha; \alpha \in A\}$ is a subset of a separable logic $\mathscr{L}$, then there is a countable subset $I \subset A$ such that $\bigvee_{\alpha \in A} a = \bigvee_{\alpha_i \in I} a_{\alpha_i}$ (similarly, $\bigwedge_{\alpha \in A} a = \bigwedge_{\alpha_i \in I} a$); see [4]. Any Boolean sub-σ-algebra of a separable logic is countably generated, so that it is the range of an observable.

Now let M be a subset of a separable logic $\mathscr{L}$. For any finite subset F of M let the commutator com (F) be defined by (4). Then

(5) $$\operatorname{com}(M) = \bigwedge_{\{F \subset M;\, F \text{ finite}\}} \operatorname{com}(F)$$

is *the commutator* of the set M. Again it was shown that M is p.c. (com (M)), see [2].

2. CONDITIONAL EXPECTATIONS

Let $(\Omega, \mathscr{S}, \mu)$ be a probability space. Let $f \in \mathscr{L}_2(\mu)$ and let $\mathscr{S}_0$ be a sub-σ-field of $\mathscr{S}$. The conditional expectation of f with respect to $\mathscr{S}_0$ is a function $g \in \mathscr{L}_2(\mu)$ such that

(i) $g^{-1}(\mathscr{B}(\mathscr{R})) \subset \mathscr{S}_0$ (i.e. g is $\mathscr{S}_0$-measurable),

(ii) $\int_B f(\omega)\, \mu(\mathrm{d}\omega) = \int_B g(\omega)\, \mu(\mathrm{d}\omega)$ for any $B \in \mathscr{S}_0$.

We shall write $g := E_\mu(f/\mathscr{S}_0)$.

Let x be an observable and m a state on $\mathscr{L}$. For $a \in \mathscr{L}$, we shall define the expression $\int_a x \, dm$ as follows:

$$\text{(6)} \qquad \int_a x \, \mathrm{d}m := \int \lambda m(x(\mathrm{d}\lambda) \wedge a),$$

if the integral on the right hand side exists. This integral makes sense if (i) $v: E \mapsto m(x(E) \wedge a)$, $E \in \mathscr{B}(\mathscr{R})$ is a measure on $\mathscr{B}(\mathscr{R})$; (ii) the function $f(\lambda) \equiv \lambda$ is integrable with respect to v.

It can be easily checked that if x is integrable with respect to m and $x \leftrightarrow a$, then the integral exists. We shall need the following lemma.

Lemma 1. *Let x and y be observables on $\mathscr{L}$ such that $x \leftrightarrow y$ (i.e. $x(E) \leftrightarrow y(F)$ for all $E, F \in \mathscr{B}(\mathscr{R})$), and let $a \in \mathscr{L}$ be such that $x \leftrightarrow a$ and $y \leftrightarrow a$. Then*

$$\text{(7)} \qquad \int_a x \, \mathrm{d}m + \int_a y \, \mathrm{d}m = \int_a (x + y) \, \mathrm{d}m .$$

Proof: By the suppositions, $M := \mathscr{R}(x) \cup \mathscr{R}(y) \cup \{a\}$ is a compatible subset of $\mathscr{L}$. This implies that there is a Boolean sub-σ-algebra $\mathscr{B}$ of $\mathscr{L}$ such that $M \subset \mathscr{B}$. Moreover, there are an observable z, Borel functions f, g and a set $A \in \mathscr{B}(\mathscr{R})$ such that $x = f(z)$, $y = g(z)$, $a = z(A)$ (see [1]). We have

$$\int_a (x + y) \, \mathrm{d}m = \int_{\mathscr{R}} \lambda m((x + y)(\mathrm{d}\lambda) \wedge a) = \int_{\mathscr{R}} \lambda m(z(f + g^{-1}(\mathrm{d}\lambda)) \wedge z(A)) =$$

$$= \int_{\mathscr{R}} \lambda m_z((f + g)^{-1}(\mathrm{d}\lambda) \cap A) .$$

Put $v(E) = m_z(E \cap A)$, $E \in \mathscr{B}(\mathscr{R})$. Clearly, v is a measure on $\mathscr{B}(\mathscr{R})$ and we have

$$\int_{\mathscr{R}} \lambda m_z((f + g)^{-1}(\mathrm{d}\lambda) \cap A) = \int_{\mathscr{R}} \lambda v((f + g)^{-1}(\mathrm{d}\lambda)) = \int_{\mathscr{R}} (f + g)(t) \, v(\mathrm{d}t) =$$

$$= \int_{\mathscr{R}} f(t) \, v(\mathrm{d}t) + \int_{\mathscr{R}} g(t) \, v(\mathrm{d}t) =$$

$$= \int_{\mathscr{R}} \lambda v(f^{-1}(\mathrm{d}\lambda)) + \int_{\mathscr{R}} \lambda v(g^{-1}(\mathrm{d}\lambda)) =$$

$$= \int_{\mathscr{R}} \lambda m_z(f^{-1}(\mathrm{d}\lambda) \cap A) + \int_{\mathscr{R}} \lambda m_z(g^{-1}(\mathrm{d}\lambda) \cap A) =$$

$$= \int_{\mathscr{R}} \lambda m(x(\mathrm{d}\lambda) \wedge a) + \int_{\mathscr{R}} \lambda m(y(\mathrm{d}\lambda) \wedge a) =$$

$$= \int_a x \, \mathrm{d}m + \int_a y \, \mathrm{d}m .$$

Let x, m, $\mathscr{L}_0$, a be an observable, a state, a sublogic and a non-zero element of a logic $\mathscr{L}$, respectively, satifying the following conditions:

(i) $\mathscr{R}(x) \cup \mathscr{L}_0$ is p.c. (a);

(ii) $m(a) = 1$;

(iii) x is integrable with respect to m.

Condition (i) implies that $(\mathscr{R}(x) \cup \mathscr{L}_0) \leftrightarrow a$ and $(\mathscr{R}(x) \cup \mathscr{L}_0) \wedge a$ is a compatible subset of $\mathscr{L}_{[0,a]}$. Let us denote by $x \wedge a$ the map $x \wedge a : E \mapsto x(E) \wedge a$, $E \in \mathscr{B}(\mathscr{R})$. Then $x \wedge a$ is an observable on the logic $\mathscr{L}_{[0,a]}$. There is a Boolean sub-σ-algebra $\mathscr{B}$ such that $(\mathscr{R}(x) \cup \mathscr{L}_0) \wedge a \subset \mathscr{B} \subset \mathscr{L}_{[0,a]}$. By the Loomis theorem [1], there is a measurable space $(\Omega, \mathscr{S})$ and a σ-homorphism h of $\mathscr{S}$ onto $\mathscr{B}$. Moreover, there is an $\mathscr{S}$-measurable function $f: \Omega \to \mathscr{R}$ such that $x \wedge a = h \circ f^{-1}$. Put $\mathscr{L}_0 \wedge a = \mathscr{B}_0$; $\mathscr{B}_0$ is a Boolean sub-σ-algebra of $\mathscr{B}$, and let $\mathscr{S}_0 = \{E \in \mathscr{S};\ h(E) \in \mathscr{B}_0\}$. If we define $\mu(E) := m(h(E))$, $E \in \mathscr{S}$, then $(\Omega, \mathscr{S}, \mu)$ is a probability space by (ii). Furthermore,

$$\int_\Omega f(\omega)\, \mu(d\omega) = \int_{\mathscr{R}} t\mu(f^{-1}(dt)) = \int_{\mathscr{R}} tm(h \circ f^{-1}(dt)) = \int_{\mathscr{R}} tm((x \wedge a)(dt)) =$$

$$= \int_{\mathscr{R}} tm(x(dt) \wedge a) = \int_{\mathscr{R}} tm_x(dt) = m(x)$$

by (ii), and by (iii) f is integrable. Hence, there is a conditional expectation $g := E_\mu(f/\mathscr{S}_0)$ of f with respect to $\mathscr{S}_0$, and $h \circ g^{-1}$ is an observable on $\mathscr{L}_{[0,a]}$ with the range in $\mathscr{B}_0$. Let us define

