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ON THE SOLUTION OF THE HEAT EQUATION WITH NONLINEAR UNBOUNDED MEMORY

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Summary. The paper deals with the question of global solution u, τ to boundary-value problem for the system of semilinear heat equation for u and complementary nonlinear differential equation for τ ("thermal memory"). Uniqueness of the solution is shown and the method of succesive approximations is used in the proof of existence of global solution provided that the condition ($\mathscr{P}$) holds. The condition ($\mathscr{P}$) is verified for some particular cases (e.g.: bounded nonlinearity, homogeneous Neumann problem (even for unbounded nonlinearities), apriori estimate of the solution holds).

1. INTRODUCTION

We shall consider the heat equation

$$\varrho(x, t) \frac{\partial u}{\partial t} = \operatorname{div} (\lambda(x, t) \operatorname{grad} u) + f , \tag{1.1}$$

where the internal heat sources are

$$f = q(\tau)\, \psi(u, \tau) , \tag{1.2}$$

with the additional equation

$$\frac{\partial \tau}{\partial t} = \psi(u, \tau) \tag{1.3}$$

for the function $\tau = \tau(x, t)$.

Such a type of equations arises in the investigation of heat conduction in concrete [1]. Then q represents the known speed of the hydration heat of cement at a certain constant reference temperature u_R. If the temperature u increases, the hydration process accelerates, which is expressed by the factor ψ in (1.2). The function τ and equation (1.3) then represent the "*thermal memory*": the internal heat sources f can be written in the form

$$f = \frac{\mathrm{d}}{\mathrm{d}t} Q(\tau) ,$$

where

$$Q(z) := \int_0^z q(s)\, ds \tag{1.4}$$

is the total hydration heat of cement at the constant reference temperature u_R. Thus the function τ can be regarded as transformed time which converts the general thermal behaviour to the constant reference temperature u_R.

The function $q \in C^{(0),1}(\langle 0, \infty))$ *) satisfies conditions

$$0 \leqq q(z) \leqq q_M < \infty\,, \quad z \geqq 0\,; \tag{1.5}$$

$$\mu := \lim_{z \to \infty} Q(z) = \int_0^\infty q(s)\, ds < \infty\,, \tag{1.6}$$

while the function $\psi \in C^{(1)}(\mathbb{R} \times \langle 0, \infty))$ can be unbounded. In general, we have only the following conditions on ψ:

$$\psi(v, z) \geqq 0\,, \quad v \in \mathbb{R}\,, \quad z \geqq 0\,; \tag{1.7}$$

$$\frac{\partial \psi}{\partial v}(v, z) \geqq 0\,, \quad v \in \mathbb{R}\,, \quad z \geqq 0\,. \tag{1.8}$$

The case of bounded ψ is naturally simpler and was solved in a similar form in [3], [4]. Here we are concerned in particular with unbounded ψ, a typical form of function ψ is ([1], [2])

$$\psi(u, z) = \alpha^{(\mu - \mu_R)/10}\,, \quad \alpha > 0\,, \tag{1.9}$$

which has an exponential form $\psi = a \exp(bu)$.

Unbounded internal sources f in (1.1) can force the solution to tend to infinity at a finite time (see Remark 4.3 and Examples 4.5, 4.6). However, in our case of f, q satisfying (1.2), (1.6), the physical reasons lead to the conjecture that the global solution exists for all $t > 0$. We shall prove it at least for the particular case of the homogeneous Neumann boundary condition (Section 4). In the case of general boundary conditions we shall prove a global existence theorem only under certain additional conditions on ψ, q (Section 6); the general case is still open.

2. FORMULATION OF THE PROBLEM

The equations (1.1)–(1.3) are considered for $t \in \langle 0, T\rangle$, $T > 0$, and for $x \in \Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz obundary $\partial\Omega$. Appropriate initial and boundary conditions are required:

$$u(x, 0) = u_0(x)\,, \quad \tau(x, 0) = 0\,, \quad x \in \Omega\,. \tag{2.1}$$

*) Here $C^{(0),1}$ denotes locally Lipschitz continuous functions — no boundedness is required.

Let

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_W \cup \partial\Omega_N \cup \partial\Omega_0 ,$$

where

$$\partial\Omega_D,\ \partial\Omega_W,\ \partial\Omega_N,\ \partial\Omega_0$$

are disjoint parts of the boundary, mes $(\partial\Omega_0) = 0$. We require

(2.2) $$u(x, t) = u_D(x, t) \quad \text{for} \quad x \in \partial\Omega_D , \quad t > 0 ;$$

(2.3) $$\lambda(x, t) \frac{\partial u}{\partial \nu}(x, t) = \alpha(x, t) \left[u_W(x, t) - u(x, t)\right] + u_N(x, t)$$
$$\text{for} \quad x \in \partial\Omega_W \cup \partial\Omega_N , \quad t > 0 .$$

The problem to find functions u, τ satisfying (1.1)–(1.3), (2.1)–(2.3) will be refered to as the *problem* $(\mathcal{M})$.

