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Ivo Vrkoč

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ANALYSIS OF INTEGRAL EQUATIONS ATTACHED TO SKIN EFFECT

Ivo Vrkoč

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The paper is a continuation of the paper [1]. In [1] the detailed description of the problem is given, the mathematical model of the skin effect is established and also a numerical method is discussed.

Nevertheless, for the convenience of the reader we formulate the problem again and sketch the derivation of the mathematical model in the introduction of the present paper.

INTRODUCTION

We investigate the skin effect of the system of n linear parallel conductors whose sections $S_1, \ldots, S_n$ are of arbitrary shapes. We assume that the currents passing through the conductors are sinusoidal functions and have the same radian frequency ω. Let γ be the conductivity of the conductors and μ_0 the permeability of both the conductors and the surrounding medium. The value γ is supposed to be constant, $\mu_0 = 4\pi 10^{-7}$ H/m. We assume that the conductors are under an effect of a planar magnetic field which is sinusoidal with respect to time t with the same radian frequency ω. We assume that the vector of the magnetic induction is parallel to the sections of the conductors.

Let x, y, z be orthogonal coordinates axes, z being parallel to the conductors, let $J(x, y)$ be the phasor of the density of the currents, $I_1, \ldots, I_n$ the phasors of the currents passing through $S_1, \ldots, S_n$, $A(x, y)$ the phasor of the vector potential of the total magnetic field.

Evidently only the coordinate $A^{(z)}(x, y)$ can be nonzero and can be expressed as a sum $A^{(z)} = A_0^{(z)} + A_{\text{out}}^{(z)}$ where A_0 is the phasor of the vector potential of the magnetic field which is induced by the currents while A_{out} is the phasor of the vector potential of the given magnetic field. Obviously

$$\Delta A_0^{(z)}(x, y) = \partial^2 A_0^{(z)}/\partial x^2 + \partial^2 A_0^{(z)}/\partial y^2 = -\mu_0\, J_0^{(z)}(x, y)$$

and hence

$$A_0^{(z)}(x, y) = \mu_0/(2\pi) \int_S J^{(z)}(\xi, \eta) \ln \frac{1}{r(x, y; \xi, \eta)} \, d\xi \, d\eta + K_i$$

$$\text{for} \quad [x, y] \in S_i, \quad i = 1, \ldots, n$$

where $r(x, y; \xi, \eta)$ is the distance between the points $[x, y]$, $[\xi, \eta]$, $S = \cup S_i$, K_i are complex constants.

Using the second Maxwell equation we obtain

$$E^{(z)} = -j\omega A^{(z)} + \partial f/\partial z$$

where f is the phasor of the scalar potential. Since $\Delta f = 0$, $\partial f/\partial x = \partial f/\partial y = 0$ we conclude $f = kz$ where k is a constant. The phasor J can be expressed as $J^{(z)} = \gamma E^{(z)}$. The above mentioned equations yield

$$J^{(z)}(x, y) = \gamma E^{(z)} = -j\gamma\omega \, A^{(z)}(x, y) + \gamma \, \partial f/\partial z =$$

$$= j\lambda \int_S J^{(z)}(\xi, \eta) \ln r(x, y; \xi, \eta) \, d\xi \, d\eta + c_i - j\gamma\omega \, A^{(z)}_{\text{out}}(x, y)$$

$$\text{for} \quad [x, y] \in S_i, \quad i = 1, \ldots, n$$

where

$$\lambda = \frac{\gamma\omega\mu_0}{2\pi}, \quad c_i = \gamma k - j\gamma\omega K_i .$$

Finally, the phasors I_k fulfil $I_k = \int_{S_k} J^{(z)}(x, y) \, dx \, dy$, $k = 1, \ldots, n$.

In the case of one conductor we rewritte the derived equations in the more general form (1), (2) while in the case of several conductors we use the form (6), (7).

In both cases the existence and unicity of the solution is proved. In the last section a numerical method is suggested. The numerical method is a modification of the usual approach which is described in [3], [4], [5]. The convergence of this method is proved under assumptions slightly different from those used in the first two sections. Nevertheless, in both cases the assumptions are sufficiently general for applications. Moreover, if we assume that the coefficients q, b_i fulfil the conditions of Theorem 2 (Theorem 1) and that the given function h has a modulus of continuity then the assumptions of Theorems 3 and 4 are satisfied.

1. THE PROBLEM WITH ONE INTEGRAL TERM

Denote by R the n-dimensional Euclidean space, $n \geqq 1$. Let S be a bounded and Lebesgue measurable set in R with a positive measure, $h(x)$ a (complex valued) function defined on S and fulfilling $\int_S |h(x)|^2 \, dx < \infty$, q a real number, I a complex constant and $V(r)$ a real function defined for $r > 0$ and such that $\int_0^A V^2(r) \, r^{n-1} \, dr < \infty$ for every $A \geqq 0$.

Setting of the problem: *we should find a (complex valued) function f defined on S, and a (complex) constant c fulfilling*

$$(1) \qquad f(x) - \mathrm{j}q \int_S f(y)\, V(|y - x|)\, \mathrm{d}y - c = h(x) \quad \text{for} \quad x \in S\,,$$

$$(2) \qquad \int_S f(x)\, \mathrm{d}x = I\,.$$

where j is the imaginary unit.