(8) $$z(E) = h(g^{-1}(E)) \vee w(E) \wedge a^\perp, \quad E \in \mathscr{B}(\mathscr{R}),$$

where

(9) $$w(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E \end{cases}$$

It can be easily checked that z is an observable on $\mathscr{L}$. Moreover, $z \leftrightarrow a$ and $\mathscr{R}(z) \wedge a \subset \mathscr{L}_0 \wedge a$. Let $b \in \mathscr{L}_0$, then

(10) $$\int_b x\, dm = \int_{\mathscr{R}} \lambda m(x(d\lambda) \wedge b).$$

As $x \leftrightarrow a$ and $b \leftrightarrow a$, we have $x(E) \wedge b \leftrightarrow a$, so that $x(E) \wedge b = (x(E) \wedge b) \wedge a \vee (x(E) \wedge b) \wedge a^\perp$, and by (ii), $m(x(E) \wedge b) = m(x(E) \wedge b \wedge a) = m((x(E) \wedge a) \wedge (b \wedge a))$. As $x(E) \wedge a \leftrightarrow b \wedge a$, the map $E \mapsto m((x(E) \wedge a) \wedge (b \wedge a)) = m(x(E) \wedge b)$ is a probability measure on $\mathscr{B}(\mathscr{R})$, so that the integral in (10) exists.

Now

(10') $$\int_b z\, dm = \int_{\mathscr{R}} \lambda m(z(d\lambda) \wedge b).$$

Using the fact that $z(E) \wedge a \in \mathscr{L}_0 \wedge a$, so that $z(E) \wedge a \leftrightarrow b \wedge a$ for $b \in \mathscr{L}_0$, we show that the integral (10′) exists. Further, we have

$$\int_b x \, \mathrm{d}m = \int_{\mathscr{R}} \lambda m(x(\mathrm{d}\lambda) \wedge a \wedge b) = \int_{\mathscr{R}} \lambda m((x \wedge a)(\mathrm{d}\lambda) \wedge (b \wedge a)) =$$

$$= \int_{\mathscr{R}} \lambda m(h \circ f^{-1}(\mathrm{d}\lambda) \wedge h(A)) = \int_A f(\omega) \, \mu(\mathrm{d}\omega) \,,$$

where $A \in \mathscr{S}_0$ is such that $h(A) = b \wedge a$.
But

$$\int_A f(\omega) \, \mu(\mathrm{d}\omega) = \int_A g(\omega) \, \mu(\mathrm{d}\omega) = \int_{\mathscr{R}} \lambda \mu(g^{-1}(\mathrm{d}\lambda) \cap A) =$$

$$= \int_{\mathscr{R}} \lambda m((z \wedge a)(\mathrm{d}\lambda) \wedge (a \wedge b)) =$$

$$= \int_{\mathscr{R}} \lambda m(z(\mathrm{d}\lambda) \wedge a \wedge b) = \int_b z \, \mathrm{d}m \,.$$

Hence

(11) $$\int_b x \, \mathrm{d}m = \int_b z \, \mathrm{d}m$$

for any $b \in \mathscr{L}_0$.

This construction enables us to introduce the following definition.

Definition 1. Let $x, m, \mathscr{L}_0, a \neq 0$ be an observable, a state, a sublogic and an element of a logic $\mathscr{L}$, respectively, such that the following conditions are satisfied:

(i) the set $(\mathscr{R}(x) \cup \mathscr{L}_0)$ is p.c. (a);
(ii) $m(a) = 1$;
(iii) x is integrabe with respect to m.

The conditional expectation of the observable x in the state m with respect to $\mathscr{L}_0$ relativized by a, denoted by $E_m(x/\mathscr{L}_0, a)$, is any observable z on $\mathscr{L}$ such that

(a) $z \leftrightarrow a$;

(b) $\mathscr{R}(z) \wedge a \subset \mathscr{L}_0 \wedge a$;

(c) $\int_b x \, \mathrm{d}m = \int_b z \, \mathrm{d}m$ for any $b \in \mathscr{L}_0$.

The above construction shows that the conditional expectation exists. To discuss the uniqueness, we need some preliminaries.

For $a, b \in \mathcal{L}$ put $a \Delta b = (a^{\perp} \wedge b) \vee (a \wedge b^{\perp})$. For the observables x, y on $\mathcal{L}$ we shall write $x \approx y\ (m)$ if

$$m(x(E) \Delta y(E)) = 0 \quad \text{for any} \quad E \in \mathcal{B}(\mathcal{R}). \tag{12}$$

Lemma 2. *Let x, y, z be observables on $\mathcal{L}$ such than $(\mathcal{R}(x) \cup \mathcal{R}(y) \cup \mathcal{R}(z))$ is p.c. (a) for some $a \in \mathcal{L}$; and let $m(a) = 1$. Then $x \approx y\ (m)$, $y \approx z\ (m)$ implies $x \approx z\ (m)$.*

Proof. First we prove the lemma in the special case $a = 1$. If b, c, d are compatible elements of $\mathcal{L}$, then $b \Delta d \leqq (b \Delta c) \vee (c \Delta d)$, so that $m(x(E) \Delta y(E)) = m(y(E) \Delta z(E)) = 0$ implies $m(x(E) \Delta z(E)) = 0$ for all $E \in \mathcal{B}(\mathcal{R})$.

Let $0 < a < 1$. Then $x \wedge a$, $y \wedge a$, $z \wedge a$ are mutually compatible observables on $\mathcal{L}_{[0,a]}$, so that, by the above part of proof, $x \approx y(m)$, $x \approx z(m)$ implies

$$m((x \wedge a)(E) \Delta (z \wedge a)(E)) = 0 \quad \text{for all} \quad E \in \mathcal{B}(\mathcal{R}).$$

But

$$m(x(E) \Delta z(E)) = m(x(E) \Delta z(E)) \wedge a) = ((x \wedge a)(E) \Delta (z \wedge a)(E)) = 0.$$

Lemma 3. *(See [5].) Let g_1, g_2 be two $\mathcal{S}_0$-measurable functions on $(\Omega, \mathcal{S}_0, \mu)$, and let*

$$\int_B g_1 \, d\mu = \int_B g_2 \, d\mu \quad \textit{for any} \quad B \in \mathcal{S}_0.$$

Then $\mu(g_1^{-1}(E) \Delta g_2^{-1}(E)) = 0$ for any $E \in \mathcal{B}(\mathcal{R})$.

Proof. Put $B_1 = \{\omega \in \Omega; g_1(\omega) > g_2(\omega)\}$, $B_2 = \{\omega \in \Omega;\ g_1(\omega) < g_2(\omega)\}$. As $B_1 \cup B_2 \in \mathcal{S}_0$, $\int_{B_0}(g_1 - g_2) \, d\mu = 0$ for any $B_0 \subset B_1 \cup B_2$, $B_0 \in \mathcal{S}_0$, hence $\mu(B_1 \cup B_2) = \mu\{\omega; g_1(\omega) \neq g_2(\omega)\} = 0$. As $g_1^{-1}(E) \Delta g_2^{-1}(E) \subset B_1 \cup B_2$, we obtain the desired result.