From now on we shall deal with the weak solutions.

Let $W^{k,p}(\Omega)$ denote the usual Sobolev spaces (k derivatives p-integrable) and let us define

$$V := \{v \in W^{1,2}(\Omega);\ v(x) = 0 \text{ for } x \in \partial\Omega_D\} ,$$
$$H := C(\langle 0, T);\ W^{1,2}(\Omega)) \cap C^{(1)}(\langle 0, T);\ L_2(\Omega)) \cap C(\langle 0, T);\ L_\infty(\Omega)) ,$$
$$H_0 := \{v \in H;\ v(t) \in V \text{ for } t \in \langle 0, T)\} .$$

We shall use the following notations:

$$(u, v) := \int_\Omega u(x)\, v(x)\, \mathrm{d}x , \quad \|u\|_0 := (u, u)^{1/2} ;$$
$$((u, v)) := (\lambda(t) \operatorname{grad} u,\ \operatorname{grad} v) ;$$
$$\langle u, v\rangle := \int_{\partial\Omega_W \cup \partial\Omega_N} u(x)\, v(x)\, \mathrm{d}S ;$$
$$\|u\|_\infty := \sup_{x\in\Omega} \operatorname{ess} |u(x)| \quad (L_\infty\text{-norm}) .$$

Coefficients ϱ, λ, α are supposed to be sufficiently smooth, bounded and, further, to satisfy

(2.4) $$\varrho(x, t) \geqq \varrho_0 > 0 , \quad \lambda(x, t) \geqq \lambda_0 > 0 , \quad (x, t) \in \Omega \times (0, T) ;$$

(2.5) $$\alpha(x, t) \geqq \alpha_0 > 0 \quad \text{for} \quad x \in \partial\Omega_W , \quad \alpha(x, t) = 0 \quad \text{for} \quad x \in \partial\Omega_N , \quad t \in (0, T) .$$

Let the given data satisty

(2.6) $$u_0 \in W^{1,2}(\Omega) , \quad u_0 = u_D(0) \text{ on } \partial\Omega_D ;$$

(2.7) $$u_W, u_N \in C^{(1)}(\langle 0, T\rangle,\ L_\infty(\partial\Omega_W \cup \partial\Omega_N)) ;$$

(2.8) $$u_D \in C^{(1)}(\langle 0, T\rangle,\ L_\infty(\partial\Omega_D) \cap W^{1,2}(\partial\Omega_D)) .$$

A weak solution of the problem $(\mathcal{M})$ *on* $\langle 0, T\rangle$ is a pair of functions

(2.9) $$u \in H\,, \quad \tau \in C^{(1)}(\langle 0, T), L_\infty(\Omega))$$

such that

(2.10) $$u(0) = u_0\,, \quad \tau(0) = 0\,;$$

(2.11) $$u(t) = u_D(t) \text{ on } \partial\Omega_D\,, \quad t \in \langle 0, T)\,;$$

(2.12) $$(\varrho(t)\, u'(t), v) + ((u(t), v)) + \langle \alpha(t)\, u(t), v\rangle = \quad \text{for all} \quad v \in V\,, \quad t \in (0, T)\,;$$

(2.13) $$\tau'(t) = \psi(u(t), \tau(t))\,, \quad t \in (0, T)\,.$$

3. UNIQUENESS

Theorem 3.1. *The problem* $(\mathcal{M})$ *has at most one weak solution, i.e., if* u_1, τ_1 *and* u_2, τ_2 *are two weak solutions on* $\langle 0, T_1)$ *and on* $\langle 0, T_2)$, *respectively, with the same data* u_0, u_D, u_W, u_N, *then*

(3.1) $$u_1(t) = u_2(t)\,, \quad \tau_1(t) = \tau_2(t)$$

for $0 \leqq t < \min(T_1, T_2)$.

Proof. We can suppose $T_1 \leqq T_2$. For an arbitrary $\varrho > 0$ there exists a constant $0 < c_1 < \infty$ such that

$$\|u_i(t)\|_\infty \leqq c_1\,, \quad \|\tau_i(t)\|_\infty \leqq c_1\,, \quad i = 1, 2\,, \quad t \in \langle 0, T_1 - \varepsilon\rangle\,,$$

and consequently

(3.2) $$\Delta(x, t) := \psi(u_1, \tau_1)\, q(\tau_1) - \psi(u_2, \tau_2)\, q(\tau_2) \leqq$$
$$\leqq c_2(|u_1(x, t) - u_2(x, t)| + |\tau_1(x, t) - \tau_2(x, t)|)\,,$$

where the constant c_2 depends on

$$\max\left\{|\psi(y, z)|, \left|\frac{\partial\psi}{\partial y}(y, z)\right|, \left|\frac{\partial\psi}{\partial z}(y, z)\right|, q(z), |q'(z)|;\ |y| \leqq c_1,\ 0 \leqq z \leqq c_1\right\}.$$