If a is a complex number then $\bar{a}$ is its complex conjugate. Re a and Im a are the real and imaginary parts of a, respectively.

Theorem 1. *Let the assumptions formulated above be fulfilled. Then the problem* (1), (2) *has a unique solution* $f(x)$, c *given by*

$$f(x) = f^{(h)}(x) + \left(I - \int_S f^{(h)}(y)\, \mathrm{d}y\right) f^{(1)}(x) \Big/ \int_S f^{(1)}(y)\, \mathrm{d}y\,,$$

$$c = \left(I - \int_S f^{(h)}(y)\, \mathrm{d}y\right) \Big/ \int_S f^{(1)}(y)\, \mathrm{d}y\,,$$

where $f^{(1)}$ *is a solution of*

$$(3) \qquad f(x) - \mathrm{j}q \int_S f(y)\, V(|y - x|)\, \mathrm{d}y = 1 \quad \textit{for} \quad x \in S$$

and $f^{(h)}$ *is a solution of*

$$(4) \qquad f(x) - \mathrm{j}q \int_S f(y)\, V(|y - x|)\, \mathrm{d}y = h(x) \quad \textit{for} \quad x \in S\,.$$

Remark 1. We shall show that the solutions $f^{(1)}, f^{(h)}$ exist and are unique.

As usual we denote by $L_2(S)$ the set of all complex valued functions fulfilling $\int_S |f(x)|^2\, \mathrm{d}x < \infty$. The set $L_2(S)$ is a Hilbert space with the scalar product

$$(f, g) = \int_S f(x)\, \bar{g}(x)\, \mathrm{d}x\,.$$

Define an operator $T: L_2(S) \to L_2(S)$ by

$$Tf(x) = \int_S f(y)\, V(|y - x|)\, \mathrm{d}y\,.$$

The operator T is symmetric, i.e. $(Tf, g) = (f, Tg)$.

Remark 2. The number (Tf, f) is real for all $f \in L_2(S)$ and the eigenvalues of T are real. (See [2].)

The proof of Theorem 1 is divided into two lemmas.

Lemma 1. *If the assumptions of Theorem* 1 *are fulfilled, then equation* (4) *has a unique solution for every* $h \in L_2(S)$.

Proof. The homogeneous adjoint equation to (4) is

$$f + \mathrm{j}qTf = 0\,. \tag{5}$$

Assume that this equation has a nontrivial solution f. Thus f is an eigenvector of T with an eigenvalue j/q (the case $q = 0$ is trivial). By Remark 2 the eigenvalue j/q has to be real. We have proved the statement that equation (5) has only trivial solution $f = 0$. Since the assumption $\int_0^A V^2(r)\, r^{n-1}\, \mathrm{d}r < \infty$ for every $A \geqq 0$ yields that the operator T is compact the assumptions of Fredholm's Theorem [2] are fulfilled and Lemma 1 is a consequence of the theorem and of the statement.

Lemma 2. *We have* $\operatorname{Re} \int_S f^{(1)}(x)\, \mathrm{d}x = (f^{(1)}, f^{(1)}) > 0$.

Proof. Multiply equation (3) by $\bar{f}(x)$ and integrate over S. We obtain $(f^{(1)}, f^{(1)}) - \mathrm{j}q(Tf^{(1)}, f^{(1)}) = \int_S \bar{f}^{(1)}(x)\, \mathrm{d}x$. Since the second term is purely imaginary (Remark 2), the real part of this equation yields the required relation.

Proof of Theorem 1. We know that $f^{(1)}$, $f^{(h)}$ exist and are unique. For given c the solution $f(x)$ of (1) can be written in the form $f = f^{(h)} + cf^{(1)}$. By Lemma 2, $\int_S f^{(1)}(x)\, \mathrm{d}x \neq 0$ and we can calculate c so that (2) is fulfilled.

2. THE PROBLEM WITH A FINITE NUMBER OF INTEGRAL TERMS

Let a finite number of bounded, disjoint and Lebesgue measurable sets S_i, $i = 1, \ldots, k$, be given. Assume $m(S_i) > 0$ for $i = 1, \ldots, k$ where m is the Lebesgue measure in R. Denote $S = \bigcup_{i=1}^{k} S_i$. Let a (complex valued) function $h(x)$ be defined on S and fulfil $\int_S |h(x)|^2\, \mathrm{d}x < \infty$. Assume that (complex) constants I_i, $i = 1, \ldots, k$, and real constants q, b_i, $i = 1, \ldots, k$, are given. We assume that b_i are nonzero and have the same sign. The function V is the same as in the previous section.

Setting of the problem: *we should find a (complex valued) function* $f(x)$ *defined on S, and complex numbers* c_i, $i = 1, \ldots, k$, *fulfilling*

$$f(x) - \mathrm{j}q \sum_{i=1}^{k} b_i \int_{S_i} f(y)\, V(|y - x|)\, \mathrm{d}y - c_p = h(x) \quad \text{for} \quad x \in S_p\,, \tag{6}$$
$$p = 1, \ldots, k\,,$$

$$\int_{S_i} f(x)\, \mathrm{d}x = I_i\,, \quad i = 1, \ldots, k\,. \tag{7}$$

Theorem 2. *Let the assumptions from the beginning of this section be fulfilled. Then there exists a unique solution* $f(x)$, c_i, $i = 1, \ldots, k$, *of* (6), (7).

