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TOWARDS A NOTION OF TESTABILITY

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Summary. The problem of testability has been undertaken many times in the context of linear hypotheses. Almost all these considerations restricted to some algebraical conditions without reaching the nature of the problem. Therefore, a general and commonly acceptable notion of testability is still wanted.

Our notion is based on a simple and natural decision theoretic requirement and is characterized in terms of the families of distributions corresponding to the null and the alternative hypothesis. Its consequences in the case of linear hypotheses are discussed. Among other it is shown that some suggestions in statistical literature are unjustified.

Keywords: general statistical hypothesis, linear hypothesis, strict unbiasedness, testability

1. INTRODUCTION

Most of well known statistical notions, such as unbiasedness or admissibility, originated from the estimation theory. Afterwards they were implanted in testing statistical hypotheses. According to the prevailing opinion, the notion of estimability should also have its counterpart in testing.

Intuitively, one can consider a statistical hypothesis to be testable if there exists a *reasonable test*, that is a test satisfying some *essential condition*. In order to reach the essential condition let us make a thorough study of the way leading to the parallel condition in the case of estimation.

Suppose we are interested in estimation (with quadratic loss) for the parameter σ^2 by using n independent observations with a normal distribution $\mathcal{N}(\mu, \sigma^2)$, where μ and σ^2 are unknown. For a change we will also consider the same problem under the assumption $\mu = 0$. It is well known that $\sum(x_i - \bar{x})^2/(n-1)$ and $\sum x_i^2/n$ are the

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minimal variance unbiased estimators in the first and the second case, respectively. Referring the first estimator to the second case we note that it is still unbiased but its variance is no longer minimal. Resuming, if we identify the *essential condition* in the estimation problem with ***unbiasedness*** then any estimator being reasonable in the initial model is also reasonable in its submodel. However, if we identify the essential condition with ***unbiasedness plus minimal variance*** then the property is not preserved. The principle that any statistical rule satisfying the essential condition in the initial model does satisfy the condition also in any submodel of the model becomes the basic prerequisite in the notion of estimability. The same principle leads us towards the notion of testability.

2. General background

Consider a statistical decision problem $(\mathscr{E}, \mathscr{D}, \mathscr{R})$, where $\mathscr{E} = (\mathscr{X}, \mathscr{A}, P_\theta; \theta \varepsilon \Theta)$ is a statistical experiment, $\mathscr{D}$ is the set of statistical rules, while $\mathscr{R} = \mathscr{R}(\theta/d)$ is the risk function of a rule $d \varepsilon \mathscr{D}$. We will also consider induced problems of the form $(\mathscr{E}_/, \mathscr{D}, \mathscr{R}_/)$ arising by restriction of the parameter set Θ to a subset $\Theta_/$ and by limiting the domain of the risk function $\mathscr{R}(\cdot/d)$ to the set $\Theta_/$.

Let C be a condition referring to statistical rules in $\mathscr{D}$ and involving, directly or indirectly, a corresponding statistical decision problem. The condition will be considered both in the initial problem $(\mathscr{E}, \mathscr{D}, \mathscr{R})$ and in the induced problems $(\mathscr{E}_/, \mathscr{D}, \mathscr{R}_/)$. Any induced problem can be also identified with a subset $\Theta_/$ of the parameter set Θ.

Definition 1. The condition C is said to be *hereditary* in the family of statistical decision problems induced by $(\mathscr{E}, \mathscr{D}, \mathscr{R})$ if any decision rule $d \varepsilon \mathscr{D}$ satisfying C in the initial problem satisfies also the condition in any nontrivial subproblem $(\mathscr{E}_/, \mathscr{D}, \mathscr{R}_/)$ induced by $(\mathscr{E}, \mathscr{D}, \mathscr{R})$.