For $v := u_1 - u_2$, $\vartheta := \tau_1 - \tau_2$ we have $v(t) \in V$, $t \in \langle 0, T_1)$ and subtracting the equations (2.12) for u_1 and u_2 (with the test function $v(t) \in V$) we obtain

(3.3) $$(\varrho(t)\, v'(t), v(t)) + ((v(t), v(t))) + \langle \alpha(t)\, v(t), v(t)\rangle = (\Delta(t), v(t))\,.$$

From (3.2), (3.3) we obtain

(3.4) $$\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho(t)\, v(t)\|_0^2 \leqq c\{\|v(t)\|_0^2 + \|\vartheta(t)\|_0^2\}\,.$$

Similarly, the equation (2.13) yields the estimates

$$|\vartheta'(t)| \leqq c_2(|v(t)| + \vartheta(t)|) ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\vartheta(t)\|_0^2 \leqq c\{\|v(t)\|_0^2 + \|\vartheta(t)\|_0^2\} . \tag{3.5}$$

If we define $w(t) := \|\varrho(t)\, v(t)\|_0^2 + \|\vartheta(t)\|_0^2$, we have $w(0) = 0$ and $w'(t) \leqq c\, w(t)$ by (3.4), (3.5). Then $w \equiv 0$ and (3.1) holds for $t \in \langle 0, T_1 - \varepsilon \rangle$. Taking $\varepsilon \to 0$ we complete the proof. ∎

4. HOMOGENEOUS NEUMANN CONDITIONS

Let us consider the particular case of homogeneous conditions of Neumann type:

$$\partial\Omega_D = \emptyset , \quad \partial\Omega_W = \emptyset , \quad \partial\Omega_N = \partial\Omega , \tag{4.1}$$

$$\lambda \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T) . \tag{4.2}$$

Moreover, let the functions ϱ, u_0 be constant (because of (2.4) we can suppose $\varrho \equiv 1$).

Then we can take u, τ independent of x and the problem reduces to the following system of ordinary differential equations:

$$u' = q(\tau)\, \psi(u, \tau) , \quad u(0) = u_0 , \tag{4.3}$$

$$\tau' = \psi(u, \tau) , \qquad \tau(0) = 0 . \tag{4.4}$$

Since $u' = q(\tau)\, \tau' = \mathrm{d}/\mathrm{d}t\,(Q(\tau))$, we have $u(t) = u_0 + Q(\tau(t))$ and consequently

$$u_0 \leqq u(t) \leqq u_0 + \mu < \infty . \tag{4.5}$$

Definition 4.1. *We shall call $\psi = \psi(u, \tau)$ bounded in τ iff*

$$\forall c > 0 \quad \exists m(c) > 0 : |u| \leqq c \Rightarrow \psi(u, \tau) \leqq m(c) , \quad \tau \geqq 0 . \tag{4.6}$$

If ψ is bounded in τ, we further obtain

$$\tau(t) = \int_0^t \psi(u(s), \tau(s))\, \mathrm{d}s \leqq m(\mu + \mu_0)\, t .$$

Thus the increasing function u, τ remain bounded on bounded intervals and therefore we have a global solution of the system (4.3), (4.4). This completes the proof of

Theorem 4.2. *Let $\psi(u, \tau)$ be bounded in τ, $\varrho \equiv$ const. Then the problem $(\mathscr{M})$ has a unique global solution for the case of a constant initial condition and a homogeneous Neumann boundary condition.*

Remark 4.3. The assumption that ψ is bounded in τ is essential: Let for example $\psi(u, \tau) = \exp(u + \tau)$. Then $\tau' \geqq \exp(u_0 + \tau)$ by (4.4), (4.5) and further $\tau(t) \geqq \vartheta(t)$, where

$$\vartheta' = \exp(u_0 + \vartheta), \quad \vartheta(0) = -\varepsilon, \quad \varepsilon > 0.$$

But $\vartheta(t) = -\ln(e^{\varepsilon} - e^{u_0}t)$ is defined only for $0 \leqq t \leqq T(\varepsilon) := \exp(\varepsilon - u_0)$ and $\lim_{t \to T(\varepsilon)} \vartheta(t) = +\infty$. Therefore τ is defined only on the interval $\langle 0, T(0))$ with $\lim_{t \to T(0)} \tau(t) = +\infty$.