Theorem 1. *Let z_1 and z_2 be two versions of conditional expectation $E_m(x/\mathcal{L}_0, a)$ by Definition 1. Then $z_1 \approx z_2\ (m)$.*

Proof. We have $z_1 \leftrightarrow a$, $z_2 \leftrightarrow a$ and $\mathcal{R}(z_1) \wedge a \subset \mathcal{L}_0 \wedge a$, $\mathcal{R}(z_2) \wedge a \subset \mathcal{L}_0 \wedge a$ As $\mathcal{L}_0 \wedge a = \mathcal{B}_0$ is a Boolean sub-σ-algebra of $\mathcal{L}_{[0,a]}$, $z_1 \wedge a \leftrightarrow z_2 \wedge a$. Let $(\Omega, \mathcal{S})$ and $h: \mathcal{S} \xrightarrow{\text{onto}} \mathcal{B}_0$ be given by the Loomis theorem, and let $g_1: \Omega \to \mathcal{R}$, $g_2: \Omega \to \mathcal{R}$ be $\mathcal{S}$-measurable functions such that $z_1 \wedge a = h \circ g_1^{-1}$ and $z_2 \wedge a = h \circ g_2^{-1}$. Then, as $z_1(E) \Delta z_2(E) \leftrightarrow a$ for any $E \in \mathcal{B}(\mathcal{R})$,

$$\begin{aligned} m(z_1 \wedge a)(E) \Delta (z_2 \wedge a)(E) &= m(h \circ g_1^{-1}(E) \Delta h \circ g_2^{-1}(E)) = \\ &= m(h \circ (g_1^{-1}(E) \Delta g_2^{-1}(E))) = \mu(g_1^{-1}(E) \Delta g_2^{-1}(E)) = 0 \end{aligned}$$

but

$$m((z_1 \wedge a)(E) \Delta (z_2 \wedge a)(E)) = m(z_1(E) \Delta z_2(E)).$$

The last but one equality follows by Lemma 3 if we apply it to the functions g_1 and g_2 on $(\Omega, \mathscr{S}, \mu)$ with $\mu := m \circ h$.

Corollary 1. *Let x, m and $\mathscr{L}_0$ be an observable, a state and a sublogic of a separable logic $\mathscr{L}$, respectively. If we put $a = \operatorname{com}(\mathscr{R}(x) \cup \mathscr{L}_0)$, and $m(a) = 1$, then the conditional expectation $E_m(x/\mathscr{L}_0, a)$ exists, provided x is integrable with respect to m.*

Proof follows by the fact that $(\mathscr{R}(x) \cup \mathscr{L}_0)$ is p.c. (a).

We note that, owing to the separability of $\mathscr{L}$, we can replace the abstract space $(\Omega, \mathscr{S}, \mu)$ by the space $(\mathscr{R}, \mathscr{B}(\mathscr{R}), m_v)$, where v is an observable on $\mathscr{L}_{[0,a]}$, when constructing conditional expectations.

We shall write

$$E_m(x/\mathscr{L}_0) := E_m(x/\mathscr{L}_0, \operatorname{com}(\mathscr{R}(x) \cup \mathscr{L}_0)) . \tag{13}$$

Lemma 4. *Let $y = E_m(x/\mathscr{L}_2, a)$ and let $\mathscr{R}(x) \wedge a \subset \mathscr{L}_1 \wedge a$, where $\mathscr{L}_1 \subset \mathscr{L}_2$. Then $\mathscr{R}(y) \wedge a \subset \mathscr{L}_1 \wedge a$.*

Proof. The lemma can be proved by repeating the construction of conditional expectations preceding Definition 1.

Theorem 4. *Let $\mathscr{L}_1 \subset \mathscr{L}_2$ be two sublogics of a separable logic $\mathscr{L}$. Let x be an observable and let $a_1 := \operatorname{com}(\mathscr{R}(x) \cup \mathscr{L}_1)$, $a_2 := \operatorname{com}(\mathscr{R}(x) \cup \mathscr{L}_2)$. Let m be a state on $\mathscr{L}$ such that x is integrable with respect to m and $m(a_1) = m(a_2) = 1$. Then*

$$E_m(E_m(x/\mathscr{L}_1, a_1))/\mathscr{L}_2, a_2) \approx E_m(x/\mathscr{L}_1, a_2) \approx$$
$$\approx E_m(E_m(x/\mathscr{L}_2, a_2)/\mathscr{L}_1, a_2)\,(m)$$

and

$$E_m(x/\mathscr{L}_1, a_2) \wedge a_2 \approx E_m(x/\mathscr{L}_1, a_1) \wedge a_2\,(m) .$$

Proof. Clearly, $a_1 \geqq a_2$. Let us denote $y_1 := E_m(x/\mathscr{L}_1, a_1)$, $y_2 := E_m(x/\mathscr{L}_2, a_2)$, $y := E_m(x/\mathscr{L}_1, a_2)$. We have $\mathscr{R}(x) \cup \mathscr{L}_1 \subset \mathscr{R}(x) \cup \mathscr{L}_2$, so that $(\mathscr{R}(x) \cup \mathscr{L}_1)$ is p.c. (a_2), and y exists. As $a_2 \leftrightarrow \mathscr{L}_2$ and $a_1 \leftrightarrow a_2$, we get that $a_2 \leftrightarrow \mathscr{R}(y_1)$. Indeed, $\mathscr{R}(y_1) \wedge \wedge\, a_1 \subset \mathscr{L}_1 \wedge a_1 \subset \mathscr{L}_2 \wedge a_1$, so that $\mathscr{R}(y_1) \wedge a_1 \leftrightarrow a_2$ and $\mathscr{R}(y_1) \wedge a_1^\perp \leqq \leqq a_1^\perp \leqq a_2^\perp$, which implies that $y_1 \leftrightarrow a_2$. Moreover, $(\mathscr{R}(y_1) \cup \mathscr{L}_2)$ is p.c. (a_2) as $\mathscr{R}(y_1) \wedge a_2 = \mathscr{R}(y_1) \wedge a_1 \wedge a_2 \subset \mathscr{L}_1 \wedge a_2 \subset \mathscr{L}_2 \wedge a_2$, and $\mathscr{L}_2$ is p.c. (a_2). Hence $y_1' := E_m(y_1/\mathscr{L}_2, a_2)$ exists. By Lemma 4, $\mathscr{R}(y_1') \wedge a_2 \subset \mathscr{L}_1 \wedge a_2$ and for $a \in \mathscr{L}_1$,

$$\int_a x \,\mathrm{d}m = \int_a y_1 \,\mathrm{d}m = \int_a y_1' \,\mathrm{d}m .$$

However, $\mathscr{R}(y) \wedge a_2 \subset \mathscr{L}_1 \wedge a_2$ as well, and for $a \in \mathscr{L}_1$,

$$\int_a x \, dm = \int_a y \, dm .$$

Hence we obtain that y_1' and y_1 are two versions of $E_m(x/\mathscr{L}_1, a_2)$, so that

$$E_m(E_m(x/\mathscr{L}_1, a_1)/\mathscr{L}_2, a_2) \approx E_m(x/\mathscr{L}_1, a_2)\,(m) .$$

Now let $y_2' = E_m(y_2/\mathscr{L}_1, a_2)$. It is defined because $\mathscr{R}(y_2) \cup \mathscr{L}_1 \subset \mathscr{R}(y_2) \cup \mathscr{L}_2$ is p.c. (a_2). Then $\mathscr{R}(y_2') \wedge a_2 \subset \mathscr{L}_1 \wedge a_2 \subset \mathscr{L}_2 \wedge a_2$, and for $a \in \mathscr{L}_1$ we have

$$\int_a y_2 \, dm = \int_a y_2' \, dm .$$

However, for $a \in \mathscr{L}_1$,

$$\int_a y_2 \, dm = \int_a x \, dm = \int_a y \, dm ,$$

so that $E_m(E_m(x/\mathscr{L}_2, a_2)/\mathscr{L}_1, a_2) = E_m(x/\mathscr{L}_1, a_2)\,(m)$.

