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Applications of Mathematics, Vol. 37 (1992), No. 5, 383–400

Persistent URL: `http://dml.cz/dmlcz/104518`

MULTIPOLAR VISCOELASTIC MATERIALS AND THE SYMMETRY OF THE COEFFICIENTS OF VISCOSITY

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(Received March 3, 1992)

Summary. The integral constitutive equations of a multipolar viscoelastic material are analyzed from the thermodynamic point of view. They are shown to be approximated by those of the differential-type viscous materials when the processes are slow. As a consequence of the thermodynamic compatibility of the viscoelastic model, the coefficients of viscosity of the approximate viscous model are shown to have an Onsager-type symmetry. This symmetry was employed earlier in the proof of the existence of solutions for the corresponding equations.

Keywords: multipolar materials, hereditary laws, Onsager's relations

AMS classification: 73B05

1. Introduction

A multipolar viscoelastic material is characterized by a functional dependence of the multipolar stress tensors on the histories of the deformation gradients up to a fixed order. The thermodynamics of such materials was analyzed by Bucháček [3] under the assumption that the stresses depend on the histories of the deformation gradients through continuous and continuously differentiable functional on the Hilbert space of histories. Here I consider a much more specific type of dependence via single-integral laws recently introduced by Gurtin and Hrusa [8], see eq. (3.1) in Sct. 3.

The reason why I consider this class of materials is that for slow processes the integral functionals can be approximated, using the Coleman and Noll idea of retardation [4], by the differential-type viscous constitutive equations of multipolar materials. These were treated by Bleustein and Green [2] (dipolar fluids) and by Nečas and Šilhavý [12] (general multipolar fluids). In a series of papers, Nečas, Novotný, Šilhavý [9, 10], Nečas and Ruzicka [11], Novotný [13], Bellout, Bloom and Gupta [1] and others showed that these materials have many nice mathematical properties. In the existence theory a crucial role is played by a certain symmetry of the coefficients of viscosity and my goal is to show that this symmetry follows from the thermodynamic properties of the original "integral" model. The method of proof is similar to the one employed in Šilhavý [14] where it was shown that the Onsager symmetry relations can be derived for Navier-Stokes viscous materials with Fourier's heat conduction.

2. Processes in multipolar materials

I refer to Green and Rivlin [5, 6] to the systematic exposition of the thermomechanics of multipolar materials. In this paper, isothermal version of the theory will be considered and a reference description will be used. Let $\Omega \subset \mathbf{R}^3$ denote the region occupied by the body in the reference configuration; the material points are identified with their positions $X = (X_A) \in \Omega$ $(A = 1, 2, 3)$ in this configuration. For simplicity the reference density is set equal to 1.

Let $P \geqslant 1$ be an integer. *A process of a multipolar body* of polarity 2^{P-1} is a collection $(\chi, \psi, \sigma^k, b)$ of $3 + P$ functions $(k = 0, \ldots, P-1)$ of $X \in \Omega$ and $t \in \mathbf{R}$ whose interpretation and tensorial nature is as follows:

(1) the vector-valued function $\chi = (\chi^i)$ $(i = 1, 2, 3)$ is the motion of the body;

(2) the scalar-valued function ψ is the specific free energy;

(3) the vector-valued function b is the specific external body force;

(4) for every $k = 0, \ldots, P-1$, the tensor-valued function $\sigma^k = (\sigma^k_{iA_1\ldots A_kA})$ $(i = 1, 2, 3, A_1, \ldots, A_k, A = 1, 2, 3)$ of order $k+2$ is the multipolar referential stress tensor; it is assumed that $\sigma^k_{iA_1\ldots A_kA}$ is symmetric with respect to the permutation of the indices $A_1, \ldots, A_k$. The symmetry is motivated by the fact that σ^k enters the basic equations only through its scalar product with the k-th referential gradient of velocity which has the same type of symmetry.

For a given motion χ and $k \geqslant 0$ an integer we introduce the *k-th deformation gradient* F^k to be tensor of order $k+2$ given by

$$F^k = \nabla^{k+1}\chi = (\chi_{i,A_1\ldots A_{k+1}}) \tag{2.1}$$

where ∇ denotes the gradient with respect to X and the comma followed by an index A (or indices) denotes the partial differentiation with respect to X_A (or the corresponding higher-order partial differentiation). F^k is symmetric in its referential indices, i.e., in $A_1, \ldots, A_{k+1}$.