However, this fact is not too restrictive for practical purposes, since while increasing temperature u accelerates the hydration process (ψ increases), increasing "transformed time" τ is rather expected to deccelerate hydration (ψ decreases). In particular, in many cases we can take $\psi = \psi(u)$ independent of τ ([1]). ■

Remark 4.4. Taking into account the physical interpretation of the problem($\mathcal{M}$), homogeneous Neumann boundary conditions represent the case of perfectly insulated body, when no heat flux across the boundary (no cooling) can cause decrease of temperature. This fact leads to the conjecture that also the solution of the general case of boundary conditions should remain bounded with no blow-up to infinity at finite time (in fact, we expect the general case to be bounded by a solution of an appropriate insulated case). However, we have no proof of this conjecture and the following examples show a possibility of blow-up of the solutions of certain equations similar to the problem ($\mathcal{M}$) with ψ bounded in τ. ■

Example 4.5. Let us solve the equations of problem ($\mathcal{M}$) with $\psi(z, t) := a \exp(bt)$, $a > 0$, $b > 0$, $\varrho \equiv 1$ and $q(y) \equiv 1$ (condition (1.6) is violated). For $u_0 = 0$ and homogeneous Neumann boundary conditions u, τ does not depend on x and $u = \tau$ is a solution of the equation $z' = a \exp(bz)$, $z(0) = 0$. Then $u(t) = -\ln(1 - abt)/b$ is defined only for $t < 1/ab$ and $u(t) \to +\infty$ for $t \to 1/ab$. ■

Example 4.6. Let $\psi(z) := a \exp(b(z + c))$, $c \in \mathbb{R}$, $a > 0$, $b > 0$. Let q satisfy (1.5), (1.6) and let us solve the following heat equation with $\varrho \equiv 1$, $\alpha > 0$, $u(0) \equiv 0$ and the homogeneous Neumann boundary condition:

$$(u'(t), z) + ((u(t), z)) = (q(\alpha at)\, \psi(u(t)), z), \quad z \in W^{1,2}(\Omega). \tag{4.7}$$

Again $u = u(t)$ is a function of t only and satisfies the ordinary differential equation

$$u' = q(\alpha at)\, a \exp(b(u + c)), \quad u(0) = 0. \tag{4.8}$$

Therefore

$$u = -\frac{1}{b} \ln\left[1 - \frac{be^{bc}}{\alpha} Q(\alpha at)\right] \tag{4.9}$$

and the solution u is defined only as far as

$$be^{bc} Q(\alpha at) < \alpha. \tag{4.10}$$

If $\mu \leqq \alpha \exp(-bc)/b$; inequality (4.10) holds for all $t > 0$ and we have a global solution. On the other hand, if $\mu > \alpha \exp(-bc)/b$, then there exists $t^* > 0$ such that $Q(\alpha a t^*) = \alpha \exp(-bc)/b$ and the solution u is defined only for $t < t^*$ with $\lim_{t \to t^*} u(t) = +\infty$. ∎

5. LINEAR PROBLEM AND POSITIVITY OF SOLUTIONS

Succesive approximations will be used in the proof of existence of a solution of the general problem $(\mathcal{M})$. This method requires the solution of the following *linear problem* $(\mathcal{L})$:

$(\mathcal{L})$: *Let u_0, u_D, u_W, u_N satisfying (2.6)–(2.8) be given and let $f = f(x, t) \in C^{(1)}(\langle 0, T\rangle, L_2(\Omega))$. We seek for a function $u \in C(\langle 0, T), W^{1,2}(\Omega)) \cap C^{(1)}(\langle 0, T), L_2(\Omega))$ such that $u(0) = u_0$, $u(t) = u_D(t)$ on $\partial\Omega_D$ and*

$$(5.1) \qquad (\varrho(t)\, u'(t), z) + ((u(t), z)) + \langle \alpha(t)\, u(t), z\rangle = (f(t), z) + \langle \alpha(t)\, u_W(t) + u_N(t), z\rangle\,, \quad z \in V\,, \quad t \in (0, T)\,.$$

Theorem 5.1. *The linear problem $(\mathcal{L})$ has a unique solution. If, moreover, $f \in C(\langle 0, T\rangle, L_\infty(\Omega))$, then $u \in C(\langle 0, T), L_\infty(\Omega))$ and for $u_0 = u_D = u_W = u_N = 0$ we have*

$$(5.2) \qquad |u(x, t)| \leqq \int_0^t \|f(s)\|_\infty \,\mathrm{d}s\,.$$

Proof. The linear problem was treated by many anthors (e.g. [5], [6]) and the first part of the theorem can be proved by the semigroup technique [7].