Now $\mathscr{R}(y_1) \wedge a_2 = \mathscr{R}(y_1) \wedge a_1 \wedge a_2 \subset \mathscr{L}_1 \wedge a_1 \wedge a_2 = \mathscr{L}_1 \wedge a_2$, and

$$\int_a x \, dm = \int_a y_1 \, dm = \int_a y \, dm$$

for any $a \in \mathscr{L}_1$. Finally, $y, y_1 \leftrightarrow a_2$, $a \leftrightarrow a_2$, $m(a_2) = 1$ imply $m(y(E) \wedge a) = m(y(E) \wedge a_2 \wedge a)$, $m(y_1(E) \wedge a) = m(y_1(E) \wedge a_2 \wedge a)$, so that

$$\int_a y_1 \, dm = \int_{a \wedge a_2} y_1 \wedge a_2 \, dm = \int_a y \, dm = \int_{a \wedge a_2} y \wedge a_2 \, dm$$

for any $a \in \mathscr{L}_1$, which implies that

$$E_m(x/\mathscr{L}_1, a_2) \wedge a_2 \approx E_m(x/\mathscr{L}_1, a_1) \wedge a_2\,(m) .$$

3. CONDITIONAL EXPECTATIONS ON SUM LOGICS

Let $\mathscr{L}$ be a logic and M a set of states on $\mathscr{L}$. M is said to be *quite full* for $\mathscr{L}$ if

(14) $\{m \in M; m(a) = 1\} \subset \{m \in M; m(b) = 1\}$ implies $a \leqq b$, $a, b \in \mathscr{L}$;

and the set M is said to be *full* for $\mathscr{L}$ if

(15) $m(a) \leqq m(b)$ for all $m \in M$ implies $a \leqq b$, $a, b \in \mathscr{L}$.

Let M be a quite full set of states. For an observable x we put $D(x) = \{m \in M; m(x^2) < \infty\}$. We say that *the observables* $x_1, x_2, \ldots, x_n$ *are summable* if the set $D(x_1) \cap D(x_2) \cap \ldots \cap D(x_n)$ is full for $\mathscr{L}$. An observable z is called *the sum* of $x_1, x_2, \ldots, x_n$ if $D(z) \supset D(x_1) \cap D(x_2) \cap \ldots \cap D(x_n)$ and $m(z) = \sum_{i=1}^{n} m(x_i)$ for all $m \in D(x_1) \cap \ldots \cap D(x_n)$. We write $z = x_1 + x_2 + \ldots + x_n$. Let $\mathscr{L}$ be a logic and M a σ-convex quite full set of states. The couple $(\mathscr{L}, M)$ is called *a sum logic* if for any finite set $x_1, \ldots, x_n$ of summable observables there is a unique sum. For the details on sum logics see [6], [9]. If summable observables $x_1, x_2, \ldots, x_n$ are compatible, then their sum according to the above definition agrees with their sum defined by the functional calculus for compatible observables.

In the sequel we shall suppose that $(\mathscr{L}, M)$ is a sum logic and the following conditions are satisfied:

(α) $a \leftrightarrow x$, $a \leftrightarrow y$ implies $a \leftrightarrow x + y$ for any summable observables x, y;
(β) if $\mathscr{R}(x) \cup \mathscr{R}(y)$ is p.c. (a), where $a \in \mathscr{L}$, $a \neq 0$, then

$$x \wedge a + y \wedge a = (x + y) \wedge a .$$

For example, the logic $\mathscr{L}(\mathscr{H})$ of all closed subspaces of a Hibert space $\mathscr{H}$ satisfies (α) and (β).

Lemma 5. *Let x, y be summable observables and let $z = x + y$. Let $a \neq 0$ be such that $(\mathscr{R}(x) \cup \mathscr{R}(y))$ is p.c.* (a). *Then the set $(\mathscr{R}(x) \cup \mathscr{R}(y) \cup \mathscr{R}(z))$ is p.c.* (a).

Proof. By (α), $a \leftrightarrow \mathscr{R}(x) \cup \mathscr{R}(y)$ implies $a \leftrightarrow \mathscr{R}(x + y)$, and by (β), $z \wedge a = x \wedge \wedge a + y \wedge a$. But $x \wedge a \leftrightarrow y \wedge a$, so that there is a Boolean sub-σ-algebra $\mathscr{B}$ of $\mathscr{L}_{[0,a]}$ such that $(\mathscr{R}(x) \cup \mathscr{R}(y)) \wedge a \subset \mathscr{B}$. This implies that $\mathscr{R}(x + y) \wedge a \subset \mathscr{B}$, i.e. $(\mathscr{R}(x) \cup \mathscr{R}(y) \cup \mathscr{R}(z))$ is p.c. (a).

Theorem 3. *Let x, y be summable observables on a sum logic $(\mathscr{L}, M)$ and let $z = x + y$. Let $a \in \mathscr{L}$, $m \in M$ and $\mathscr{L}_0 \subset \mathscr{L}$ be such that $m \in D(x) \cap D(y)$, $(\mathscr{L}_0 \cup \cup \mathscr{R}(x) \cup \mathscr{R}(y))$ is p.c.* (a) *and $m(a) = 1$. Then*

(i) $\int_b E_m(x/\mathscr{L}_0, a)\, \mathrm{d}m + \int_b E_m(y/\mathscr{L}_0, a)\, \mathrm{d}m = \int_b E_m(z/\mathscr{L}_0, a)\, \mathrm{d}m$ *for any* $b \in \mathscr{L}_0$;

(ii) *if $\mathscr{L}$ is separable and $a = \operatorname{com}(\mathscr{R}(x) \cup \mathscr{R}(y) \cup \mathscr{L}_0)$, then*

$$\int_b E_m(x/\mathscr{L}_0, a_1)\, \mathrm{d}m + \int_b E_m(y/\mathscr{L}_0, a_2)\, \mathrm{d}m = \int_b E_m(z/\mathscr{L}_0, a_3)\, \mathrm{d}m ,$$

where $a_1 = \operatorname{com}(\mathscr{R}(x) \cup \mathscr{L}_0)$; $a_2 = \operatorname{com}(\mathscr{R}(y) \cup \mathscr{L}_0)$, $a_3 = \operatorname{com}(\mathscr{R}(z) \cup \mathscr{L}_0)$, *or with respect to* (13),

$$\int_b E_m(x/\mathscr{L}_0)\, \mathrm{d}m + \int_b E_m(y/\mathscr{L}_0)\, \mathrm{d}m = \int_b E_m(z/\mathscr{L}_0)\, \mathrm{d}m .$$

Proof. Similarly as in the proof of Lemma 5. we show that $(\mathscr{R}(x) \cup \mathscr{R}(y) \cup \cup \mathscr{R}(z) \cup \mathscr{L}_0)$ is p.c. (a), so that $E_m(z/\mathscr{L}_0, a)$ exists. Let us define $v_1 := E_m(x/\mathscr{L}_0, a)$, $v_2 := E_m(y/\mathscr{L}_0, a)$, $v_3 := E_m(z/\mathscr{L}_0, a)$; the inclusions $\mathscr{R}(v_i) \wedge a \subset \mathscr{L}_0 \wedge a$, $i = 1, 2, 3$, imply that $v_i \wedge a$, $i = 1, 2, 3$, are mutually compatible. Let $\mathscr{B} = (\mathscr{R}(x) \cup \cup \mathscr{R}(y) \cup \mathscr{L}_0) \wedge a$, $\mathscr{B}_0 = \mathscr{L}_0 \wedge a$, and let $(\Omega, \mathscr{S})$ and $h: \mathscr{S} \xrightarrow{\text{onto}} \mathscr{B}$ be defined by the Loomis theorem. Let $x = h \circ f^{-1}$, $y = h \circ g^{-1}$, where $g, f: \Omega \to \mathscr{R}$ are measurable functions. Let $h^{-1}(\mathscr{B}_0) = \mathscr{S}_0$. Then $v_1 \wedge a = h \circ E_\mu(f/\mathscr{S}_0)^{-1}$, $v_2 \wedge a = h \circ E_\mu(g/\mathscr{S}_0)^{-1}$, $v_3 \wedge a = h \circ E_\mu(f + g/\mathscr{S}_0)^{-1}$; μ is defined by $\mu(B) = m(h(B))$, $B \in \mathscr{S}$. Hence we obtain that

$$v_1 \wedge a + v_2 \wedge a \approx v_3 \wedge a\,(m)\,,$$

which implies that

$$\int_b v_1 \,\mathrm{d}m + \int_b v_2 \,\mathrm{d}m = \int_b v_3 \,\mathrm{d}m \quad \text{for any} \quad b \in \mathscr{L}_0\,.$$

This shows (i). We note that as follows from the construction preceding Definition 1, there are compatible versions of $v_1 = E_m(x/\mathscr{L}_0, a)$ and $v_2 = E_m(y/\mathscr{L}_0, a)$, so that $v_1 + v_2$ exists and we can write $v_1 + v_2 \approx v_3\,(m)$.