The basic equations are the *equation of balance of linear momentum*

$$\ddot{\chi} = \operatorname{Div} \sigma^0 + b, \tag{2.2}$$

and the reduced form of the *equation of balance of angular momentum*

$$(\sigma^0 + \operatorname{Div} \sigma^1) F^T = F(\sigma^0 + \operatorname{Div} \sigma^1)^T, \tag{2.3}$$

where Div denotes the referential divergence and the superscript T the transposition. For isothermal processes the second law of thermodynamics takes the form of the *dissipation inequality*

$$\left(\psi + \frac{1}{2} v^2\right) \leqslant \operatorname{Div} \left(\sum_{k=0}^{P-1} \sigma^k . \nabla^k v \right) + v \cdot b, \tag{2.4}$$

where $v = \dot{\chi}$ is the velocity, and the product $\sigma^k \cdot \nabla^k v$ is a referential vector with components

$$(\sigma^k \cdot \nabla^k v)_A = \sigma^k_{iA_1 \ldots A_k A} v_{i, A_1 \ldots A_k}. \tag{2.5}$$

A combination of (2.2) with (2.4) provides the reduced dissipation inequality

$$\dot{\psi} \leqslant \sum_{k=0}^{P-1} (\sigma^k + \operatorname{Div} \sigma^{k+1}) \cdot \nabla^{k+1} v, \tag{2.6}$$

where we set $\sigma^m = 0$ for $m \geqslant P$; in indices,

$$\dot{\psi} \leqslant \sum_{k=0}^{P-1} (\sigma^k_{iA_1 \ldots A_{k+1}} + \sigma^{k+1}_{iA_1 \ldots A_{k+1} A, A}) v_{i, A_1 \ldots A_{k+1}}. \tag{2.7}$$

In view of the symmetry of $\nabla^{k+1} v$ in its referential indices, we see that only the symmetric part of $\sigma^k + \operatorname{Div} \sigma^{k+1}$ is relevant to the dissipation inequality (2.6), (2.7). Noting that $\operatorname{Div} \sigma^{k+1} = (\sigma^{k+1}_{iA_1 \ldots A_{k+1} A, A})$ is symmetric in $A_1, \ldots, A_{k+1}$, we introduce the tensor $\beta^k = (\beta^k_{iA_1 \ldots A_{k+1}})$ by

$$\beta^k = \operatorname{Sym} \sigma^k + \operatorname{Div} \sigma^{k+1}, \tag{2.8}$$

where Sym denotes the symmetrization with respect to the referential indices. The dissipation inequality then takes the form

$$\dot{\psi} \leqslant \sum_{k=0}^{P-1} \beta^k \cdot \dot{F}^k \tag{2.9}$$

where we have also used that

$$\nabla^{k+1} v = \dot{F}^k. \tag{2.10}$$

For a future use, we divide the stresses σ^k into the regular and singular parts $\sigma^{k,R}$, $\sigma^{k,S}$, respectively. By definition, we set $\sigma^{k,R} = 0$ for $k \geqslant P$. Then we define $\sigma^{P-1,R}$ by

$$\sigma^{P-1} = \beta^{P-1} \tag{2.11}$$

and for $k = N-2,\ N-2,\ \ldots,\ 0$ recursively by

$$\sigma^{k,R} = \beta^k - \operatorname{Div} \sigma^{k+1,R}. \tag{2.12}$$

Hence $\sigma^{k,R}$ is symmetric in its referential indices for every k. The singular part is defined by

$$\sigma^k = \sigma^{k,R} + \sigma^{k,S}. \tag{2.13}$$

As a consequence,

$$\operatorname{Sym} \sigma^{k,S} + \operatorname{Div} \sigma^{k+1,S} = 0, \tag{2.14}$$

$k = 0,\ 1,\ \ldots$.

We shall see that the regular part of the stress is completely determined by the free-energy functional for the material, while the singular part is completely arbitrary (subject to (2.14)). The following proposition shows that the singular part does not contribute to the balance of linear momentum; it can contribute only to the boundary conditions.

Proposition 2.1. *In every process compatible with the balance equations,*

$$\operatorname{Div} \sigma^{0,S} = 0. \tag{2.15}$$

P r o o f. Follows from (2.14), using the considerations described in [12], eqs. (5.28), (5.29), (5.42), (5.43). □

Consider a scalar-, vector,- or tensor-valued quantity μ which is determined by the history of the notion. We say that μ is given by a *single-integral law of grade* L ($L \geqslant 0$ an integer) if the value $\mu(X,t)$ of μ at $X \in \Omega$, $t \in \mathbf{R}$ is given by

$$(3.1) \qquad \begin{aligned} \mu(X,t) = M\big(f^0(X,t), \dots, F^{L-1}(X,t)\big) \\ + \int_0^\infty m\big(F^0(X,t), \dots, F^{L-1}(X,t), F^0(x,t-s), \dots, \\ F^{L-1}(X,t-s), s\big)\,\mathrm{d}s \end{aligned}$$

where the functions M and m are subject to the requirements to be stated below (cf. [8]). To simplify the notation, when dealing with the expressions like (3.1), we shall write μ for $\mu(X,t)$, F^0, ..., F^{L-1} for $F^0(X,t)$, ..., $F^{L-1}(X,t)$, and H^0, ..., H^{L-1} for $F^0(X,t-s)$, ..., $F^{L-1}(X,t-s)$, $(s>0)$, respectively. The typical argument of m is thus $(F^0,\dots,F^{L-1},H^0,\dots,H^{L-1},s)$ and the corresponding derivatives of m will be denoted by