Remark 3. Theorem 2 is valid even if some coefficients b_i equal zero.

The method of solving the problem is similar to that used in the previous section. Denote

$$P f(x) = \int_S f(y)\, b(y)\, V(|y - x|)\, \mathrm{d}y\,,$$

where the function b is defined by $b(x) = b_i$ for $x \in S_i$, $i = 1, \ldots, k$. Similarly we can define a function $c(x)$ by $c(x) = c_i$ for $x \in S_i$, $i = 1, \ldots, k$. The set of identities (6) can be rewritten as

$$f - \mathrm{j}\, qPf - c = h\,. \tag{8}$$

Lemma 3. *Let the assumptions of Theorem 2 be fulfilled. Then*

$$f - \mathrm{j}qPf = h \tag{9}$$

has a unique solution for every $h \in L_2(S)$.

Proof. Define an operator

$$P^* f(x) = b(x) \int_S V(|y - x|)\, f(y)\, \mathrm{d}y \quad \text{for} \quad x \in S\,.$$

The operator P^* is adjoint to the operator P in the sense $(Pf, g) = (f, P^*g)$. The homogeneous equation adjoint to (9) is

$$f + \mathrm{j}qP^*f = 0\,. \tag{10}$$

First, we prove a statement that (10) has only the trivial solution. Assume, on the contrary, that (10) has a nontrivial solution f. Identity (10) can be rewritten as

$$\int_S V(|y - x|)\, f(y)\, \mathrm{d}y = j\, f(x)/(q b_i) \quad \text{for} \quad x \in S_i$$

(the case $q = 0$ is trivial). If we multiply this equation by $\bar{f}(x)$ and integrate over S we obtain

$$(Tf, f) = \mathrm{j} \sum_{i=1}^{k} \int_{S_i} |f(x)|^2\, \mathrm{d}x/(q b_i)\,.$$

The expression on the left-hand side is real and the expression on the right-hand side is purely imaginary and nonzero. The statement is proved. Now Lemma 3 is a consequence of Fredholm's theorem and of the statement (as in the first section).

Let $f^{(i)}$ be a solution of (9) where $h(x) = 1$ on S_i and $h(x) = 0$ on the other S_p's ($p = 1, \ldots, k,\ p \neq i$). Lemma 3 yields that these solutions exist and are unique.

Denote

$$a_{pr} = \int_{S_p} f^{(r)}(x)\, \mathrm{d}x \quad \text{for} \quad p, r = 1, \ldots, k\,,$$

and let A be the $k \times k$ matrix with the elements a_{pr}.

Lemma 4. *We have* $\det A \neq 0$.

Proof. Assume $\det A = 0$. Then $Az = 0$ has a nontrivial solution z. Denote by z_i the i-th component of z and $f^{(0)} = \sum_{i=1}^{k} f^{(i)} z_i$. Certainly the function $f^{(0)}$ is again a solution of (9) where the right- hand side $h(x)$ is a function which equals z_i on S_i. We denote this function by $h^{(z)}(x)$. Moreover, we have

$$\int_{S_i} f^{(0)}(x)\, \mathrm{d}x = 0 \quad \text{for} \quad i = 1, \ldots, k\,.$$

Consider equation (9) with $h = h^{(z)}$. Multiplying this equation by $b_i \bar{f}^{(0)}(x)$ and integrating over S_i we obtain

$$b_i \int_{S_i} |f^{(0)}|^2\, \mathrm{d}x - jq \int_{S_i} P f^{(0)}(x)\, b_i \bar{f}^{(0)}(x)\, \mathrm{d}x = z_i b_i \int_{S_i} \bar{f}^{(0)}(x)\, \mathrm{d}x = 0\,.$$

If we sum these equations over all i's we obtain

$$jq \int_S \int_S V(|x - y|)\, b(x) f^{(0)}(x)\, \overline{b(y) f^{(0)}}(y)\, \mathrm{d}x\, \mathrm{d}y = \sum_{i=1}^{k} b_i \int_{S_i} |f^{(0)}(x)|^2\, \mathrm{d}x\,.$$

The right-hand side of the identity is nonzero real. The left-hand side is purely imaginary since the integral expression can be written as $(Tbf^{(0)}, bf^{(0)})$ and thus is real (Remark 2). The contradiction proves Lemma 4.

Proof of Theorem 2. The solution $f(x)$ of (6) can be written as $f(x) = f^{(h)}(x) + \sum_{i=1}^{k} c_i f^{(i)}(x)$, where $f^{(h)}$ is the solution of (9). Since $\det A \neq 0$ we can calculate c_i such that condition (7) is fulfilled.

3. NUMERICAL SOLUTION

Let S_i be disjoint, bounded and Lebesgue measurable sets, $m(S_i) > 0$ for $i = 1, \ldots, k$. The function V is the same as in Section 1.

Remark 4. If $m(S_i) = 0$ then also the corresponding I_i has to be zero.