(ii) Clearly, $a \leqq a_i$ for $i = 1, 2, 3$. By Theorem 3, $E_m(x/\mathscr{L}_0, a) \wedge a \approx \approx E_m(x/\mathscr{L}_0, a_1) \wedge a\,(m)$, $E_m(y/\mathscr{L}_0, a_2) \wedge a \approx E_m(y/\mathscr{L}_0, a) \wedge a\,(m)$, $E_m(z/\mathscr{L}_0, a_3) \wedge \wedge a \approx E_m(z/\mathscr{L}_0, a) \wedge a\,(m)$, which together with (i) implies (ii).

4. MEASURABLE SUBSPACES

Let $(\mathscr{L}, M)$ be a sum logic with the properties (α) and (β). *A* sublogic $\mathscr{L}_0$ of $\mathscr{L}$ will be called a *sum sublogic* if $\mathscr{R}(x_1) \cup \mathscr{R}(x_2) \cup \ldots \cup \mathscr{R}(x_n) \subset \mathscr{L}_0$ implies $\mathscr{R}(x_1 + x_2 + \ldots + x_n) \subset \mathscr{L}_0$ for any sumable observables $x_1, x_2, \ldots, x_n$ on $\mathscr{L}$.

For $m \in M$, we denote by $X_m(\mathscr{L})$ the set of all square integrable observables, i.e.

$$X_m(\mathscr{L}) = \{x; m(x^2) < \infty\}\,. \tag{16}$$

By the definition of the sums, $D(x_1 + \ldots + x_n) \subset D(x_1) \cap D(x_2) \cap \ldots \cap D(x_n)$, so that $x_1 + x_2 + \ldots x_n \in X_m(\mathscr{L})$ provided $x_1, x_2, \ldots, x_n \in X_m(\mathscr{L})$ and they are sumable. We shall call $X_m(\mathscr{L})$ *a measurable space.*

Let $\mathscr{L}_0 \subset \mathscr{L}$ be a sum sublogic. We put

$$X_m(\mathscr{L}_0) = \{x \in X_m(\mathscr{L})\,;\;\; \mathscr{R}(x) \subset \mathscr{L}_0\} \tag{17}$$

and we shall call $X_m(\mathscr{L}_0)$ *a measurable subspace* of $X_m(\mathscr{L})$.

For sumable observables x, y we put

$$M(x, y) = \tfrac{1}{2}(x + y) + \tfrac{1}{2}|x - y|\,, \tag{18}$$

where by $|x|$ we denote the function $f(x)$ of x with $f(t) = |t|$, $t \in \mathscr{R}$. If $x \leftrightarrow y$ and $x = g(z)$, $y = h(z)$ for an observable z, then $M(x, y) = z \circ (\frac{1}{2}(h + g) + \frac{1}{2}|h - g|)^{-1} = z \circ (\max(h, g))^{-1}$. It can be easily seen that $M(x, y)$ exists for any sumable x, y. We recall that a sequence $\{x_n\}_{n=1}^{\infty}$ of observables converges to an observable x everywhere if

$$\limsup_{n \to \infty} (x_n - x)([-\varepsilon, \varepsilon]^c) = 0 \tag{19}$$

for any $\varepsilon > 0$ (see [7]).

Lemma 6. *Let $\{x_n\}_{n=1}^{\infty}$, x be mutually compatible. If $x_n \to x$ everywhere, then for any functional representation $x_n = f_n(z), x = f(z), f_n(t) \to f(t)\ \forall t \notin M$, $z(M) = 0$. On the other hand, if $f_n \to f$ everywhere for some representation, then $x_n \to x$ everywhere. If $x_n \to x$ and $x_n \to y$ everywhere and $\{x_n\}_{n=1}^{\infty}$, x, y are mutually compatible, then $x = y$.*

The proof of this lemma is straightforward.

For $a \in \mathscr{L}$, let x_a denote the simple observable such that $x_a\{1\} = a$, $x_a\{0\} = a^{\perp}$.

The following theorem gives a characterization of measurable subspaces analogous to the characterization of measurable subspaces in the probability theory (see [8], Theorem 3).

Theorem 4. *A system $Y \subset X_m(\mathscr{L})$ is a measurable subspace if and only if the following conditions hold:*

(i) *If $x_1, x_2, \ldots, x_n \in Y$ are summable, then*

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \in Y \quad \text{for any} \quad \alpha_1, \ldots, \alpha_n \in \mathscr{R}\,.$$

(ii) *The unit observable $x_1 \in Y$.*

(iii) *If $\{x_n\}_{n=1}^{\infty} \subset Y$ are mutually compatible, $\mathscr{R}(z)$ is generated by $\bigcup_{i=1}^{\infty} \mathscr{R}(x_i)$, and $x_n = f_n(z)$, $n = 1, 2, \ldots$ for measurable functions f_n such that $f_n \to f$ in $\mathscr{L}_2(\mathscr{R}, \mathscr{B}(\mathscr{R}), m_z)$ (i.e. $\int_{\mathscr{R}} (f_n - f)^2 (\lambda)\, m(z(\mathrm{d}\lambda)) \to 0$), then $f(z) \in Y$.*

(iv) *If $x, y \in Y$ are summable, then $M(x, y) \in Y$.*

Proof. I. Let Y be a measurable subspace, i.e. $Y = X_m(\mathscr{L}_0)$ for a sum sublogic $\mathscr{L}_0$ of $\mathscr{L}$. Then (i), (ii) and (iv) follow immediately. To prove (iii), observe that $\bigcup_{n=1}^{\infty} \mathscr{R}(x_n) \subset \mathscr{L}_0$ implies $\mathscr{R}(z) \subset \mathscr{L}_0$, hence $f(z) \in Y$.

II. Let Y satisfy the above conditions (i)–(iv). We denote by $\mathscr{L}_0$ the system of all elements $a \in \mathscr{L}$ such that $x_a \in Y$. If $x_a \in Y$, then $x_{a^{\perp}} = x_1 - x_a \in Y$, i.e. $a \in \mathscr{L}_0$ implies $a^{\perp} \in \mathscr{L}_0$. Let $a \perp b$, a, $b \in \mathscr{L}_0$. It can be easily seen that $x_a + x_b = x_{a \vee b}$, so that $a \vee b \in \mathscr{L}_0$. Now let $\{a_n\}_{n=1}^{\infty} \subset \mathscr{L}_0$ be a sequence of mutually orthogonal elements. We have $\bigvee_{i=1}^{k} a_i \in \mathscr{L}_0$, $k = 1, 2, \ldots$. As x_{a_n}, $n = 1, 2. \ldots$, are mutually compatible, their ranges generate a Boolean sub-σ-algebra $\mathscr{B} \subset \mathscr{L}$. Let $\mathscr{B} = \mathscr{R}(z)$, and let A_n, $n = 1, 2, \ldots$ be Borel sets such that $x_{a_n} = \chi_{A_n}(z)$, $n = 1, 2, \ldots$. The observables $x_{\bigvee_{i=1}^{k} a_i} = \chi_{\bigcup_{i=1}^{k} A_i}(z)$ are in Y. As $\chi_{\bigcup_{i=1}^{k} A_i} \mapsto \chi_{\bigcup_{i=1}^{\infty} A_i}$ pointwise and they are majorized by 1, they converge also in $\mathscr{L}_2(\mathscr{R}, \mathscr{B}(\mathscr{R}), m_z)$. This implies by (iii) that $\chi_{\bigcup_{i=1}^{\infty} A_i}(z) = x_{\bigvee_{i=1}^{\infty} a_i} \in$ $\in Y$, i.e. $\bigvee_{i=1}^{\infty} a_i \in \mathscr{L}_0$.