$$(3.2) \qquad \frac{\partial m}{\partial F^0}, \; \dots, \; \frac{\partial m}{\partial F^{L-1}}, \; \frac{\partial m}{\partial H^0}, \; \dots, \; \frac{\partial m}{\partial H^{L-1}}, \; m',$$

respectively. We shall assume that the functions M, m are infinitely differentiable on their domains (in the case of m we take the open interval $(0,+\infty)$ as the domain of s) and that the following condition holds: let w stand for any partial derivative of m (of arbitrary order, including order 0), then we require that for every compact set C of $2L$-tuples $(F^0,\dots,H^{L-1})$ there exists an integrable function f on $(0,+\infty)$ such that

$$(3.3) \qquad (1+s)|w(F^0,\dots,H^{L-1},s)| \leqslant f(s)$$

for every $s>0$ and every $(F^0,\dots,H^{L-1}) \in C$. Finally we impose a normalization condition on m which requires that "m vanishes on the diagonal", i.e., that

$$(3.4) \qquad m(F^0,\dots,F^{L-1},F^0,\dots,F^{L-1},s) = 0$$

for every $(F^0,\dots,F^{L-1})$ and every $s>0$.

We shall consider only motions which are C^∞ and bounded on every interval $(-\infty,t)$ ($t \in \mathbf{R}$ arbitrary) in the sense that for every $t \in \mathbf{R}$ and $X \in \Omega$ there exists a neighbourhood Ω_0 of X, $\Omega_0 \subset \Omega$ such that every space-time derivative of χ is bounded on $\Omega_0 \times (-\infty, t)$. This in combination with the decay conditions on m implies that the integral in (3.1) converges and gives a C^∞ function of X and t.

Proposition 3.1. *If μ is given by a single integral law of grade L, then the referential gradient $\nabla\mu$ is given by a single-integral law of grade $L+1$,*

$$\nabla\mu = \tilde{M}(F^0, \dots, F^L) + \int_0^\infty \tilde{m}(F^0, \dots, F^L, H^0, \dots, H^L, s)\,\mathrm{d}s \tag{3.5}$$

where M and m are given by the chain rules

$$\tilde{M}(F^0, \dots, F^L) = \sum_{k=0}^{L-1} \frac{\partial M}{\partial F^k} F^{k+1} \tag{3.6}$$

$$\tilde{m}(F^0, \dots, F^L, H^0, \dots, H^L, s) = \sum_{k=0}^{L-1} \Big(\frac{\partial m}{\partial F^k} F^{k+1} + \frac{\partial m}{\partial H^k} H^{k+1} \Big). \tag{3.7}$$

This is obvious.

4. Constitutive equations and thermodynamic consequences

We now assume that the free energy and the multipolar stresses are given by the single-integral laws of grade L:

$$\psi = P(F^0, \dots, F^{L-1}) + \int_0^\infty p(F^0, \dots, F^{L-1}, H^0, \dots, H^{L-1}, s)\,\mathrm{d}s, \tag{4.1}$$

$$\sigma^k = S^k(F^0, \dots, F^{L-1}) + \int_0^\infty s^k(F^0, \dots, F^{L-1}, H^0, \dots, H^{L-1}, s)\,\mathrm{d}s, \tag{4.2}$$

$k = 0, 1, \dots, P-1$, with given constitutive functions

$$P,\ p,\ S^k,\ s^k. \tag{4.3}$$

For monopolar materials ($P = L = 1$) the constitutive equations of the above form are typical for models of viscoelastic type with relaxation. For multipolar materials constitutive equations qualitatively similar to (4.1), (4.2), but more general, were studied by Bucháček [3].

We note that as a consequence of eqs. (4.1), (4.2) and of Proposition 3.1 also the regular and singular parts of the stress and the quantity β^k is given by single-integral laws, generally of grade higher than L, since the differentiation is involved in the definitions of these quantities. In case of β^k only one divergence is involved and thus the grade of the single-integral law for β^k is $L+1$, and we shall write

$$\beta^k = B^k(F^0,\dots,F^L) + \int_0^\infty b^k(F^0,\dots,F^L,H^0,\dots,H^L,s)\,\mathrm{d}s \tag{4.4}$$

The grade of the single-integral laws for $\sigma^{k,R}$, $\sigma^{k,S}$ is generally $L+P-k-1$, as is easily checked by counting the numbers of differentiations in the recursive definitions (2.11)–(2.13). Below we shall obtain a more detailed information about the grades of the single-integral laws for β^k and $\beta^{k,R}$, but the grade of $\sigma^{k,S}$ will be seen to be arbitrary. With this in view we denote by N the lowest number such that $\beta^m = 0$ identically in all processes for $m \geqslant N$ and β^{N-1} not identically 0. This is also the lowest order such that the regular stresses of order $m+2$ vanish identically for $m \geqslant N$ and the stress $\sigma^{N-1,R}$ is not identically 0. We shall also denote by Q the lowest possible number m such that the constitutive functions P, p for the free energy are independent of F^m, F^{m+1}, ..., and at least one of P, p depends of on F^{m-1}. (The number Q is not related to the dependence of P, p on H^0, ..., H^{L-1}.)