Remark 1. We realize that the term "nontrivial" used in the definition needs specification. It would be difficult to settle this in general but it is easy to do it in each particular case. For instance, in the estimation problem, all nontrivial subproblems are determined by those subsets $\Theta_/$ which contain at least two elements. Similarly, in the problem of testing a hypothesis $H: \theta \varepsilon \Theta_H$ against $K: \theta \varepsilon \Theta_K$, all nontrivial subproblems correspond to such subsets $\Theta_/$ of Θ that both intersections $\Theta_H \cap \Theta_/$ and $\Theta_K \cap \Theta_/$ are nonempty.

Example 1. Condition of unbiasedness in estimation of a parameter θ with quadratic risk. Recall that an estimator $d \varepsilon \mathscr{D}$ is unbiased if the expectation $E_\theta d(X) = \theta$ for all θ. The condition is hereditary.

E x a m p l e 2. Condition of unbiasedness in testing a hypothesis $H: \theta \varepsilon \Theta_H$ against $K: \theta \varepsilon \Theta_K$. Recall that a randomized test φ is unbiased if $\sup_{\theta \varepsilon \Theta_H} E_\theta \varphi(X) \leqslant \inf_{\theta \varepsilon \Theta_K} E_\theta \varphi(X)$. The condition is hereditary.

E x a m p l e 3. Condition of admissibility in a statistical decision problem $(\mathscr{E}, \mathscr{D}, \mathscr{R})$. Recall that a rule $d \varepsilon \mathscr{D}$ is admissible if there is no $d' \varepsilon \mathscr{D}$ such that $\mathscr{R}(\theta/d') \leqslant \mathscr{R}(\theta/d)$ for all θ with the strict inequality for some θ_0. In general, this condition is not hereditary (cf. Stępniak, 1987, for an illustration of the fact in linear estimation).

3. Notion of testability

Let X be the observation vector in a statistical experiment $\mathscr{E} = (\mathscr{X}, \mathscr{A}, P_\theta, \theta \varepsilon \Theta)$. Consider the problem of testing a hypothesis $H: \theta \varepsilon \Theta_H$, where Θ_H is a nontrivial subset of Θ, against an alternative $K: \theta \varepsilon \Theta_K$, where $\Theta_K = \Theta \backslash \Theta_H$. The problem can be treated as a statistical decision problem $(\mathscr{E}, \mathscr{D}, \mathscr{R})$, where $\mathscr{D}$ is the set of all randomized tests φ, i.e. measurable functions from the space $(\mathscr{X}, \mathscr{A})$ into the interval $[0, 1]$, and

$$\mathscr{R}(\theta/\varphi) = \begin{cases} a_0 \int_{\mathscr{X}} \varphi(x) \, \mathrm{d}P_\theta(x) & \text{if } \theta \in \Theta_H \\ a_1 \int_{\mathscr{X}} [1 - \varphi(x)] \, \mathrm{d}P_\theta(x) & \text{if } \theta \in \Theta_K, \end{cases}$$

where a_0 and a_1 are given positive scalars.

As we have mentioned in Section 1, our problem of testability reduces to some *essential condition* C on φ. We have already postulated that such a condition should be *hereditary*. In particular, any test φ satisfying the condition C should also satisfy C for any pair of simple hypotheses $H: \theta = \theta_0$ against $K: \theta = \theta_1$, where $\theta_0 \varepsilon \Theta_H$ and $\theta_1 \varepsilon \Theta_K$. Let us consider the particular case with a greater care.

In this case any test φ may be characterized by two numbers $E_{\theta_0} \varphi(X)$ and $E_{\theta_1} \varphi(X)$ corresponding to the expected probability of rejection of the hypothesis H under the condition that H is true of false, respectively. It is clear that any reasonable test should satisfy the condition

$$E_{\theta_0} \varphi(X) < E_{\theta_1} \varphi(X).$$

(By the Fundamental Neyman-Pearson Lemma such a test exists if and only if $P_{\theta_0} \neq P_{\theta_1}$.)

The consideration leads us to the following definition.