For $f \in C(\langle 0, T\rangle, L_\infty(\Omega))$ let c_1 be a common bound for the L_∞-norms of u_0, u_D, u_W, u_N/α_0 and let

$$v(x, t) := c_1 + \int_0^t \|f(s)\|_\infty \,\mathrm{d}s - u(x, t)\,.$$

Then $v'(t) = \|f(t)\|_\infty - u'(t)$, grad v = grad u, and for all $z \in V$ we have

$$(\varrho(t)\, v'(t), z) + ((v(t), z)) + \langle \alpha(t)\, v(t), z\rangle = (\|f(t)\|_\infty - f(t), z) + \langle h(t), z\rangle\,,$$

where

$$h(t) := \alpha\left(\int_0^t \|f(s)\|_\infty \,\mathrm{d}s + c_1 - u_W\right) - u_N \geqq 0\,.$$

Since $v(0) \geqq 0$ and $v(t) \geqq 0$ on $\partial\Omega_D$, Theorem 5.3 below yields $v \geqq 0$ and the rest of the theorem holds. ∎

Remark 5.2. Though we have the estimate (5.2), we can have a function $f(x, t) \geqq 0$ with

$$\int_0^T f(x, s)\,\mathrm{d}s \leqq c < \infty \quad \forall x \in \Omega$$

which generates an unbounded solution u of the problem $(\mathscr{L})$. For the case of one-dimensional Cauchy problem

$$u_t - u_{xx} = f(x, t), \quad u(x, 0) = 0$$

we can take

$$f(x, t) := \frac{\partial}{\partial t}\left[-\exp\left(\frac{-x^2}{4(1-t)}\right)\right], \quad t < 1 .$$

Then

$$\int_0^1 f(x, s)\, ds = \exp(-x^2/4) < 1 .$$

The solution can be expressed in terms of the Green function ([5]), in particular,

$$u(0, t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{(\pi(t-s))}} \frac{x^2}{4(1-s^2)} \exp\left(\frac{-2x^2}{4(1-s)}\right) dx\, ds ,$$

and we can see that $\lim_{t \to 1} u(0, t) = +\infty$. ∎

Theorem 5.3. *Let* $u_0 \geqq 0$, $u_D \geqq 0$, $u_N \geqq 0$, $u_W \geqq 0$, $f \geqq 0$. *Then the solutions of problems* $(\mathscr{L}), (\mathscr{M})$ *are nonnegative*:

$$u(x, t) \geqq 0 \quad \text{a.e. in} \quad \Omega \times (0, T) .$$

Proof. Since $q \geqq 0$, $\psi \geqq 0$ in the problem $(\mathscr{M})$, it is sufficient to prove the theorem for the problem $(\mathscr{L})$.

Here we can use a technique from [8]:

Let $w := u^- := \max(0, -u)$. Then $w = 0$ on $\partial\Omega_D$, and taking in (5.1) $z = w$ we obtain

$$-(\varrho w', w) - ((w, w)) - \langle \alpha w, w \rangle = \langle \alpha u_W + u_N, w \rangle + (f, w) \geqq 0 .$$

Let $\varphi(t) := (\varrho(t)\, w(t), w(t))$. Then $\varphi(0) = 0$ and $\varphi'(t) \leqq \max(|\varrho'|/\varrho_0)\, \varphi(t)$, which implies $\varphi(t) \equiv 0$. ∎

6. EXISTENCE OF SOLUTION

Let $v \in H$ be the solution of the linear problem $(\mathscr{L})$ with $f \equiv 0$. This notation will be used throughout the whole Section 6.

Let $w \in H_0$, $\tau \in C^{(1)}(\langle 0, T), L_2(\Omega))$ be a solution of the following *problem* $(\mathscr{M}_0)$ with zero initial and boundary data:

(6.1) $$w(0) = 0, \quad \tau(0) = 0 ;$$

(6.2) $$(\varrho(t)\, w'(t), z) + ((w(t), z)) + \langle \alpha(t)\, w(t), z \rangle = \\ = (q(\tau(t))\, \psi(w(t) + v(t), \tau(t)), z), \quad z \in V ;$$

(6.3) $$\tau'(t) = \psi(w(t) + v(t), \tau(t)) .$$

Then evidently the functions $\mu := w + v \in H, \tau$ are the solution of our problem $(\mathcal{M})$. Thus we can confine ourselves to the "reduced" problem $(\mathcal{M}_0)$, to which we shall apply the method of successive approximations:

Starting from arbitrary $u_0 \in H_0$, $\vartheta_0 \in C^{(1)}(\langle 0, T \rangle, L_2(\Omega))$ we define for $i \geqq 0$ the subsequent approximations v_{i+1}, ϑ_{i+1} as follows:

$$\vartheta_{i+1}(t) := \int_0^t \psi(v_i(s) + v(s), \vartheta_i(s))\, ds\,; \tag{6.4}$$

$v_{i+1} \in H_0$ is the solution of the linear problem

$$(\varrho v_{i+1}, z) + ((v_{i+1}, z)) + \langle \alpha v_{i+1}, z \rangle = \tag{6.5}$$

$$= (q(\vartheta_i)\, \psi(v_i + v, \vartheta_i), z)\,, \quad z \in V\,,$$

$$v_{i+1}(0) = 0\,. \tag{6.6}$$

Existence of the function v_{i+1} follows from Theorem 5.1, since for the composite function

$$f_i(x, t) := q(\vartheta_i(x, t))\, \psi(v_i(x, t) + v(x, t), \vartheta_i(x, t))$$

we have $f_i \in H$ (see e.g. [9]).