The material specified by the constitutive functions (4.3) is said to be *compatible with thermodynamics* if the dissipation inequality (2.9) holds for every motion χ and every ψ, σ^k given by the constitutive equations (4.1), (4.2).

Proposition 4.1. *The material (4.3) is compatible with thermodynamics if and only if the following three assertions hold:*

(1) *The numbers N, Q defined above satisfy $N = Q$, B^k depends only on F^0, ..., F^{N-1}, and b^k depends only on F^0, ..., F^{N-1}, H^0, ..., H^{L-1}, s;*

(2) *the equations*

$$B^k = \frac{\partial P}{\partial F^k}, \quad b^k = \frac{\partial p}{\partial F^k} \tag{4.5}$$

fold for $k = 0$, ..., $N-1$ throughout the domains of the functions involved;

(3) *everywhere on the domain of p,*

$$p' \leqslant 0. \tag{4.6}$$

The assertion of this proposition is a direct specialization of the results of Gurtin and Hrusa [8] on the consequences of the abstract dissipation inequality.

Hence, if the material is compatible with thermodynamics, then the regular stress $\sigma^{k,R}$ is given by a single-integral law of the form

$$\begin{aligned}\sigma^{k,r} &= S^{k,R}(F^0,\dots,F^{2N-k-2}) \\ &\quad + \int_0^\infty s^{k,R}(F^0,\dots,F^{2N-k-2},H^0,\dots,H^{N+L-k-2},s)\,\mathrm{d}s\end{aligned} \tag{4.7}$$

where the constitutive functions $S^{k,R}$, $s^{k,R}$ are completely determined by the constitutive functions P, p via Proposition 4.1, Proposition 3.1 and the definitions (2.11), (2.12) of the regular stresses. In contrast to this, the constitutive functions for the singular stresses are completely arbitrary, subject only to the condition that (2.14) hold identically in all processes. Also note that Proposition 4.1 does not specify the number of the tensors H^k on which p may depend; this number may generally exceed N.

Proposition 4.2. *If the material is compatible with thermodynamics then the following three assertions hold:*

(1) *Everywhere on the domain of p, we have*

$$p \geqslant 0; \tag{4.8}$$

(2) *the equations*

$$\frac{\partial p}{\partial F^k} = 0 \quad (k = 0, \ldots, N-1), \tag{4.9}$$

$$\frac{\partial p}{\partial H^m} = 0 \quad (m = 0, \ldots, L-1), \tag{4.10}$$

$$\frac{\partial^2 p}{\partial F^k \partial H^m} = -\frac{\partial^2 p}{\partial F^k \partial F^m} \quad (k, m = 0, \ldots, N-1), \tag{4.11}$$

$$\frac{\partial^2 p}{\partial F^k \partial H^m} = 0 \quad (k = 0, \ldots, N-1, m = N, \ldots, L-1) \tag{4.12}$$

hold whenever the argument

$$(F^0, \ldots, F^{N-1}, H^0, \ldots, H^{L-1}, s) \tag{4.13}$$

is of the diagonal form

$$F^0 = H^0, \quad \ldots, \quad F^{N-1} = H^{N-1} \tag{4.14}$$

with H^N, …, H^{L-1} arbitrary;

(3) *the inequality*

$$\sum_{k,m=0}^{N-1} M^k \cdot \frac{\partial^2 p}{\partial F^k \partial H^m} \cdot M^m \geqslant 0 \tag{4.15}$$

holds for every diagonal argument and every collection M^0, …, M^{N-1} of tensor of orders 2, …, $N+1$, respectively.

Also these results are specializations of the results of Gurtin and Hrusa [8]. I only note that (4.8) is obtained by integrating (4.6). Combining (4.8) with the normalization condition that p vanish whenever the argument (4.13) is diagonal ((4.14)), we see that for every s the function $(F^0, \ldots, F^{N-1}, H^0, \ldots, H^{L-1}) \longmapsto p(F^0, \ldots, F^{N-1}, H^0, \ldots, H^{L-1}, s)$ has the minimum at every argument satisfying (4.14). Eqs. (4.9), (4.10) express the vanishing of the first partial derivatives at the minimum. Eqs. (4.11), (4.12) then follow by differentiating (4.9), (4.10) with respect to the obvious arguments. Finally, (4.15) is the positive-definite character of the second differential at the minimum.