Definition. A finite system of sets $\{X_s\}$ is called a *covering of* $\{S_i\}$ if X_s are disjoint, Lebesgue measurable sets, every X_s is a subset of some S_i, $m(X_s) > 0$, $m(S - \bigcup_s X_s) = 0$ and if in every X_s one point x_s is chosen. These points will be called *distinguished points.*

The norm of $\{X_s\}$ is defined by

$$d(\{X_s\}) = \max_s \sup \{|u - v| : u, v \in X_s\} .$$

Definition. A function $e(r)$ is a *modulus of continuity* if it is defined for $r \geqq 0$, $e(0) = 0$, e is continuous and nondecreasing. If, moreover, $|h(u) - h(v)| \leqq e(|u - v|)$ for $u, v \in S_i$, $i = 1, \ldots, k$ (h a function defined on S) then e is a *modulus of continuity of the function h.*

If a covering $\{X_s\}$ of $\{S_i\}$ is given we consider the system of algebraic linear equations (11_i), (12), $i = 1, \ldots, k$, for constants $f_1, \ldots, f_\varkappa, c_1, \ldots, c_k$, where $\varkappa$ is the number of elements of the covering $\{X_s\}$:

$$(11_i) \qquad f_p - jq \sum_{t=1}^{k} b_t \sum_{X_s \subset S_t} f_s \int_{X_s} V(|x_p - y|)\, \mathrm{d}y - \tilde{c}_i = h(x_p) \quad \text{for all} \quad p$$

$$\text{fulfilling} \quad X_p \subset S_i \quad \text{and for} \quad i = 1, \ldots, k ,$$

$$(12) \qquad \sum_{X_s \subset S_i} f_s\, m(X_s) = I_i \quad \text{for} \quad i = 1, \ldots, k .$$

If a solution of (11_i), (12), $i = 1, \ldots, k$, is given we can define $\tilde{f}(x) = f_p$ for $x \in X_p$, $\tilde{f}(x) = 0$ for $x \in S - \bigcup_s X_s$ and $\tilde{c}(x) = \tilde{c}_i$ for $x \in S_i$, $i = 1, \ldots, k$. We denote by Sol $(h, I_i, \{X_s\})$ the couple $[\tilde{f}(x), \tilde{c}(x)]$.

The function $\tilde{f}(x)$ and the constants $\tilde{c}_i$ will be considered an approximation of the solution $f(x)$, c_i, $i = 1, \ldots, k$, of (6), (7).

We shall use Hypothesis (H) in this section:

(H) *System* (6), (7) *with* $h(x) = 0$, $I_i = 0$ *for* $i = 1, \ldots, k$, *has only the trivial solution* $f(x) = 0$, $c_i = 0$ *for* $i = 1, \ldots, k$.

Theorem 3. *Let the sets S_i be disjoint, bounded, Lebesgue measurable with $m(S_i) > 0$ for $i = 1, \ldots, k$, let I_i, $i = 1, \ldots, k$, be given (complex) constants, let a function $h(x)$ be defined on S and let the function V be the same as in Section* 1. *If h has a modulus of continuity and if Hypothesis* (H) *is fulfilled, then system* (6), (7) *has a unique solution.*

Theorem 4. *If the assumptions of Theorem* 3 *are fulfilled then for every $\varepsilon > 0$ there exists $\delta > 0$ so that system* (11_i), (12), $i = 1, \ldots, k$, *has a unique solution if the covering $\{X_s\}$ fulfils $d(\{X_s\}) < \delta$ and moreover*

$$|f(x) - \tilde{f}(x)| < \varepsilon , \quad |c_i - \tilde{c}_i| < \varepsilon , \quad i = 1, \ldots, k ,$$

where $f(x)$, c_i is the solution given by Theorem 3 *and* $[\tilde{f}(x), \tilde{c}(x)] = \mathrm{Sol}\,(h, I_i, \{X_s\})$.

These two theorems will be proved simultaneously but the proof will be divided into a series of auxiliary results.

Theorem 5. (auxiliary). *Let Hypothesis* (H) *be fulfilled. Let the following assumptions be fulfilled*

(i) *a sequence of coverings* $\{X_s^{(r)}\}$ *fulfils* $d(\{X_s^{(r)}\}) \to 0$ *for* $r \to \infty$,

(ii) $I_i^{(r)}, I^{(r)}, i = 1, \ldots, k, r = 1, \ldots,$ *are complex numbers with* $\lim\limits_{r\to\infty} I_i^{(r)} = I_i$,

(iii) $h^{(r)}(x), h(x)$ *are functions defined on S having the same modulus of continuity* $e(r)$ *and fulfilling* $h^{(r)}(x) \to h(x)$ *uniformly.*

Assume that $\mathrm{Sol}\,(h^{(r)}, I_i^{(r)}, \{X_s^{(r)}\})$ *exists for every r and that there exists a constant M such that*

$$\int_S |\tilde{f}^{(r)}(x)|^2 \,\mathrm{d}x + \int_S |\tilde{c}^{(r)}(x)|^2 \,\mathrm{d}x \leqq M , \tag{13}$$

where $[\tilde{f}^{(r)}, \tilde{c}^{(r)}] = \mathrm{Sol}\,(h^{(r)}, I_i^{(r)}, \{X_s^{(r)}\})$.
Then such a subsequence of $\{X_s^{(r)}\}$ *can be chosen that the corresponding* $\tilde{f}^{(r)}$, $\tilde{c}_i^{(r)}$ *converge uniformly to a sollution* $f(x)$, c_i *of* (6), (7).

