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OSCILLATIONS OF A NONLINEARLY DAMPED EXTENSIBLE BEAM

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Summary. It is proved that any weak solution to a nonlinear beam equation is eventually globally oscillatory, i.e., there is a uniform oscillatory interval for large times.

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1. INTRODUCTION

A possible model for vibrations of a beam may be written in the form

$$\begin{aligned}(1.1)\quad u_{tt}+\alpha\, u_{xxxx}-\Big(\beta+\gamma\int_0^\ell u_\xi^2\,\mathrm{d}\xi+\delta\int_0^\ell u_\xi\, u_{t\xi}\,\mathrm{d}\xi\Big)\,u_{xx}&\\ +g(u)\,u_t+f(u)=0,\qquad t\in\mathbf{R}^+,\ x\in(0,\ell),&\end{aligned}$$

$$(1.2)\qquad u(t,0)=u(t,\ell)=u_{xx}(t,0)=u_{xx}(t,\ell)=0,\qquad t\in\mathbf{R}^+,$$

$$(1.3)\qquad u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),\qquad x\in(0,\ell).$$

We assume that

$$(1.4)\qquad \alpha,\beta,\gamma,\delta \textit{ are real constants},\ \alpha>0,$$

$$(1.5_1)\qquad g \textit{ is a locally Lipschitz continuous function on } \mathbf{R},$$

$$(1.5_2) \qquad g(u) \geqslant g_0 > 0, \qquad u \in \mathbf{R},$$

$$(1.6_1) \qquad f \textit{ is a locally Lipschitz continuous function on } \mathbf{R},$$

$$(1.6_2) \qquad F(u) = \int_0^u f(z)\,\mathrm{d}z \geqslant 0, \qquad u \in \mathbf{R},$$

$$(1.6_3) \qquad u\,f(u) - F(u) \geqslant 0, \qquad u \in \mathbf{R}.$$

The term $-(\beta + \cdots)\,u_{xx}$ corresponds to the effect of the presumed extensibility of the beam, see e.g. Woinowski-Krieger (1950), Eisley (1964), Ball (1973a,b), Kopáčková-Vejvoda (1977), Lovicar (1977), Lunardi (1987). The function g represents the external damping coefficient depending on possible non-homogeneous surrounding media (water-air), cf. Feireisl-Herrmann-Vejvoda (1991), and f is the restoring force caused by the foundation or by external constraints, cf. McKenna-Walter (1987).

The boundary conditions (1.2) describe the hinged beam (which is, in fact, the only situation we are able to cope with). The corresponding eigenvalue problem

$$(1.7_1) \qquad v_{xxxx} = \lambda\, v, \qquad x \in (0, \ell),$$

$$(1.7_2) \qquad v(0) = v(\ell) = v_{xx}(0) = v_{xx}(\ell) = 0$$

is easily solvable, viz.

$$(1.8) \qquad \lambda_k = \left(\frac{k\pi}{\ell}\right)^2, \qquad v_k = \sin\frac{k\pi x}{\ell}, \qquad k \in \mathbf{N}.$$

A measurable function $u\colon \mathbf{R}^+ \times (0,\ell) \to \mathbf{R}$ is said to be *eventually globally oscillatory* if there exists $T \geqslant 0$ and $\Theta > 0$ such that for any interval $J \subset [T, +\infty)$ the length of which is greater than Θ we have either $u \equiv 0$ on $\mathbf{R}^+ \times (0,\ell)$ or simultaneously

$$\operatorname{meas}\{(t,x) \in J \times (0,\ell) \mid u(t,x) > 0\} > 0 \quad \text{and}$$
$$\operatorname{meas}\{(t,x) \in J \times (0,\ell) \mid u(t,x) < 0\} > 0.$$

Our goal is to prove the following result.

Theorem 1.1. *Let the hypotheses* (1.4)–(1.6) *be satisfied and, moreover, let*

$$(1.9) \qquad \beta > -\alpha\lambda_1,$$

$$(1.10) \qquad \frac{g(0)}{2} < \sqrt{\lambda_1\alpha + \sqrt{\lambda_1}\,\beta}.$$

Then any weak solution to (1.1)–(1.3) *is eventually globally oscillatory.*

The weak solution is defined in the usual way; for the precise definition see Sec. 2. There is a great amount of papers dealing with oscillatory properties of solutions to undamped equations, e.g. Kreith (1973), Kreith-Kusano-Yoshida (1984), Cazenave-Haraux (1984), (1987), (1988), Haraux-Zuazua (1988). The cases of the damping of the form $h(u_t)$ and $h(\|u_t\|)\,u_t$ are treated by Zuazua (1990); here and hereafter $\|\cdot\|$ stands for the norm in $L_2 = L_2(0,\ell)$. The problem (1.1)–(1.3) with $\delta = g = 0$ is studied by Yoshida (1988).