Let us consider an arbitrary $x \in Y$ and denote by a the element $a = x([0, \infty))$. The observable $M(x, x_0) = g(x)$, where $g(t) = \max(t, 0)$, is also an element of Y. Furthermore, for each $n = 1, 2, \ldots$ we define the functions

$$g_n(t) = n \,.\, \min\left(g(t), \frac{1}{n}\right).$$

From the conditions assumed it follows that with two summable observables x, y, Y contains also the observable $\mu(x, y) = 1/2(x + y) - 1/2|x - y|$, therefore $g_n(x) = n \,.\, \mu(g(x), (1/n)\, x_1) \in Y$. It is easily seen that $g_n(t) \to \chi_A(t)$ pointwise, where $A = [0, \infty)$. As $0 \leqq g_n(t) \leqq 1$ for all $t \in \mathscr{R}$, they converge also in $\mathscr{L}_2(\mathscr{R}, \mathscr{B}(\mathscr{R}), m_x)$, i.e. $m((g_n(x) - \chi_A(x))^2) \to 0$ $(n \to \infty)$.

Now let z be the observable such that $\mathscr{R}(z)$ is generated by $\bigcup_{n=1}^{\infty} \mathscr{R}(g_n(x))$, and let measurable functions f_n be such that $g_n(x) = f_n(z)$, $n = 1, 2, \ldots$, By Lemma 6, $\{f_n\}$ converges pointwise to some function f, (with a possible exception of a set B such that $z(B) = 0$) and $f(z) = \chi_A(x)$. This implies that $f_n \to f$ in $\mathscr{L}_2(\mathscr{R}, \mathscr{B}(\mathscr{R}), m_z)$, so that (iii) yields $f(z) \in Y$, i.e. $\chi_A(x) = x_a \in Y$. Hence $a \in \mathscr{L}_0$.

If c is an arbitrary real number, we denote $f_c(t) = t - c$, $t \in \mathscr{R}$. Then $f_c(x) = x - cx_1 \in Y$ provided $x \in Y$, and $b = x([c, \infty)) = x(f_c^{-1}[0, \infty)) = f_c(x)([0, \infty))$. From the previous part of the proof, we conclude that $b \in \mathscr{L}_0$.

By [9], on the sum logic $(x_a + x_b)\{2\} = a \wedge b$. Let $a, b \in \mathscr{L}_0$. Then $a \wedge b = (x_a + x_b)(\{2\}) = (x_a + x_b)([2, \infty)) \in \mathscr{L}_0$. Summing up, we have proved that $\mathscr{L}_0$ is a sublogic of $\mathscr{L}$. The fact that $x([c, \infty)) \in \mathscr{L}_0$ for all $c \in \mathscr{R}$ provided $x \in Y$ implies that $\mathscr{R}(x) \subset \mathscr{L}_0$. Hence $Y \subset X_m(\mathscr{L}_0)$. The theorem will be proved if we show that $Y = X_m(\mathscr{L}_0)$. For each simple observable $x \in X_m(\mathscr{L}_0)$ we have $x \in Y$. Any observable x can be written as $x = f(x)$, where $f(t) = t$ or 0 if $t \in \sigma(x)$ or $t \notin \sigma(x)$ and $\sigma(x)$ is the spectrum of x. Using the fact that a characteristic function $\chi_A(x)$, $A \in \mathscr{B}(\mathscr{R})$, of the observable x is a simple observable, we show step by step that simple functions, non-negative functions and eventually the functions of $\mathscr{L}_2(\mathscr{R}, \mathscr{B}(\mathscr{R}), m_x)$ of the observable x are elements of Y provided $x \in X_m(\mathscr{L}_0)$. Then also $x = f(x) \in Y$.

In what follows we shall need some lemmas.

Lemma 7. *Let x, y be such observables that $\mathscr{R}(x) \cup \mathscr{R}(y)$ is p.c. (a) and $m(a) = 1$. Then $x \approx y\,(m)$ iff $m((x \wedge a - y \wedge a)^2) = 0$. If, in addition, x and y are summable, then $x \approx y\,(m)$ iff $m((x - y)^2) = 0$.*

Proof. Let $x \approx y\ (m)$. This means that $m(x(E)\,\Delta y(E)) = 0$ for all $E \in \mathscr{B}(\mathscr{R})$. Then $0 = m(x(E)\,\Delta y(E)) = m((x(E)\,\Delta y(E)) \wedge a) = m(h(f^{-1}(E)\,\Delta g^{-1}(E)))$ for any $E \in \in \mathscr{B}(\mathscr{R})$, where $(\Omega, \mathscr{S})$, $h: \mathscr{S} \to \mathscr{B}$ exist by the Loomis theorem, $\mathscr{B}$ is a Boolean sub-σ-algebra of $\mathscr{L}_{[0,a]}$ which contains the ranges of $x \wedge a$ and $y \wedge a$, and $x \wedge a = = h \circ f^{-1}$, $y \wedge a = h \circ g^{-1}, f, g: \Omega \to \mathscr{R}$ are measurable functions. But $m(h(f^{-1}(E)\,\Delta g^{-1}(E))) = 0$ for all $E \in \mathscr{B}(\mathscr{R})$ implies that $m(h\{\omega; f(\omega) \neq g(\omega)\}) = 0$, and this means that

$$m((x \wedge a - y \wedge a)^2) = \int_{\mathscr{R}} \lambda^2 m((x \wedge a - y \wedge a)(\mathrm{d}\lambda)) =$$

$$= \int_{\mathscr{R}} \lambda^2 m(h(f - g)^{-1}(\mathrm{d}\lambda)) = \int_{\Omega} (f - g)^2\, m(h(\mathrm{d}\omega)) = 0\,.$$

The converse statement can be proved similarly. If x and y are summable, then

$$m((x - y)^2) = \int_{\mathscr{R}} \lambda^2 m((x - y)(\mathrm{d}\lambda)) = \int_{\mathscr{R}} \lambda^2 m((x - y)(\mathrm{d}\lambda) \wedge a) =$$

$$= \int_{\mathscr{R}} \lambda^2 m((x \wedge a - y \wedge a)(\mathrm{d}\lambda)) = m(x \wedge a - y \wedge a)^2\,.$$

Lemma 8. *Let $\mathscr{L}_0$ be a sum sublogic of $\mathscr{L}$, which is p.c.* (a), *and $m(a) = 1$. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be two n-tuples of summable observables in $X_m(\mathscr{L}_0)$. If $x_i \approx y_i\ (m)$, $i = 1, \ldots, n$, then $x_1 + x_2 + \ldots + x_n \approx y_1 + y_2 + \ldots + y_n\ (m)$.*

Proof. As $x_i \approx y_i\,(m)$, $i = 1, \ldots, n$, we have by Lemma 7 that $m((x_i \wedge a - y_i \wedge \wedge a)^2) = 0$. The statement can be proved by using the functional representation for $x_i \wedge a$ and $y_i \wedge a$, $i = 1, \ldots, n$, as in Lemma 7.

Lemma 9. *Let $x \approx y\ (m)$ and let $f: \mathscr{R} \to \mathscr{R}$ be a Borel function. Then $f(x) \approx \approx f(y)\,(m)$.*

Proof. From $x \approx y\,(m)$ we have $m(x(E)\,\Delta y(E)) = 0$ for any $E \in \mathscr{B}(\mathscr{R})$. This implies that $m(f(x)\,(E)\,\Delta f(y)\,(E)) = m(x(f^{-1}(E))\,\Delta y(f^{-1}(E))) = 0$ for any $E \in \mathscr{B}(\mathscr{R})$, i.e. $f(x) \approx f(y)\,(m)$.