5. Retardation theorem

For a given motion χ and $\alpha \in (0, 1]$ we define *α-related motion* χ_α by setting

$$\chi_\alpha(X, t) = \chi(X, \alpha t), \tag{5.1}$$

for all X and t. The deformation gradient F^k_α corresponding to χ_α then satisfies

$$F^k_\alpha(t) = F^k(\alpha t), \tag{5.2}$$

for all $k = 0,\ 1,\ \ldots$ and all t. For the general single-integral law (3.1) the evolution μ_α of the quantity μ corresponding to χ_α is given by

$$\begin{aligned} \mu_\alpha(t) = M\big(F^0_\alpha(t), \ldots, F^{L-1}_\alpha(t)\big) \\ + \int_0^\infty m\big(F^0_\alpha(t), \ldots, F^{L-1}_\alpha(t), F^0_\alpha(t-s), \ldots, F^{L-1}_\alpha(t-s), s\big)\, \mathrm{d}s, \end{aligned}$$

where the argument X has been suppressed.

The k-th *kinetic coefficient* K^k ($k = 0, \ldots, L-1$) for the single-integral law (3.1) of grade L is defined to be the function K^k of $F^0, \ldots, F^{L-1}$ given by

$$K^k(F^0, \ldots, F^{L-1}) = -\int_0^\infty \frac{\partial m}{\partial H^k}(F^0, \ldots, F^{L-1}, F^0, \ldots, F^{L-1}, s) s\, \mathrm{d}s. \tag{5.4}$$

The following proposition, the retardation theorem (cf. [4]), shows that for slow motions χ_α, $\alpha \longrightarrow 0$, the integral law (3.1) can be approximated by a differential-type law involving the kinetic coefficients. I refer to [14] for the proof of the retardation theorem within the context of the single-integral laws.

Proposition 5.1. *If χ is an arbitrary motion, then*

$$\mu_\alpha(t) = M\left(F_\alpha^0(t), \ldots, F_\alpha^{L-1}(t)\right) + \sum_{k=0}^{L-1} K^k\left(F_\alpha^0(t), \ldots, F_\alpha^{L-1}(t)\right) F_\alpha^k(t) + o(\alpha, t),$$

where

$$\frac{O(\alpha, \alpha t)}{\alpha} \to 0 \tag{5.6}$$

for $\alpha \to 0$ and every $t \in \mathbf{R}$.

The proposition thus associates with the "exact" constitutive equation (3.1) the approximate differential-type constitutive equation

$$\bar{\mu} = M(F^0, \ldots, F^{L-1}) + \sum_{k=0}^{L-1} K^k(F^0, \ldots, F^{L-1}) \dot{F}^k, \tag{5.7}$$

which approximates (3.1) in slow processes. Alternatively, (5.7) can be written

$$\bar{\mu} = M(F^0, \ldots, F^{L-1}) + \sum_{k=0}^{L-1} K^k(F^0, \ldots, F^{L-1}) \nabla^{k+1} v. \tag{5.8}$$

The evolutions according to the approximate constitutive equations will be systematically distinguished by superimposed bars. The first two terms on the right-hand side of (5.5) is just the evolution according to the approximate law (5.7) corresponding to the retarded motion χ_α,

$$\bar{\mu}_\alpha = M(F_\alpha^0, \ldots, F_\alpha^{L-1}) + \sum_{k=0}^{L-1} K^k(F_\alpha^0, \ldots, F_\alpha^{L-1}) \dot{F}_\alpha^k. \tag{5.9}$$

The rest of the paper is devoted to the study of the properties of the approximate constitutive equations.

We know (cf. Proposition 3.1) that if μ is given by the single-integral law (3.1) of grade L, then $\nabla\mu$ is given by a single-integral law (3.5) of grade $L + 1$.

Proposition 5.2. *The kinetic coefficients $\tilde{K}^0$, ..., $\tilde{K}^L$ of the single-integral law* (3.5) *are given by*

$$\tilde{K}^k = \nabla K^k + \mathrm{Sym}(K^{k-1} \otimes 1), \tag{5.10}$$

$k = 0, \ldots, L$. *Here* K^k $(k = 0, \ldots, L-1$ *are the kinetic coefficients of* (3.1) *and we have set* $K^{-1} = 0$, $K^L = 0$ *in* (5.10); ∇K^k *is the formal gradient of* K, *defined to be the function of* $F^0, \ldots, F^L$ *given by the chain rule*

$$\nabla K^k = \sum_{m=0}^{L-1} \frac{\partial K^k}{\partial F^m} F^{m+1}. \tag{5.11}$$

The proof is a direct computation of $\tilde{K}^k$ using the definition (5.4) and the formula (3.7) for the function $\tilde{m}$. The details are omitted.

Proposition 5.3. *If, for a given motion* χ, *the function* μ_α *is defined by (5.3) and the function* $\bar{\mu}_\alpha$ *by* (5.9), *then*

$$\nabla \mu_\alpha(t) = \nabla \bar{\mu}_\alpha(t) + o(\alpha, t), \tag{5.12}$$

where

$$\frac{o(\alpha, \alpha t)}{\alpha} \to 0 \tag{5.13}$$

for $\alpha \to 0$ *and* $t \in \mathbf{R}$.

That is, the approximation corresponding to the single-integral law for the gradient is the gradient of the approximation of the original single-integral law. This follows from (5.10).