Definition 6.1. *We shall say that the condition $(\mathcal{P})$ holds if there exists $M < \infty$ such that for $i = 1, 2, \ldots, t \in \langle 0, T \rangle$ and a.a. $x \in \Omega$ we have*

$$|f_{i+1}(x, t) - f_i(x, t)| + |\psi(v_{i+1} + v, \vartheta_{i+1}) - \psi(v_i + v, \vartheta_i)| \leqq \tag{6.7}$$

$$\leqq M(|v_{i+1}(x, t) - v_i(x, t)| + |\vartheta_{i+1}(x, t) - \vartheta_i(x, t)|)\,.$$

From now on we shall suppose that condition $(\mathcal{P})$ holds, since we are concerned with existence of a global solution on the whole interval $\langle 0, T \rangle$. If the validity of condition $(\mathcal{P})$ is not ensured, we can modify our proof in such a way (similarly as in [9]), that we prove existence of a local solution (on an interval $\langle 0, \Delta \rangle \subset \langle 0, T \rangle$ for a sufficiently small $\Delta > 0$).

For $i = 1, 2, \ldots$ let us denote

$$w_i := v_i - v_{i-1}\,, \quad \Theta_i := \vartheta_i - \vartheta_{i-1}\,, \quad \varphi_i(t) := \|w_i(t)\|_\infty + \|\Theta_i(t)\|_\infty\,.$$

Subtracting equations (6.4) and (6.5) for i, $i + 1$ and using Theorem 5.3 and condition $(\mathcal{P})$ we deduce

$$\varphi_i(t) \leqq c \int_0^t \varphi_{i-1}(s)\, ds\,, \tag{6.8}$$

where the constant c is independent of i. Then

$$\varphi_i(t) \leqq \max \varphi_1(t)\, (ct)^{i-1}/(i - 1)!\,, \quad i = 1, 2, \ldots\,. \tag{6.9}$$

Consequently, v_i, ϑ_i are Cauchy sequences in $C(\langle 0, T\rangle, L_\infty(\Omega))$ and there exist their limits u, $\vartheta \in C(\langle 0, T\rangle, L_\infty(\Omega))$. Passing to the limit in (6.4) we immediately obtain

$$\vartheta(t) = \int_0^t \psi(u(s) + v(s), \vartheta(s))\, \mathrm{d}s\,,$$

so $\vartheta \in C^{(1)}(\langle 0, T\rangle, L_\infty(\Omega))$ and the second equation (6.3) of the problem $(\mathscr{M}_0)$ holds.

Let $\omega(t, h) := \exp(t^2/(t^2 - h^2))$, $|t| < h$ and $\omega(t, h) := 0$, $|t| \geqq h$, denote $\varkappa := \int \omega(t, 1)\, \mathrm{d}t$ and define

$$v_{i,h}(x, t) := \frac{1}{\varkappa h} \int v_i(x, s)\, \omega(t - s, h)\, \mathrm{d}s\,,$$

where we take $v_i(s) = 0$ for $s \notin \langle 0, T\rangle$. Then

$$v_{i,h} \in C^\infty((0, T), W^{1,2}(\Omega))\,, \quad (v_{i,h})' = (v_i')_{,h}$$

and

$$(6.10) \qquad (w'_{i,h}, z) + ((w_{i,h}, z)) + \langle \alpha w_{i,h}, z\rangle = ((f_{i-1} - f_{i-2})_{i,h}, z)\,, \quad z \in V.$$

(For the sake of simplicity we suppose $\varrho \equiv 1$, λ, α are constant. In the general case we should add to (6.10) terms of the type $\langle (\alpha w_i)_{,h} - \alpha w_{i,h}, z\rangle$ which conerge to zero as $h \to 0$; therefore we should obtain the same results.)

We can differentiate (6.10) with respect to t:

$$(6.11) \qquad ((w_{i,h})'', z) + ((w'_{i,h}, z)) + \langle \alpha w'_{i,h}, z\rangle = ((f_{i-1} - f_{i-2})'_{i,h}, z)\,.$$

By virtue of condition $(\mathscr{P})$ we derive

$$(6.12) \qquad |((f_{i-1} - f_{i-2})'_{,h}, z)| \leqq \frac{M}{\varkappa h} \int_\Omega |z(x)|\, \mathscr{I}(x, t)\, \mathrm{d}x\,,$$

where

$$\mathscr{I}(x, t) := \int_0^T \{|w_{i-1}(x, s)| + |\Theta_{i-1}(x, s)|\} \left|\frac{\partial \omega}{\partial s}(t - s, h)\right| \mathrm{d}s\,.$$