Let $\mathscr{L}_0$ be a sum sublogic of a sum logic $(\mathscr{L}, M)$, which is p.c. (a) for some $a \in \mathscr{L}_0$, and let $m \in M$ be such that $m(a) = 1$. (We may put $a = \mathrm{com}\,(\mathscr{L}_0)$ if it exists). Lemma 2 implies that the relation $x \approx y\,(m)$ is an equivalence relation on $X_m(\mathscr{L}_0)$. We shall denote by $\tilde{X}_m(\mathscr{L}_0)$ the set of all equivalence classes, i.e.

$$\tilde{X}_m(\mathscr{L}_0) = \{\tilde{x};\, x \in X_m(\mathscr{L}_0)\}\,. \tag{20}$$

We define the sum on $\tilde{X}_m(\mathscr{L}_0)$ as follows: we shall say that elements $\tilde{x}_1, \ldots, \tilde{x}_n \in \tilde{X}_m(\mathscr{L}_0)$ are summable if there are summable representants $x_1, \ldots, x_n$ of $\tilde{x}_1, \ldots, \tilde{x}_n$, respectively; and we put

$$\tilde{x}_1 + \tilde{x}_2 + \ldots + \tilde{x}_n = \tilde{x}\,, \tag{21}$$

where $x = x_1 + \ldots + x_n$. Lemma 8 implies that the sums are well defined.

For $\tilde{x} \in \tilde{X}_m(\mathscr{L}_0)$ we put $m(\tilde{x}(E)) = m(x(E))$, $E \in \mathscr{B}(\mathscr{R})$, where x is any representant of $\tilde{x}$.

Lemma 10. *The map $E \to m(\tilde{x}(E))$, $E \in \mathscr{B}(\mathscr{R})$, does not depend on the choice of the representant $x \in \tilde{x}$.*

Proof. Let $x, y \in \tilde{x}$. Then for any $E \in \mathscr{B}(\mathscr{R})$, $0 = m(x(E)\,\Delta y(E)) = m((x(E) \wedge a)\,\Delta(y(E) \wedge a))$. As $x(E) \wedge a \leftrightarrow y(E) \wedge a$, this implies that $m(x(E) \wedge a) = m(y(E) \wedge a)$ i.e. $m(x(E)) = m(y(E))$.

The map $E \to m(x(E))$ is a probability measure on $\mathscr{B}(\mathscr{R})$. It can be treated as the probability distribution of the element $\tilde{x} \in \tilde{X}_m(\mathscr{L}_0)$. For any Borel function f we have $m(\tilde{f}(x)\,(E)) = m(f(x)\,(E)) = m(x(f^{-1}(E))) = m(\tilde{x}(f^{-1}(E)))$, where $f(x) \in \tilde{f}(x)$ is any representant. By Lemma 9, we may put

$$\tilde{f}(x) = f(\tilde{x})\,. \tag{22}$$

The following theorem gives the characterization of conditional expectations as transformations of measurable subspaces (see [8], Theorem 6).

Theorem 5. *Let $(\mathscr{L}, M)$ be a sum logic. Let Q be a sum sublogic of $\mathscr{L}$,* let $a \in Q$ *be such that Q is p.c. (a) and $m \in M$ such that $m(a) = 1$. A transformation T of $\tilde{X}_m(Q)$ into itself is a relative conditional expectation (with respect to a sum sublogic $\mathscr{L}_0 \subset Q$ such that $a \in \mathscr{L}_0$) if and only if it satisfies the following conditions:*

(i) *T is idempotent (i.e. $T^2 \equiv T$);*

(ii) *$T\tilde{x}_1 = \tilde{x}_1$ and $T\tilde{x}_a = \tilde{x}_a$;*

(iii) *if $\tilde{x}_1, \ldots, \tilde{x}_n \in \tilde{X}_m(Q)$ are summable, then*

$$T(\alpha_1\tilde{x}_1 + \ldots + \alpha_n\tilde{x}_n) = \alpha_1 T\tilde{x}_1 + \ldots + \alpha_n T\tilde{x}_n \quad \textit{for any} \quad \alpha_1, \ldots, \alpha_n \in \mathscr{R}\,;$$

(iv) *if $\tilde{x}, \tilde{y} \in \tilde{X}_m(Q)$ are summable, then $T(M(T\tilde{x}, T\tilde{y})) = M(T\tilde{x}, T\tilde{y})$;*

(v) *if $\{\tilde{x}_n\}_{n=1}^{\infty}, \tilde{x} \in \tilde{X}_m(Q)$ and $m((\tilde{x}_n - \tilde{x})^2) \to 0$ ($\tilde{x}_n, \tilde{x}$* are *supposed to be summable), then $m((T\tilde{x}_n - T\tilde{x})^2) \to 0$.*

Proof. I. Let us consider the conditional expectation $E_m(x/\mathscr{L}_0, a)$ for $x \in X_m(Q)$, where $\mathscr{L}_0 \subset Q$ is a sum sublogic such that $a \in \mathscr{L}_0$. It is easily seen that there is a version of $E_m(x/\mathscr{L}_0, a)$ with the range in $\mathscr{L}_0$, thus we can suppose that $E_m(x/\mathscr{L}_0, a) \in X_m(\mathscr{L}_0)$. (Using the functional representation we can show that the conditional expectation of a square integrable observable is square integrable.) We shall show that $x_1 \approx x_2\,(m)$ implies $E_m(x_1/\mathscr{L}_0, a) \approx E_m(x_2/\mathscr{L}_0, a)\,(m)$. Let $y_1 = E_m(x_1/\mathscr{L}_0, a)$,

$y_2 = E_m(x_2/\mathscr{L}_0, a)$. $x_1 \approx x_2\,(m)$ implies that $\int_b x_1\,dm = \int_b x_2\,dm$ for any $b \in \mathscr{L}_0$. Hence we obtain that $\int_b y_1\,dm = \int_b y_2\,dm$ for any $b \in \mathscr{L}_0$, i.e. $y_1 \approx y_2\,(m)$. If we put $T\tilde{x} = E_m(x/\mathscr{L}_0, a)$, then T is the map from $\tilde{X}_m(Q)$ into $\tilde{X}_m(\mathscr{L}_0)$. Now we shall prove that T has the properties (i)$-$(v):

(i) If $x \in X_m(\mathscr{L}_0)$, then clearly $E_m(x/\mathscr{L}_0, a) \approx x(m)$. This implies that the map T is onto and it is idempotent.

(ii) This follows from the fact that $x_1, x_a \in X_m(\mathscr{L}_0)$ and from (i).

(iii) It can be easily checked that $E_m(\alpha x/\mathscr{L}_0, a) = \alpha E_m(x/\mathscr{L}_0, a)$, $\alpha \in \mathscr{R}$. If $x_1, \ldots, x_n$ are summable elements of $X_m(Q)$, then there are summable versions of $E_m \cdot (x_i/\mathscr{L}_0, a)$, $i = 1, \ldots, n$. (iii) follows by Theorem 3 (the generalization of this theorem to any finite set of observables is straightforward) and Lemma 8.

(iv) Let x, y be summable observables from $X_m(Q)$. As $E_m(x/\mathscr{L}_0, a)$ and $E_m(y/\mathscr{L}_0, a)$ have the ranges in $\mathscr{L}_0$, $M(E_m(x/\mathscr{L}_0, a), E_m(y/\mathscr{L}_0, a))$ also has the range in $\mathscr{L}_0$. This implies that $E_m(M(E_m(x/\mathscr{L}_0, a), E_m(y/\mathscr{L}_0, a))/\mathscr{L}_0, a)) \approx M(E_m(x/\mathscr{L}_0, a), E_m(y/\mathscr{L}_0, a))\,(m)$. Lemmas 8 and 9 imply that for $x_1 \approx x_2\,(m)$ and $y_1 \approx y_2\,(m)$ we have $M(x_1, y_1) \approx M(x_2, y_2)\,(m)$. Hence $T(M(T\tilde{x}, T\tilde{y})) = M(T\tilde{x}, T\tilde{y})$.