6. The coefficients of viscosity

We now associate with the viscoelastic material of Sct. 4 the approximate material via the retardation theorem. As above, the values of the quantities given by the approximate constitutive equations will be denoted by the superimposed bars. Throughout the present and the subsequent sections it is assumed that the viscoelastic material (4.3) is compatible with thermodynamics.

We introduce the kinetic coefficients K^{km} $(m = 0, \ldots, L-1)$ for the constitutive equations for the tensors β^k $(k = 1, \ldots, N-1)$. We shall call them the *coefficients of viscosity*. According to the general definition, K^{km} is given by

$$K^{km}(F^0, \ldots, F^{L-1}) = -\int_0^\infty \frac{\partial b^k}{\partial H^m}(F^0, \ldots, F^{N-1}, F^0, \ldots, F^{L-1}, s) s \, ds. \tag{6.1}$$

Using the thermodynamical relation $(4.5)_2$ this can be rewriten as

$$K^{km} = -\int_0^\infty \frac{\partial^2 p}{\partial F^k \partial H^m}(F^0, \dots, F^{N-1}, F^0, \dots, F^{L-1}, s) s \, ds. \tag{6.2}$$

If $m \geqslant N$, then (4.12) tells us that the integral vanishes and hence

$$K^{km} = 0. \tag{6.3}$$

On the other hand, if $m = 0, \dots, N-1$, then the use of (4.11) gives finally

$$K^{km} = -\int_0^\infty \frac{\partial^2 p}{\partial F^k \partial F^m}(F^0, \dots, F^{N-1}, F^0, \dots, F^{L-1}, s) s \, ds. \tag{6.4}$$

This implies, among other things,

Proposition 6.1. *The approximate constitutive equations for β^k are*

$$\bar{\beta}^k = B^k(F^0, \dots, F^{N-1}) + \sum_{m=0}^{N-1} K^{km}(F^0, \dots, F^{L-1}) \dot{F}^m \tag{6.5}$$

for $k = 0, \dots, N-1$ and

$$\bar{\beta}^k = 0 \tag{6.6}$$

for $k \geqslant N$.

The interchangeability of the second partial derivatives in (6.4) implies a symmetry of the coefficients of viscosity. To state the result conveniently, we introduce, for each $F^0, \dots, F^{L-1}$, the bilinear form $\langle . , . \rangle$ as follows:

$$\langle V, W \rangle = \sum_{k,m=0}^{N-1} V^k \cdot K^{km} \cdot W^m \tag{6.7}$$

where V, W are N-tuples $V = (V^0, \dots, V^{N-1})$, $W = (W^0, \dots, W^{N-1})$ with entries V^k, W^k ($k = 0, \dots, N-1$) the tensors of order $k+2$, symmetric in the last $k+1$ indices. In indices,

$$\langle V, W \rangle = \sum_{k,m=0}^{N-1} K^{km}_{iA_1 \dots A_{k+1}; jB_1 \dots B_{m+1}} V^k_{iA_1 \dots A_{k+1}} W^m_{jB_1 \dots B_{m+1}}. \tag{6.8}$$

The symmetry resulting from the interchangeability of the partial derivatives in (6.4) says that

$$K^{km}_{jB_1\ldots B_{m+1};iA_1\ldots A_{k+1}} = K^{km}_{iA_1\ldots A_{k+1};jB_1\ldots B_{m+1}}. \tag{6.9}$$

Proposition 6.2. *Let F^0, …, F^{L-1} be given. Then the bilinear form $\langle .,. \rangle$ is symmetric and positive-semidefinite, i.e.,*

$$\langle V, W\rangle = \langle W, V\rangle \tag{6.10}$$

and

$$\langle V, V\rangle \geqslant 0 \tag{6.11}$$

for each V, W.

Proof. The symmetry is proved above. The positive-semidefiniteness follows from (6.4) and Proposition 4.2(3). □

Note that the positive-semidefiniteness of $\langle .,. \rangle$ can be obtained by a direct application of the dissipation inequality to the approximate constitutive equations [12]. However, the symmetry cannot be obtained in this way. The symmetry plays crucial role in the existence theory [9, 10].

Proposition 6.3. *For every $k = 0$, …, $N-1$,*

(1) *the approximate constitutive equations for the regular part of the stress have the form*

$$\bar\sigma^{k,R} = \bar\sigma^{k,R,E} + \bar\sigma^{k,R,V}, \tag{6.12}$$

where $\bar\sigma^{k,R,E}$, called the equilibrium part of the regular stress, depends only on F^0, …, F^{2N-k-2}, and $\bar\sigma^{k,R,V}$, called the viscous part of the regular stress, depends linearly on $\dot F^0$, …, $\dot F^{2N-k-2}$ with the coefficients depending on F^0, …, $F^{L+N-k-2}$;

(2) *for each motion,*

$$\bar\sigma^{k,R,E} + \operatorname{Div}\bar\sigma^{k+1,R,E} = B^k(F^0,\ldots,F^{N-1}), \tag{6.13}$$

$$\bar\sigma^{k,R,V} + \operatorname{Div}\bar\sigma^{k+1,R,V} = \sum_{m=0}^{N-1} K^{km}(F^0,\ldots,F^{L-1})\dot F^m. \tag{6.14}$$

In particular, we can deduce from (1) that $\bar\sigma^{k,R,V}$ depends linearly on ∇v, …, $\nabla^{2N-k-1}v$. This is an analogue of the results by Nečas and Šilhavý [12, § 5].