This theorem is proved in Appendix.

Lemma 5. (on regularity). *If the assumptions of Theorem* 4 *are fulfilled, then there exists* $\delta_0 > 0$ *such that the linear system of algebraic equations* (11_i), (12), $i = 1, \ldots$ $\ldots, k$, *is regular if* $d(\{X_s\}) < \delta_0$.

Remark 5. The number δ_0 is independent of $h(x)$ and I_i.

Proof of Lemma 5. Assume, on the contrary, that for every $\delta_r = 1/r$ there exists a covering $d\{X_s^{(r)}\} < \delta_r$ such that the determinant of (11_i), (12), $i = 1, \ldots, k$, is zero. Thus this system for $h(x) = 0, I_i = 0$ has a nontrivial solution to which there corresponds a nontrivial $\tilde{f}^{(r)}(x)$, $\tilde{c}^{(r)}(x)$. Since the system is homogeneous we can assume

$$\int_S |\tilde{f}^{(r)}(x)|^2 \,\mathrm{d}x + \int_S |\tilde{c}^{(r)}(x)|^2 \,\mathrm{d}x = 1 .$$

Hence the assumptions of Theorem 5 are fulfilled and there exists a subsequence of $\tilde{f}^{(r)}(x)$, $\tilde{c}^{(r)}(x)$ converging uniformly to a solution $f(x)$, c_i of (6), (7). Since the convergence is uniform we conclude

$$\int_S |f(x)|^2 \,\mathrm{d}x + \int_S |c(x)|^2 \,\mathrm{d}x = 1$$

and we obtain a contradiction with Hypothesis (H).

Lemma 6. (boundedness). *Let Hypothesis* (H) *and conditions* (i) *to* (iii) *be fulfilled. There exist numbers* $\delta_1 > 0$, $M_1 > 0$ *such that*

$$\int_S |\tilde{f}^{(r)}(x)|^2 \, dx + \int_S |\tilde{c}^{(r)}(x)|^2 \, dx \leqq M_1 \quad \text{if} \quad d(\{X_s^{(r)}\}) < \delta_1 \, .$$

where $[\tilde{f}^{(r)}, \tilde{c}^{(r)}] = \mathrm{Sol}\,(h^{(r)}, I_i^{(r)}, \{X_s^{(r)}\})$.

Proof. Assume, on the contrary, that there exists a subsequence of $\{X_s^{(r)}\}$ (which we denote by $\{X_s^{(r)}\}$ again) such that $d(\{X_s^{(r)}\}) \to 0$ and

$$\int_S |\tilde{f}^{(r)}(x)|^2 \, dx + \int_S |\tilde{c}^{(r)}(x)|^2 \, dx \to \infty \quad \text{for} \quad r \to \infty \, .$$

We can multiply $\tilde{f}^{(r)}$, $\tilde{c}^{(r)}$ by numbers $z_r > 0$, $z_r \to 0$ such that

$$z_r \left(\int_S |\tilde{f}^{(r)}(x)|^2 \, dx + \int_S |\tilde{c}^{(r)}(x)|^2 \, dx \right) = 1 \, .$$

The multiplied functions belong to $\mathrm{Sol}\,(z_r h, z_r I_i, \{X_s^{(r)}\})$. Thus the assumptions of Theorem 5 are fulfilled and at the same time $z_r I_i \to 0$. As in the previous proof we can choose a subsequence converging to a nontrivial solution of (6), (7).

Proof of Theorem 3. Consider a sequence of coverings $\{X_s^{(r)}\}$ fulfilling (i). By Lemma 5 and Lemma 6 there exists a sequence $\mathrm{Sol}\,(h, I_i, \{X_s^{(r)}\})$ which is bounded in the sense of (13). An application of Theorem 5 yields the existence of a solution of (6), (7).

Proof of Theorem 4. Let $f(x)$, $c(x)$ be the solution of (6), (7) given by Theorem 3. Assume, on the contrary, that there is a sequence of coverings $\{X_s^{(r)}\}$ fulfilling (i) so that $\tilde{f}^{(r)}(x)$, $\tilde{c}^{(r)}(x)$ given by (11_i), (12) does not converge uniformly to $f(x)$, $c(x)$ for $r \to \infty$. Certainly there exist a number $\varepsilon_0 > 0$ and a subsequence of $\{X_s^{(r)}\}$ (which we denote by $\{X_s^{(r)}\}$ again) so that

$$\sup_{x \in S} \{|f(x) - \tilde{f}^{(r)}(x)| + |c(x) - \tilde{c}^{(r)}(x)|\} \geqq \varepsilon_0 > 0 \, . \tag{14}$$

By the same reasoning as in the proof of Theorem 3 we conclude that there exists a subsequence of $\{X_s^{(r)}\}$ so that the corresponding $\tilde{f}^{(r)}(x)$, $\tilde{c}^{(r)}(x)$ converge uniformly to a solution $\hat{f}(x)$, $\hat{c}(x)$ of (6), (7). Inequality (14) yields

$$\sup_{x \in S} \{|f(x) - \hat{f}(x)| + |c(x) - \hat{c}(x)|\} \geqq \varepsilon_0 > 0 \, . \tag{15}$$

The existence of two different solutions of (6), (7) is a contradiction with Hypothesis (H).