Because $w_{i-1} \in W^{1,2}(\Omega \times (0, T))$, the functions $|w_{i-1}(x, \cdot)|$, $|\Theta_{i-1}(x, \cdot)|$ are absolutely continuous for a.a. $x \in \Omega$. Therefore we can apply integration by parts to $\mathscr{I}(x, t)$ separately on $(0, t)$ (where $\partial\omega/\partial s \geqq 0$) and on (t, T) (where $\partial\omega/\partial s \leqq 0$). By means of the inequality $|\,|\varphi|'| \leqq |\varphi'|$ we obtain, after some calculation,

$$(6.13) \qquad \frac{1}{\varkappa h} \mathscr{I}(x, t) \leqq \frac{2}{\varkappa h} \int_0^T \{|w'_{i-1}(x, s)| + |\Theta'_{i-1}(x, s)|\}\, \omega(t - s, h)\, \mathrm{d}s \leqq$$

$$\leqq 2\{(w'^2_{i-1})_{,h} + (\Theta'^2_{i-1})_{,h}\}^{1/2}\,.$$

If we take $z = w'_{i,h}(t) \in V$ in (6.11), we derive by means of (6.12), (6.13):

$$(6.14) \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w'_{i,h}(t)\|_0^2 \leqq c\left\{\|w'_{i,h}(t)\|_0^2 + \int_\Omega \{(w'^2_{i-1})_{,h} + (\Theta'^2_{i-1})_{,h}\}(x,t)\,\mathrm{d}x\right\}.$$

Integration of (6.14) with respect to t gives, after $h \to 0$, the inequality

$$(6.15) \qquad \|w'_i(t)\|_0^2 \leqq c\int_0^t \{\|w'_i(s)\|_0^2 + \|w'_{i-1}(s)\|_0^2 + \|\Theta'_{i-1}(s)\|_0^2\}\,\mathrm{d}s.$$

Making use of the Gronwall lemma we obtain from (6.15)

$$(6.16) \qquad \|w'_i(t)\|_0^2 \leqq c\int_0^t (\|w'_{i-1}(s)\|_0^2 + \|\Theta'_{i-1}(s)\|_0^2)\,\mathrm{d}s.$$

Since $|\Theta'_{i-1,h}| \leqq M(|w_{i-2}|_{,h} + |\Theta_{i-2}|_{,h})$ and (6.9) holds we can show

$$(6.17) \qquad \|\Theta'_{i-1,h}(t)\|_0^2 \leqq c_1(c_2t)^{i-1}/(i-1)!\,.$$

Now the convergence of the sequence v'_j in $C(\langle 0, T\rangle, L_2(\Omega))$ follows from (6.16), (6.17).
Finally, if we take $z = w_i(t) \in V$ in the equation for w_i, we can derive

$$((w_i(t), w_i(t))) + \langle \alpha(t)\, w_i(t), w_i(t)\rangle \leqq c_1(c_2t)^{i-1}/(i-1)!\,.$$

Therefore $\{v_i\}$ converges in $C(\langle 0, T), W^{1,2}(\Omega))$ and we can pass to the limit $i \to \infty$ in (6.5). But the limit is the first equation (6.2) of the problem $(\mathscr{M}_0)$. Thus we have proved the following

Theorem 6.2. *Let condition $(\mathscr{P})$ hold. Then there exists a solution of the problem $(\mathscr{M})$.*

The following assertion is evident:

Theorem 6.3. *Let q, ψ be uniformly Lipwchitz continuous and bounded functions. Then there exists a global solution of the problem $(\mathscr{M})$.*

The forthcoming assertions are based on a detailed investigation of the previously derived iteration process (6.4)–(6.6) under some additional conditions. Therefore we shall keep the notation v, v_i, ϑ_i used in the proof of Theorem 6.2.

Definition 6.4. *We shall say that the apriori estimate holds if there exist $A > 0$, $v_0 \in H_0$, $\vartheta_0 \in C^{(1)}(\langle 0, T\rangle. L_2(\Omega))$ such that the iteration process (6.4), (6.5) satisfies the estimate*

$$(6.18) \qquad \|v_i(t)\|_\infty \leqq A\,, \quad i = 0, 1, 2, \ldots, \quad t \in \langle 0, T\rangle\,.$$

Theorem 6.5. *Let ψ be bounded in τ (Definition 4.1) and let the apriori estimate hold. Then there exists a global solution of the problem $(\mathscr{M})$.*

Proof. Since $\|v_i + v\|_\infty \leqq A + \|v\|_\infty =: B$, we have by (6.4),

$$(4.6) \qquad \|\vartheta_i(t)\|_\infty \leqq m(B)\, T \quad \text{and} \quad \vartheta_i\,, \quad i = 1, 2, \dots$$

are bounded, too. The condition $(\mathscr{P})$ then follows from the uniform Lipschitz condition for ψ, q on bounded sets and Theorem 6.3 yields a global solution of $(\mathscr{M})$. ■

Theorem 6.7. *Let $\psi = \psi(u)$, $v_\infty := \max \|v(t)\|_\infty$ and let q be nondecreasing on $\langle 0, z_0 \rangle$, $z_0 > 0$. Let $u(t)$, $\tau(t)$ satisfy*

$$u' = q(\tau)\, \psi(u + v_\infty)\,, \quad u(0) = 0\,,$$
$$\tau' = \psi(u + v_\infty)\,, \qquad \tau(0) = 0\,,$$

and let T_0 be such that $\tau(T_0) = z_0$. Then the apriori estimate holds on the interval $\langle 0, T_0 \rangle$ and consequently, the problem $(\mathscr{M})$ has a solution at least for $t < T_0$.