(v) To prove (v) we shall use the functional representation. The set $\mathscr{B} = Q \wedge a$ is a Boolean sub-σ-algebra of $\mathscr{L}_{[0,a]}$. Let $\mathscr{B}_0 = \mathscr{L}_0 \wedge a \subset \mathscr{B}$. By the Loomis theorem, there is a measurable space $(\Omega, \mathscr{S})$ and a σ-homomorphism $h\colon \mathscr{S} \to \mathscr{B}$. Furthermore, for any observable $x \in X_m(Q)$ there is a measurable function $f_x\colon \Omega \to \mathscr{R}$ such that $x \wedge a = h \circ f_x^{-1}$. Let $\mathscr{S}_0 = h^{-1}(\mathscr{B}_0)$ and $\mu = m \circ h$. Let $\{x_n\}, x \subset X_m(Q)$ and let x_n, x be summable for $n = 1, \ldots$. Then

$$m((x_n - x)^2) = \int_{\mathscr{R}} \lambda^2 m((x_n - x)(d\lambda)) = \int_{\Omega} (f_{x_n} - f_x)^2\,d\mu \to 0\,(n \to \infty)$$

implies that

$$\int (E_\mu(f_{x_n}/\mathscr{S}_0) - E_\mu(f/\mathscr{S}_0))^2\,d\mu = m((E_m(x_n/\mathscr{L}_0, a) - E_m(x/\mathscr{L}_0 a))^2) \to 0$$

$(n \to \infty)$. Lemma 10 then shows that $m((T\tilde{x}_n - T\tilde{x})^2) \to 0\,(n \to \infty)$.

II. Let T be a transformation of $\tilde{X}_m(Q)$ into itself with the properties $(i)$$-$(v). Let us put $Y = \{x \in X_m(Q)\colon T\tilde{x} = \tilde{x}\}$.

We shall show that Y is a measurable subspace. We have to show the properties (i)$-$(iv) from Theorem 4.

If $x_1, \ldots, x_n \in Y$ and they are summable, then by (iii), $T(\alpha_1\tilde{x}_1 + \ldots + \alpha_n\tilde{x}_n) = \alpha_1 T\tilde{x}_1 + \ldots + \alpha_n T\tilde{x}_n = \alpha_1\tilde{x}_1 + \ldots + \alpha_n\tilde{x}_n$, so that $\alpha_1 x_1 + \ldots + \alpha_n x_n \in Y$ for any $\alpha_1, \ldots, \alpha_n \in \mathscr{R}$. (ii) implies that $x_1 \in Y$.

If $\{x_n\}_{n=1}^{\infty} \subset Y$ and $m((x_n - y)^2) \to 0$ for some $y \in X_m(Q)$ then $m((\tilde{x}_n - \tilde{y})^2) \to 0$ by Lemma 10. This implies by (v) that $m((T\tilde{x}_n - T\tilde{y})^2) \to 0$. It can be easily checked by that $\tilde{y} = T\tilde{y}$, i.e. $y \in Y$. If $x, y \in Y$ are summable, (iv) implies that $M(x, y) \in Y$. This shows that Y is a measurable subspace, i.e. there is a sum sublogic $\mathscr{L}_0 \subset Q$ such that $Y = X_m(\mathscr{L}_0)$. $x_a \in Y$ implies that $a \in \mathscr{L}_0$.

To show that T is the conditional expectation, we use the functional representation introduced in part I of this proof. For $\tilde{x} \in \tilde{X}_m(Q)$ we put $Tf_{\tilde{x}} = f_{T\tilde{x}}$. Thus we get a transformation of $\mathscr{L}_2(\Omega, \mathscr{S}, \mu)$ into itself. It can be easily checked that this transformation has the properties (0), (1), (2) from Theorem 6 in [8]; hence $f_{T\tilde{x}}$ is the conditional expectation of $f_{\tilde{x}}$. This implies that T is the conditional expectation.

5. CONCLUDING REMARKS

1. In [11] and [12], another approach to the characterization of conditional expectations on probability spaces is given. In these papers, the operation of multiplying two functions is used. For the observables on the quantum logic no product is defined unless the observables are compatible. On the sum logics, *the Segal* [9] *product* could be used, i.e. the operation defined by

$$x \circ y = \tfrac{1}{4}((x + y)^2 - (x - y)^2)\,, \tag{23}$$

where x, y are observables on $\mathscr{L}$. But this operation is well defined only for bounded observables and, moreover, it may be non-distributive with respect to the addition of observables. For these reasons, the approach in [8] is much more suitable for our purposes.

2. The observables x and y are said to *have a joint distribution of type* 1 in a state m if there is a measure on the Borel subsets $\mathscr{B}(\mathscr{R}^2)$ of $\mathscr{R}^2$ such that

$$\mu(E \times F) = m(x(E) \wedge y(F)) \tag{24}$$

for any rectangle set $E \times F \in \mathscr{B}(\mathscr{R}^2)$ (see [13], [14], [15]). The following theorem gives a relation between joint distributions and conditional expectations.

Theorem 6. *Let x and y be observables on a separable logic $\mathscr{L}$. Let a state m on $\mathscr{L}$ be such that x is integrable with respect to m. Then $E_m(x/\mathscr{R}(y))$ exists iff x and y have a joint distribution in m.*

Proof. By [2] and [15], a joint distribution of x and y in the state m exists iff $m(\text{com}\,(\mathscr{R}(x) \cup \mathscr{R}(y)) = 1$. This implies the statement of our theorem.

3. If $\mathscr{B}_0$ is a discrete Boolean sub-σ-algebra of $\mathscr{L}$ generated by mutually orthogonal elements $\{b_i\}_{i=1}^{\infty}$, then the conditional expectation of an observable x in a state m, if it exists, is of the form

$$E_m(x/\mathscr{B}_0) = \sum_{\{i;m(b_i)\neq 0\}} \frac{1}{m(b_i)} \left(\int_{b_i} x \,\mathrm{d}m\right) x_{b_i}\,, \tag{25}$$

as can be easily checked. In the Hilbert space formulation this gives, for a bounded s.a. operator A and mutually orthogonal projectors $B_1, B_2, \ldots$ generating $\mathscr{B}_0$,

$$E_m(A/\mathscr{B}_0) = \sum_{\{i;m(B_i)\neq 0\}} \frac{1}{m(B_i)}\, m(B_iAB_i)\, B_i\,. \tag{26}$$

This agrees, in our special case, with the conditional expectation considered in [16].

4. There are several definitions of conditional expectations in the non-commutative probability theory, see e.g. [16]–[21]. It is resonable to expect that, if y is a conditional expectation of an observable x with respect to a sublogic $\mathscr{L}_0$ of $\mathscr{L}$ in a state m by any definition, which for the compatible case agrees with the usual form of conditional expectations in the probability theory, then $y \approx E_m(x/\mathscr{L}_0)(m)$ provided $m(\mathrm{com}\,(\mathscr{R}(x) \cup \mathscr{L}_0) = 1$. In this sense, our definition of conditional expectations on a logic $\mathscr{L}$ has a "general character".

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Súhrn

RELATIVE CONDITIONAL EXPECTATIONS ON A LOGIC

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V práci bol zavedený pojem relativizovanej podmienenej strednej hodnoty na logike vzhľadom k podlogike a prvku a z logiky, pre ktorý platí $m(a) = 1$, kde m je stav na logike. Obor hodnôt pozorovatelnej a daná podlogika sú čiastočne kompatibilné vzhľadom k a. Tento pojem podmienenej strednej hodnoty je analogický k relativizovanej podmienenej strednej hodnote integrovatelnej funkcie na pravdepodobnostnom priestore. Bolo ukázané, že relativizovaná podmienená stredná hodnota na logike spĺňa všetky základné vlastnosti podmienenej strednej hodnoty na pravdepodobnostnom priestore.

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