In this section we shall prove the auxiliary Theorem 5 and inequality (16).

Lemma 7. *Let $D = \operatorname{diam} S$. There exists a function $\tilde{\varepsilon}(r)$ such that $\tilde{\varepsilon}$ is a modulus of continuity and*

$$\int_S [V(|u-y|) - V(|v-y|)]^2 \, dy \leqq \tilde{\varepsilon}^2(|u-v|) \quad \text{for} \quad u, v \in R, \ |u|, |v| \leqq D$$

is valid.

Proof. First we prove a statement: For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_S [V(|u-y|) - V(|v-y|)]^2 \, dy < \varepsilon^2 \quad \text{for} \quad |u-v| < \delta, \ |u|, |v| \leqq D.$$

Let $\varepsilon > 0$ be given. There exists a continuous function $V_\varepsilon(r)$ such that

$$\int_0^{2D} (V(r) - V_\varepsilon(r))^2 \, r^{n-1} \, dr < \varepsilon^2,$$

and since V_ε is continuous there exists $\delta > 0$ such that

$$|V_\varepsilon(a) - V_\varepsilon(b)| < \varepsilon \quad \text{for} \quad 0 \leqq \min(a, b) \leqq \max(a, b) \leqq 2D, \ |a - b| < \delta.$$

We have

$$\sqrt{\int_S [V(|u-y|) - V(|v-y|)]^2 \, dy} \leqq \sqrt{\int_S [V(|u-y|) - V_\varepsilon(|u-y|)]^2 \, dy} +$$

$$+ \sqrt{\int_S [V(|v-y|) - V_\varepsilon(|v-y|)]^2 \, dy} + \sqrt{\int_S [V_\varepsilon(|u-y|) - V_\varepsilon(|v-y|)]^2 \, dy} \leqq$$

$$\leqq 2 \sqrt{\left(a_{n-1} \int_0^{2D} [V(r) - V_\varepsilon(r)]^2 \, r^{n-1} \, dr\right)} + \varepsilon \sqrt{m(S)} \leqq$$

$$\leqq \varepsilon(2\sqrt{(a_{n-1})} + \sqrt{m(S)}) \quad \text{for} \quad |u-v| < \delta, \quad |u|, |v| \leqq 2D,$$

where a_n is the volume of the n-dimensional unit sphere. The statement is proved and Lemma 7 is its easy consequence.

Lemma 8. *There exists a constant M_2 such that*

$$|\tilde{f}^{(r)}(x)| \leqq M_2 \quad \text{for} \quad x \in S,$$

where $[\tilde{f}^{(r)}, \tilde{c}^{(r)}] = \operatorname{Sol}(h^{(r)}, I_i^{(r)}, \{X_s^{(r)}\})$ is given in the auxiliary Theorem 5.

Proof. System (11_i) can be rewritten as

$$\tilde{f}^{(r)}(x_p^{(r)}) = \tilde{c}^{(r)}(x_p^{(r)}) + h^{(r)}(x_p^{(r)}) + \mathrm{j}q \sum_t b_t \int_{S_t} \tilde{f}^{(r)}(y)\, V(|x_p^{(r)} - y|)\, \mathrm{d}y\,.$$

Using the Hölder inequality,

$$|\tilde{f}^{(r)}(x_p^{(r)})| \leqq |\tilde{c}^{(r)}(x_p^{(r)})| + |h^{(r)}(x_p^{(r)})| +$$

$$+ |q| \max |b_t| \sqrt{\int_S |f^{(r)}(y)|^2\, \mathrm{d}y} \sqrt{\int_S [V(|x_p^{(r)} - y|)]^2\, \mathrm{d}y}\,.$$

Since $h^{(r)}(x)$ converge uniformly to $h(x)$ and $h(x)$ is bounded, $\sup |h^{(r)}(x_p^{(r)})| < \infty$. By Lemma 7, $\sup_{x\in S} \int_S [V(|x - y|)]^2\, \mathrm{d}y < \infty$. If we take into consideration (13) and $m(S_i) > 0$ we conclude that $\sup_r |\tilde{c}^{(r)}(x_p^{(r)})| < \infty$ and Lemma 8 is valid.

Lemma 9. *There exists a modulus of continuity $e^+(r)$ such that*

$$|f^{(r)}(x_p^{(r)}) - f^{(r)}(x_t^{(r)})| \leqq e^+(|x_p^{(r)} - x_t^{(r)}|) \quad \textit{for} \quad x_p^{(r)},$$

$x_t^{(r)} \in S_i$, $i = 1, \ldots, k$, *where* $x_p^{(r)}$, $x_t^{(r)}$ *are distinguished points of* $\{X_s^{(r)}\}$.