Proof. Let $v_0 \equiv 0$, $\vartheta_0 \equiv 0$ and suppose

$$(6.19) \quad 0 \leqq v_i(x, t) \leqq u(t)\,, \quad 0 \leqq \vartheta_i(x, t) \leqq \tau(t)\,, \quad t \in \langle 0, T_0 \rangle\,, \quad i = 0, 1, 2, \dots, n$$

(this is true for $n = 0$). Then

$$q(\vartheta_n(x, t))\, \psi(v_n(x, t) + v(x, t)) \leqq q(\tau(t))\, \psi(u(t) + v_\infty)$$

since $\vartheta_n \leqq \tau \leqq z_0$ and q is nondecreasing.

Therefore $w := u - v_{n+1}$ satisfies

$$(\varrho w', z) + ((w, z)) + \langle \alpha w, z \rangle \geqq 0\,, \quad z \in V\,,$$

and by Theorem 5.3 we have $w \geqq 0$, i.e. $v_{n+1} \leqq u$.

Evidently

$$\vartheta_{n+1}(x_1, t) = \int_0^t \psi(v_n(x, s) + v(x, s))\, \mathrm{d}s \leqq \int_0^t \psi(u(s) + v_\infty)\, \mathrm{d}s = \tau(t)\,,$$

and therefore (6.19) holds for $n + 1$ as well. But $u(t) \leqq \mu$ and the apriori estimate holds. ■

Theorem 6.8. *Let $\psi(u) = a \exp(bu)$, $a > 0$, $b > 0$, and let q be nonincreasing. Let*

$$\gamma_1 := \min v(x, t)\,, \quad \gamma_2 := \max \dot{v}(x, t)\,, \quad (x, t) \in \Omega \times (0, T)\,,$$

and let T^* be such that

$$Q(a e^{b\gamma_1} T^*) = e^{b(\gamma_1 - \gamma_2)}/b$$

(if $\mu \leqq e^{b(\gamma_1 - \gamma_2)}/b$ we define $T^* := +\infty$).

Then the apriori estimate holds on $\langle 0, T^* - \varepsilon \rangle$ for all $\varepsilon > 0$ and consequently, the problem $(\mathscr{M})$ has a solution at least for $t < T^*$.

Proof. Let $v_0 \equiv 0$, $\vartheta_0 \equiv 0$ and let $u(t)$ be the solution of the equation (4.7) from Example 4.6 with $\alpha := \exp(b\gamma_1)$, $c := \gamma_2$; u exists on $\langle 0, T^*)$. Suppose

$$0 \leqq v_i(x, t) \leqq u(t), \quad i = 0, 1, \ldots, n, \tag{6.20}$$

which is true for $n = 0$. Then

$$\vartheta_n(x, t) = \int_0^t \psi(u_{n-1}(x, s) + v(x, s))\,\mathrm{d}s \geqq a\mathrm{e}^{b\gamma_1}t$$

and because of the monotonicity of q,

$$q(\vartheta_n(x, t))\,\psi(v_n(x, t) + v(x, t)) \leqq q(\alpha a t)\,\psi(u(t) + \gamma_2).$$

Making use of Theorem 5.3 we see that (6.20) holds for all n. ∎

Theorem 6.8 ensure global existence for the problem $(\mathscr{M})$ (for all $t \geqq 0$) only for sufficiently small b ($T^* = +\infty$). However, numerical experiments for the particular case $Q(z) := \operatorname{arctg}(z)$ show that the solution of the problem remains bounded even in the case $T^* < \infty$.

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Souhrn

O ŘEŠENÍ ROVNICE PRO VEDENÍ TEPLA S NELINEÁRNÍ NEOMEZENOU PAMĚTÍ

ALEXANDR DOKTOR

Práce se zabývá otázkou globálního řešení u, τ okrajové úlohy pro soustavu semilineární rovnice vedení tepla pro u a doplňkovou nelineární diferenciální rovnici

pro τ („tepelná paměť"). Je dokázána jednoznačnost řešení a dále je metodou postupných aproximací dokázána existence globálního řešení za předpokladu podmínky ($\mathscr{P}$). Podmínka ($\mathscr{P}$) je ověřena pro některé speciální případy (např.: omezené nelineární funkce, homogenní Neumannova úloha (i v případě neomezených nelinearit), platí-li apriorní odhad pro řešení).

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