Proof. The definition (2.12) of $\sigma^{k,R}$ implies that the single-integral laws for $\sigma^{k,R}$, $\sigma^{k+1,R}$ and β^k satisfy

$$\sigma^{k,R} + \operatorname{Div} \sigma^{k+1,R} = \beta^k. \tag{6.15}$$

Using the interchangeability of the passage to the approximation via the retardation theorem with taking the gradients, as stated in Proposition 5.3, the approximate constitutive equation for $\sigma^{k,R} + \operatorname{Div} \sigma^{k+1,R}$ is seen to be

$$\bar{\sigma}^{k,R} + \operatorname{Div} \bar{\sigma}^{k+1,R}, \tag{6.16}$$

where $\bar{\sigma}^{k,R}$ and $\bar{\sigma}^{k+1,R}$ are given by the approximations to the single-integral laws for $\sigma^{k,R}$ and $\sigma^{k+1,R}$, respectively. On the other hand (6.15) implies that (6.16) must be equal to the approximate constitutive equation (6.5) for β^k. This gives

$$\bar{\sigma}^{k,R} + \operatorname{Div} \bar{\sigma}^{k+1,R} = B^k(F^0, \ldots, F^{N-1}) + \sum_{m=0}^{N-1} K^{km}(F^0, \ldots, F^{L-1}) \dot{F}^m, \tag{6.17}$$

$k = 0, \ldots, N-1$. We have $\sigma^{N,R} = 0$ by definition and hence $\bar{\sigma}^{N,R} = 0$. (6.17) for $k = N-1$ gives

$$\bar{\sigma}^{N-1,R} = B^N(F^0, \ldots, F^{N-1}) + \sum_{m=0}^{N-1} K^{km}(F^0, \ldots, F^{L-1}) \dot{F}^m, \tag{6.18}$$

Hence $\bar{\sigma}^{N-1,R}$ is of the form claimed in Assertion (1). Proceeding inductively, we obtain, omitting the obvious details, Assertion (1).

Having proved (1), we insert (6.12) into (6.17) to obtain

$$\begin{aligned} &\bar{\sigma}^{k,R,E} + \operatorname{Div} \bar{\sigma}^{k+1,R,E} + \bar{\sigma}^{k,R,V} + \operatorname{Div} \bar{\sigma}^{k+1,R,V} \\ &\quad = B^k(F^0, \ldots, F^{N-1}) + \sum_{m=0}^{N-1} K^{km}(F^0, \ldots, F^{L-1}) \dot{F}^m, \end{aligned} \tag{6.19}$$

This equation must hold for every motion.Since the time derivatives $\dot{F}^m$ can be chosen independently of the values of F^m, the equation splits into the parts linear in $\dot{F}^m$ and into the absolute term, which are (6.14) and (6.13) respectively. □

Analogous reasoning gives

Proposition 6.4. *The approximate constitutive equations for the singular stresses have the following properties:*

(1) *$\bar{\sigma}^{k,S}$ is the form*

$$\bar{\sigma}^{k,S} = \bar{\sigma}^{k,S,E} + \bar{\sigma}^{k,S,V}, \tag{6.20}$$

where $\bar{\sigma}^{k,S,E}$, called the equilibrium part of the singular stress, depends on $F^0, \ldots, F^{L+P-k-2}$, and $\bar{\sigma}^{k,S,V}$, called the viscous part of the singular stress, depends linearly on $\dot{F}^0, \ldots, \dot{F}^{L+P-k-2}$ with the coefficients depending on $F^0, \ldots, F^{L+P-k-2}$;

(2) *for each motion,*

$$\operatorname{Sym}\bar{\sigma}^{k,S,E} + \operatorname{Div}\bar{\sigma}^{k+1,S,E} = 0 \tag{6.21}$$

$$\operatorname{Sym}\bar{\sigma}^{k,S,V} + \operatorname{Div}\bar{\sigma}^{k+1,S,V} = 0 \tag{6.22}$$

$$\sum_{k=0}^{P-1} \operatorname{Div}(\bar{\sigma}^{k,S,E} \cdot \nabla^k v) = 0 \tag{6.23}$$

$$\sum_{k=0}^{P-1} \operatorname{Div}(\bar{\sigma}^{k,S,V} \cdot \nabla^k v) = 0 \tag{6.23}$$