2. Existence, uniqueness and energy decay

We will look for a solution pair u, u_t of the problem (1.1)–(1.3) belonging, for all $t \in \mathbf{R}^+$, to the natural energy space

$$E = H^2 \cap H_0^1 \times L_2.$$

It is well-known that the operator

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\alpha\, u_{xxxx} \end{pmatrix}, \qquad \mathscr{D}(L) = \widehat{H}^4 \times H^2 \cap H_0^1,$$

where

$$\widehat{H}^4 = \{v \in H^4 \mid v \ \text{ satisfies } \ (1.7_2)\},$$

generates a group of linear operators on E. Consequently, setting $v = u_t$ we may solve the problem (at least locally) via the variation-of-constants formula and the fixed point argument. To this end, we have to show that the nonlinear operators

$$\begin{aligned} u &\mapsto \left(\beta + \gamma \int_0^\ell u_\xi^2 \,\mathrm{d}\xi\right) u_{xx}, \\ u &\mapsto f(u) \end{aligned}$$

acting on $H^2 \cap H_0^1$ and ranging in L_2 are locally Lipschitz continous and, similarly, that

$$\begin{aligned} (u,v) &\mapsto \int_0^\ell u_{\xi\xi} v \,\mathrm{d}\xi\, u_{xx}, \\ (u,v) &\mapsto g(u)v \end{aligned}$$

are locally Lipschitz continuous from E into L_2. But this is ensured by the hypotheses (1.5_1) and (1.6_1).

Thus the classical existence theory gives rise to the following local result.

Proposition 2.1. *For any pair $(u_0, u_1) \in E$ there is a life time $s > 0$ such that the problem* (1.1)–(1.3) *possesses a weak solution u determined uniquely in the class*

$$u \in C([0,s), H^2 \cap H_0^1) \cap C^1([0,s), L_2). \tag{2.1}$$

As we need global existence, some a priori estimates are of interest. We multiply Eq. (1.1) by u_t and integrate by parts to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|u_t\|^2 + \alpha\|u_{xx}\|^2 + \beta\|u_x\|^2 + \frac{1}{2}\gamma\|u_x\|^4 + \int_0^\ell F(u)\,\mathrm{d}x\Big] + \delta\left(\int_0^\ell u_{xx}u_t\,\mathrm{d}x\right)^2 + \int_0^\ell g(u)u_t^2\,\mathrm{d}x \leqslant 0. \tag{2.2}$$

Consequently, having the hypotheses (1.5_2) and (1.6_2) in mind, we see that if $(u_0, u_1) \in B \subset E$ where B is bounded, then any local solution remains in a bounded set $\zeta(B)$ for all t. Thus, in fact, any local solution may be prolonged for all $t \to +\infty$.

Proposition 2.2. *Assume that $(u_0, u_1) \in B \subset E$, B bounded. Then there exists a unique global weak solution (u, u_t) of the problem* (1.1)–(1.3) *satisfying*

$$(u, u_t) \in \zeta(B), \qquad t \in \mathbf{R}^+. \tag{2.3}$$

Remark 2.1. We have tacitly assumed that all functions in question are smooth enough so that all integrations may be justified. Using the regularization technique due to Lions-Magenes (1968) or the arguments of Temam (1988) we may show that the energy inequality (2.2) holds, in fact, for any weak solution. The same remark applies to integrations in what follows.

Finally, we shall prove the decay of solutions in the energy space. Suppose that all the hypotheses (1.4)–(1.6) are satisfied. Multiplying (1.1) by εu and adding to (2.2) we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|u_t\|^2 + \alpha\,\|u_{xx}\|^2 + \beta\,\|u_x\|^2 + \frac{1}{2}\gamma\|u_x\|^4 + \int_0^\ell F(u)\,\mathrm{d}x + 2\varepsilon\int_0^\ell u_t u\,\mathrm{d}x\Big]\\ + \int_0^\ell (g(u)-\varepsilon)u_t^2\,\mathrm{d}x + \delta\Big(\int_0^\ell u_{xx}u_t\,\mathrm{d}x\Big)^2 + \varepsilon\left(\beta + \gamma\|u_x\|^2\right)\,\|u_x\|^2\\ + \varepsilon\delta\|u_x\|^2\int_0^\ell u_{xx}u_t\,\mathrm{d}x + \varepsilon\int_0^\ell g(u)u_t u\,\mathrm{d}x + \varepsilon\int_0^\ell (f(u)u - F(u))\,\mathrm{d}x \leqslant 0.\end{aligned} \tag{2.4}$$