Definition 2. A hypothesis $H: \theta \in \Theta_H$ is said to be *testable* under an alternative $K: \theta \in \Theta_K$ if there exists a test φ such that

$$\text{(C)} \qquad E_\theta\varphi(X) < E_{\theta'}\varphi(X) \quad \text{for any } \theta \in \Theta_H \text{ and } \theta' \in \Theta_K.$$

Remark 2. The same conclusion is drawn in Stępniak (1980) in another way.

Remark 3. Any test φ satisfying the condition (C) is said to be strictly unbiased (ibid.).

It was shown by Stępniak (1980, Th. 1) that a necessary condition for testability of the hypothesis $H: \theta \in \Theta_H$ against $K: \theta \in \Theta_K$ is that the convex hulls of the distributions P_θ with $\theta \in \Theta_H$ and the distributions $P_{\theta'}$ with $\theta' \in \Theta_K$ are disjoint. This is also a sufficient condition for testability under some additional assumption, for instance if the set Θ is finite.

Let us end the section by two examples.

Example 4. Testability in the Behrens-Fisher problem. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ $(m, n \geqslant 2)$ be independent random variables with two normal distributions $\mathcal{N}(\mu, \sigma^2)$ and $\mathcal{N}(\eta, \tau^2)$, where μ, η, σ^2 and τ^2 are unknown. Consider a hypothesis $H: \eta = \mu$ against $K: \eta \neq \mu$.

Let us define $Z_i = Y_i - X_i$, $i = 1, \ldots, k$, where $k = \min(m, n)$. Let $\varphi = \varphi(z)$ be the usual t-test for the hypothesis H_0: $E(Z_i) = 0$, $i = 1, \ldots, k$. We note that the test is strictly unbiased in the initial problem. Therefore the hypothesis considered in the Behrens-Fisher problem is testable.

Example 5. Testing non-centrality of χ^2 distribution. Suppose a random variable X has a χ^2 distribution with n degrees of freedom and the non-centrality parameter λ, where the positive integer n and $\lambda \geqslant 0$ are unknown. Let us consider a hypothesis $H: \lambda = 0$. It is known that the density function $f(x)$ of the χ^2 distribution with n degree of freedom and the non-centrality parameter λ can be presented in the form

$$f(x) = \sum_{k=0}^{\infty} p_k(\lambda) f_{2k+r}(x)$$

where $p_k(\lambda) = \frac{(\lambda^2/2)^k e^{-\lambda^2/2}}{k!}$ and f_p is the density function of the central χ^2 distribution with p degrees of freedom (cf. Lehmann, 1959, Sec. VII, Problem 1). Therefore the non-central χ^2 is a mixture of central χ^2's. Hence the hypothesis H is not testable.

In this section the usual matrix notation will be used. Among other, if M is a matrix, then M', $r(M)$, $R(M)$ and P_M will denote the transpose, the rank, the range (column space) of M and the orthogonal projector on $R(M)$, respectively.

Let us consider a linear normal model $Y \to \mathcal{N}(A\beta, \sigma^2 I)$, where Y is an observable random vector in R^n, while A is the design matrix of size $n \times p$ such that $r(A) < p$. Roy & Roy (1959) noted that some hypotheses of the form $H: C\beta = 0$ against $K: C\beta \neq 0$ do not possess a reasonable test in such a model. A more complete interpretation of the fact was given by Seely (1977). His argument throws a lot of light on the nature of the problem and, in consequence, shows that some suggestions in statistical literature are unjustified.

It appears that the sets $\{A\beta: C\beta = 0\}$ and $\{A\beta: C\beta \neq 0\}$ of possible means of the random vector Y under H and K, respectively, are not disjoint, unless $R(C') \subseteq R(A')$. Conversely, if $R(C') \subseteq R(A')$ then the hypothesis H is testable in the sense of Definition 2 because the usual F-test is strictly unbiased.