We conclude this section with noting that for various boundary conditions the form $\langle .\,, . \rangle$ is related to the weak form of the equation of balance of linear momentum (2.2). For instance, if one considers a fixed configuration of the body and two velocity fields v, w such that their gradients up to order $N-1$ vanish on the boundary of Ω, then the regular viscous stresses $\bar{\sigma}^{k,R,V}(v)$ and $\bar{\sigma}^{k,R,V}(w)$ corresponding to v, w satisfy

$$\int_\Omega w \cdot \operatorname{Div}\bar{\sigma}^{0,R,V}(v)\,\mathrm{d}V = -\int_\Omega \langle Dv, Dw\rangle\ \mathrm{d}V = \int_\Omega v \cdot \operatorname{Div}\bar{\sigma}^{0,R,V}(w)\,\mathrm{d}V, \tag{6.25}$$

where $Dv = (\nabla v, \ldots, \nabla^N v)$, $Dw = (\nabla w, \ldots, \nabla^N w)$. Hence the first term in (6.25), which is obtained upon multiplying the equation of balance of linear momentum with the virtual velocity field w and integrating, can eventually be transformed into the symmetric form $\langle .\,, . \rangle$. The same applies to the boundary conditions considered in [9, 10]. (6.25) is obtained by repeated use of Green's formula, the boundary conditions, and (6.14).

Proposition 7.1. *The approximate constitutive equation for the energy is*

$$\bar{\psi} = P(F^0, \ldots, F^{N-1}) \tag{7.1}$$

That is, the kinetic coefficients for the single-integral law for the free energy vanish.

Proof. According to the general formula (5.4), the kinetic coefficients K_ψ^k for ψ are given by

$$K_\psi^k = -\int_0^\infty \frac{\partial p}{\partial H^k}(F^0, \ldots, F^{N-1}, F^0, \ldots, F^{L-1}, s) s \, ds, \tag{7.2}$$

$k = 0, \ldots, N-1$ and the derivative in the integrand vanishes in view of (4.10).

In the text proposition we set

$$\bar{\sigma}^{k,E} = \bar{\sigma}^{k,R,E} + \bar{\sigma}^{k,S,E}, \tag{7.3}$$

$$\bar{\sigma}^{k,V} = \bar{\sigma}^{k,R,V} + \bar{\sigma}^{k,S,V}. \tag{7.4}$$

□

Proposition 7.2. *Let χ be any motion and denote $Dv = (\nabla v, \ldots, \nabla^N v)$. Then the following relations hold:*

$$\dot{\bar{\psi}} = \sum_{k=0}^{N-1} \operatorname{Div}(\bar{\sigma}^{k,R,E} \cdot \nabla^k v) = \sum_{k=0}^{P-1} \operatorname{Div}(\bar{\sigma}^{k,E} \cdot \nabla^k v), \tag{7.5}$$

$$\langle Dv, Dv \rangle = \sum_{k=0}^{N-1} \operatorname{Div}(\bar{\sigma}^{k,R,V} \cdot \nabla^k v) = \sum_{k=0}^{P-1} \operatorname{Div}(\bar{\sigma}^{k,V} \cdot \nabla^k v), \tag{7.6}$$

$$\dot{\bar{\psi}} + \langle Dv, Dv \rangle = \sum_{k=0}^{N-1} \operatorname{Div}(\bar{\sigma}^{k,R} \cdot \nabla^k v) = \sum_{k=0}^{P-1} \operatorname{Div}(\bar{\sigma}^{k} \cdot \nabla^k v). \tag{7.7}$$

Eq. (7.5) is the generalized Gibbs equation for multipolar materials, (7.6) gives the connection of the form $\langle , \rangle$ with the power of viscous stresses, and (7.7) identifies $\langle Dv, Dv \rangle$ with the production of entropy. In the terminology of the linear thermodynamics of irreversible processes, the viscous part of the stress is given by a linear expression in the non-equilibrium parameters Dv. The symmetry of the form $\langle , \rangle$ is then the Onsager symmetry of the kinetic coefficients, a famous postulate of irreversible thermodynamics. Within the present scheme this is a consequence of the thermodynamic compatibility of the viscoelastic model.

Proof. (7.5) is obtained by combining (4.5), with (6.13) and (6.21). (7.6) follows from (6.14), (6.22) and the definition of $\langle .,. \rangle$. Finally, (7.7) is the sum of (7.5) and (7.6). $\square$

Acknowledgment. This research was supported by Grant No 11911 of the Czechoslovak Academy of Sciences.

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Souhrn

MULTIPOLÁRNÍ VISKOLEASTICKÉ MATERIÁLY A SYMETRIE KOEFICIENTŮ VISKOZITY

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V článku jsou z termodynamického hlediska analyzovány integrální konstitutivní rovnice multipolárních viskoelastických materiálů. Je ukázáno, že je lze aproximovat konstitutivními rovnicemi diferenciálního viskózního materiálu při pomalých dějích. Jako důsledek thermodynamických vlastností viskoelastického modelu je dokázáno, že koeficienty viskozit mají onsangerovskou symetrii. Tato symetrie byla dříve uplatněna při důkazu existence řešení příslušných rovnic.

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