Proof. We have

$$\tilde{f}^{(r)}(x_p^{(r)}) - \tilde{f}^{(r)}(x_t^{(r)}) =$$

$$= h(x_p^{(r)}) - h(x_t^{(r)}) + \mathrm{j}q \sum_\tau b_\tau \int_{S_\tau} \tilde{f}^{(r)}(y) [V(|x_p^{(r)} - y|) - V(|x_t^{(r)} - y|)]\, \mathrm{d}y$$

so that similarly as in the proof of the previous lemma

$$|\tilde{f}^{(r)}(x_p^{(r)}) - \tilde{f}^{(r)}(x_t^{(r)})| \leqq e(|x_p^{(r)} - x_t^{(r)}|) +$$

$$+ |q| \max_\tau |b_\tau| \sqrt{\int_S |f^{(r)}(y)|^2\, \mathrm{d}y} \sqrt{\int_S [V(|x_p^{(r)} - y|) - V(|x_t^{(r)} - y|)]^2\, \mathrm{d}y} \leqq$$

$$\leqq e(|x_p^{(r)} - x_t^{(r)}|) + |q| \max_\tau |b_\tau| M\tilde{e}(|x_p^{(r)} - x_t^{(r)}|)\,.$$

The statement of the lemma is valid for $e^+(r) = e(r) + |q| \max_\tau |b_\tau|\, M\tilde{e}(r)$.

Proof of Theorem 5. Let $\bar{S}_i$ be the closure of S_i. Since S_i are bounded the sets $\bar{S}_i$ are compact.

We define functions $f^{(r)}(x; i)$ for $x \in \bar{S}_i$ $i = 1, \ldots, k$:

$$\operatorname{Re} f^{(r)}(x; i) = \min \{\operatorname{Re} \tilde{f}^{(r)}(x_p^{(r)}) + e^+(|x_p^{(r)} - x|): \text{ for all distinguished points } x_p^{(r)} \text{ of } \{X_s^{(r)}\} \text{ from } S_i\}\,,$$

$$\operatorname{Im} f^{(r)}(x; i) = \min \{\operatorname{Im} \tilde{f}^{(r)}(x_p^{(r)}) + e^+(|x_p^{(r)} - x|): \text{ for all distinguished points } x_p^{(r)} \text{ of } \{X_s^{(r)}\} \text{ from } S_i\}\,.$$

Remark 6. By Lemma 9 we have $f^{(r)}(x_p^{(r)}; i) = \tilde{f}^{(r)}(x_p^{(r)})$ for the distinguished points $x_p^{(r)}$ from S_i.

Lemma 10. *The functions $f^{(r)}(x; i)$ are uniformly bounded:*

$$|f^{(r)}(x; i)| \leqq (M_2 + e^+(\operatorname{diam} S_i))\sqrt{2} \quad \textit{for} \quad x \in \bar{S}_i\,,$$

and the functions $f^{(r)}(x; i)$ have a common modulus of continuity:

$$|f^{(r)}(x; i) - f^{(r)}(y; i)| \leqq e^*(|x - y|) \quad \textit{for} \quad x, y \in \bar{S}_i\,, \quad i = 1, \ldots, k\,,$$

where $e^*(r) = \sqrt{(2)} \sup \{e^+(\alpha) - e^+(\beta) : \alpha \leqq \beta + r,\ \beta \leqq \operatorname{diam} S_i\}$ *and* $\operatorname{diam} S_i = \sup \{|u - v| : u, v \in S_i\}$.

Proof. The bound for $\operatorname{Re} f^{(r)}(x; i)$ and $\operatorname{Im} f^{(r)}(x; i)$ is a consequence of Lemma 8 and Lemma 9. Consider now $x, y \in \bar{S}_i$. Let the minimum in the definition of $\operatorname{Re} f^{(r)}(x; i)$ be realized by a distinguished point $x_p^{(r)}$, $x_p^{(r)} \in S_i$. We conclude

$$\operatorname{Re} f^{(r)}(y; i) - \operatorname{Re} f^{(r)}(x; i) =$$
$$= \min \{\operatorname{Re} \tilde{f}^{(r)}(x_t^{(r)}) - \operatorname{Re} \tilde{f}^{(r)}(x_p^{(r)}) + e^+(|x_t^{(r)} - y|) -$$
$$- e^+(|x_p^{(r)} - x|) : x_t^{(r)} \in S_i\} \leqq e^+(|x_p^{(r)} - y|) - e^+(|x_p^{(r)} - x|) \leqq e^*(|x - y|).$$

Since this bound is symmetric with regard to x, y and is valid also for the imaginary parts of $f^{(r)}(x; i)$, Lemma 10 is proved.

Since the functions $\tilde{c}^{(r)}(x)$ are constant on S_1, bounded ((13) and $m(S_1) > 0$) we can choose a subsequence of $\{X_s^{(r)}\}$ such that the subsequence of $\tilde{c}^{(r)}(x)$ converges to a constant c_1. Since $f^{(r)}(x; 1)$ fulfil the conditions of Arzela's Lemma we can choose a new subsequence $f^{(r)}(x; 1)$ converging uniformly to a function $f(x; 1)$. We repeat this construction for $S_2, \ldots, S_k$. We denote $f(x) = f(x; i)$ for $x \in S_i$, $i = 1, \ldots, k$.

Lemma 11. *The function $f(x)$ and the constants c_i, $i = 1, \ldots, k$, form a solution of* (6), (7).