The difficulty concerning the disjointness can be overcome by considering a testable hypothesis which is *implied* by H (cf. Stępniak, 1984). Any such implication is of the form $H_0 : C_0\beta = 0$, where C_0 is an arbitrary matrix with p columns such that $R(C_0') \subseteq R(A') \cap R(C')$. In particular, if $AC' = 0$ then any testable hypothesis implied by H is trivial (i.e. $0\beta = 0$); otherwise one can construct a non-trivial testable hypothesis H_0. In fact we are interested in a ***maximal*** testable hypothesis implied by H (for definition see Stępniak, 1984). Such a hypothesis can be presented in the form

$$H^*: [I_n - P_{A(I_p - P_{C'})}]A\beta = 0. \tag{1}$$

Moreover, the uniformly most powerful invariant test for H^* has the rejection region defined by

$$\frac{\|[P_A - P_{A(I_p - P_{C'})}]Y\|}{\|(I_n - P_A)Y\|} > C(\alpha), \tag{2}$$

where $\|\cdot\|$ is the usual norm in R^n and $C(\alpha)$ is a given constant dependent on the significance level α.

It is worth noting that the formulae (1) and (2) simplify essentially when testing a hypothesis of the form

$$H: \beta^{(1)} = 0, \tag{3}$$

where $\beta^{(1)}$ is a subvector of β. To be more precise let us consider a partitioned model $Y \rightarrow \mathscr{N}(A_1\beta^{(1)} + A_2\beta^{(2)}, \sigma^2 I_n)$. In this case the formulae (1) and (2) for the hypothesis (3) can be replaced by

$$(1') \qquad H^*: (I_n - P_{A_2})A\beta = 0$$

and

$$(2') \qquad \frac{\|(P_A - P_{A_2})Y\|}{\|(I_n - P_A)Y\|} > C(\alpha),$$

respectively.

In 1986 Peixoto tried to extend the notion of testability to *all* hypotheses H having a nontrivial testable implication (cf. his Definition 2.1). According to him any such hypothesis is ***equivalent*** to the maximal testable hypothesis H^* implied by H (cf. his Definition 2.2). Really, the sets $\mathscr{P}_H$ and $\mathscr{P}_{H^*}$, corresponding to the possible distributions of the random vector Y under H and H^* coincide but the difference is just hidden in the sets corresponding to the alternative hypotheses.

The usual (logical) negation of $C\beta = 0$ is $C\beta \neq 0$. Let C^* be the coefficient matrix in the maximal testable hypothesis H^* implied by H. Consider the sets $\Theta_K = \{A\beta\colon C\beta \neq 0\}$ and $\Theta_{K^*} = \{A\beta\colon C^*\beta \neq 0\}$. Then $\Theta_K \neq \Theta_{K^*}$ unless $R(C') \subseteq R(A')$. To throw more light on the question let us consider the following example.

Example 6. Testing the hypothesis

$$H: \beta_1 = 0 \quad \text{and} \quad \beta_2 = 0$$

in the linear normal model

$$(4) \qquad Y \rightarrow \mathscr{N}(A\beta, \sigma^2 I),$$

where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The hypothesis can be written in the form (3) with $\beta^{(1)} = (\beta_1, \beta_2)$. Now we get

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$P_{A_2} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$P_A = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and, by (1′),

$$C^* = [0, 1, 0].$$

Moreover, we have

$$\Theta_K = \{A\beta\colon \beta_1 \neq 0 \text{ or } \beta_2 \neq 0\},$$
$$\Theta_{K^*} = \{A\beta\colon \beta_2 \neq 0\}.$$

In particular, the vector $v = (1, 0, 1)'$ does not belong to Θ_{K^*} and belongs to Θ_K. Moreover, the uniformly most powerful invariant test for the maximal testable hypothesis H^* implied by H has the rejection region defined by

$$\frac{Y_2^2}{(Y_1 - Y_3)^2} > C_1(\alpha). \tag{4}$$

It is worth noting that the distribution of the test function (4) does not depend on the parameters β_1 and β_3.

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