Proof. First, we denote the subsequence of $f^{(r)}(x; i)$ which converges uniformly to $f(x; i)$ by $f^{(r)}(x; i)$ again. By Remark 6 we have $f^{(r)}(x_p^{(r)}; i) = \tilde{f}^{(r)}(x_p^{(r)}) = f_p^{(r)}$. Thus using (11_i), we have

$$f^{(r)}(x_p^{(r)}, i) - jq \sum_{t=1}^{k} b_t \int_{S_t} f^{(r)}(y; t)\, V(|x_p^{(r)} - y|)\, \mathrm{d}y - \tilde{c}_i^{(r)} =$$

$$= h(x_p^{(r)}) - jq \sum_{t=1}^{k} b_t \sum_{X_s^{(r)} \subset S_t} \int_{X_s^{(r)}} [f^{(r)}(y; t) - f^{(r)}(x_p^{(r)}; t)]\, V(|x_p^{(r)} - y|)\, \mathrm{d}y$$

for $x_p^{(r)} \in S_i$, $i = 1, \ldots, k$ and

$$f^{(r)}(x; i) - \mathrm{j}q \sum_{t=1}^{k} b_t \int_{S_t} f^{(r)}(y; t)\, V(|x - y|)\, \mathrm{d}y - \tilde{c}_i^{(r)} = h(x) + J^{(r)},$$

where

$$J^{(r)} = h(x_p^{(r)}) - h(x) + f^{(r)}(x; i) - f^{(r)}(x_p^{(r)}; i) +$$

$$+ \mathrm{j}q \sum_t b_t \int_{S_t} f^{(r)}(y; t)\, [V(|x_p^{(r)} - y|) - V(|x - y|)]\, \mathrm{d}y -$$

$$- \mathrm{j}q \sum_t b_t \sum_{X_s^{(r)} \subset S_t} \int_{X_s^{(r)}} [f^{(r)}(y; t) - f^{(r)}(x_p^{(r)}; t)]\, V(|x_p^{(r)} - y|)\, \mathrm{d}y,$$

where $x_p^{(r)}$ is the distinguished point of $\{X_s^{(r)}\}$ in S_i nearest to x. Assume for a while that $S = \bigcup_s X_s^{(r)}$ for all r. Then $|x - x_p^{(r)}| \to 0$ for $r \to \infty$. By (iii), Lemma 10 and Lemma 7 we conclude that $J^{(r)} \to 0$ for $r \to \infty$. Thus $f(x)$, c_i fulfil equation (6). Equation (7) for $f(x)$, c_i can be proved in a similar way.

It remain to prove that $\tilde{f}^{(r)}(x)$ converge to $f(x)$ uniformly.

Since $\tilde{f}^{(r)}(x)$ is constant on $X_s^{(r)}$ and $\tilde{f}^{(r)}(x) = \tilde{f}^{(r)}(x_p^{(r)}) = f_p$, where $x_p^{(r)}$ is the distinguished point in $X_p^{(r)}$, we have

$$|\tilde{f}^{(r)}(x) - f^{(r)}(x; i)| \leqq e^*(|x - x_p^{(r)}|) \leqq e^*(d(\{X_s^{(r)}\})) \to 0$$

for $r \to \infty$. This proves that also $\tilde{f}^{(r)}(x)$ converge uniformly to $f(x)$. If $S \neq \bigcup_s X_s^{(r)}$ we put $\hat{S}_i = \bigcap_r (\bigcup_s X_s^{(r)} \cap S_i)$. We have proved that $f(x)$, c_i fulfil (6), (7) with S_i replaced by $\hat{S}_i$. But $m(S_i - \hat{S}_i) = 0$, $i = 1, \ldots, k$, and we can extend $f(x)$ by (6) onto S. Theorem 5 is proved.

Lemma 12. *Let the assumptions of Theorem* 3 *be fulfilled. Then*

$$|f(u) - f(v)| \leqq e(|u - v|) + |q| \max_i |b_i|\, \tilde{e}(|u - v|) \sqrt{\int_S |f(y)|^2\, \mathrm{d}y} \tag{16}$$

$$\text{for} \quad u, v \in S_i, \quad i = 1, \ldots, k.$$

Proof. Let $u, v \in S_i$, then using (6) we can derive

$$f(u) - f(v) = h(u) - h(v) + \mathrm{j}q \sum_i b_i \int_{S_i} f(y)\, [V(|u - y|) - V(|v - y|)]\, \mathrm{d}y.$$

By the Hölder inequality and Lemma 7 we obtain the statement of Lemma 12.

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Souhrn

ANALÝZA INTEGRÁLNÍCH ROVNIC POPISUJÍCÍCH POVRCHOVÝ JEV

IVO VRKOČ

Článek navazuje na práci [1], kde je odvozen a diskutován matematický model povrchového jevu. V případě jednoho vodiče v proměnném magnetickém poli je fázor hustoty elektrického proudu ve vodiči určen rovnicemi (1), (2). V případě více rovnoběžných vodičů obdržíme systém (6), (7). V první a druhé části článku jsou řešeny problémy existence a jednoznačnosti řešení daných rovnic. V třetí části článku je navržena numerická metoda výpočtu řešení a je dokázána její konvergence.

Author's address: RNDr. *Ivo Vrkoč*, DrSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.