According to (2.3) we can estimate

$$\left|\int_0^\ell g(u)u_t u\,\mathrm{d}x\right| \leqslant C_1(u_t,u) \qquad \text{as} \quad u\in H^2\cap H_0^1 \hookrightarrow C. \tag{2.5}$$

Similarly the remaining nonlinear terms may be estimated. Consequently, for any $B\subset E$ bounded there is $\tilde{\varepsilon}(B)>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t)+\tilde{\varepsilon}\mathscr{E}(t)\leqslant 0 \tag{2.6}$$

where

$$\mathscr{E}=\|u_t\|^2+\alpha\|u_{xx}\|^2+\beta\|u_x\|^2+\frac{1}{2}\gamma\|u_x\|^4+\int_0^\ell F(u)\,\mathrm{d}x+2\varepsilon\int_0^\ell u_t u\,\mathrm{d}x$$

with u, u_t solving the problem for the data from B.

As (2.6) implies exponential decay of the energy, we conclude:

Propositon 2.3. *For any bounded $B\subset E$ there is $\varepsilon=\varepsilon(B)>0$ such that*

$$\|u_{xx}(t,\cdot)\|^2+\|u_t(t,\cdot)\|^2\leqslant C_1(B)\exp(-\varepsilon t), \qquad t\in\mathbf{R}^+ \tag{2.7}$$

for any solution pair (u,u_t) originating from B.

3. Oscillation

In this section we will use notation and results concerning the summit function ϑ and the universal comparison function C from the paper Herrmann (1991). For the reader's convenience we recall some of these topics in Appendix. The comparison function, in its particular form, was used in the study of oscillations for the first time by Zuazua (1990).

Proposition 3.1. *Let the hypotheses* (1.9) *and* (1.10) *be satisfied. Then for any weak solution u of* (1.1)–(1.3) *and any couple $(q,p)\in\mathbf{R}^2$ satisfying*

$$\frac{g(0)}{2}<-p<\sqrt{q}<\sqrt{\lambda_1\alpha+\sqrt{\lambda_1}\,\beta} \tag{3.1}$$

there exists $T=T(u,q,p)$ such that the following implication holds:

$$J\subset[T,+\infty), \quad |J|>\vartheta_p^q+\vartheta_0^q, \qquad u\geqslant 0 \quad (\leqslant 0) \quad \textit{on} \quad J\times(0,\ell) \quad \Longrightarrow$$

$$\Longrightarrow \quad u \equiv 0 \quad on \quad \mathbf{R}^+ \times (0, \ell).$$

Proof. Let J be an interval with end points t_1, t_2 and let its length $|J| = t_2 - t_1$ be greater than $\vartheta_p^q + \vartheta_0^q$, where q, p satisfy (3.1). Let u be a weak solution which does not change the sign on $J \times (0, \ell)$. Proposition 2.3 yields

$$\|u\|_{L_\infty} \to 0, \qquad t \to +\infty, \tag{3.2}$$

$$\|u_t\|,\ \|u_x\|,\ \|u_{xx}\| \to 0, \qquad t \to +\infty. \tag{3.3}$$

Thus, defining

$$\widetilde{G}(t,x) = \begin{cases} \dfrac{1}{u(t,x)} \displaystyle\int_0^{u(t,x)} g(z)\,\mathrm{d}z, & u(t,x) \neq 0, \\ g(0), & u(t,x) = 0, \end{cases}$$

$$\tilde{b}(t) = \gamma \int_0^\ell u_\xi^2(t,\xi)\,\mathrm{d}\xi + \delta \int_0^\ell u_\xi(t,\xi)\,u_{t\xi}(t,\xi)\,\mathrm{d}\xi,$$

we have

$$\|\widetilde{G}(t,\cdot) - g(0)\|_{L_\infty} \to 0, \qquad t \to +\infty, \tag{3.4}$$

$$\tilde{b}(t) \to 0, \qquad t \to +\infty. \tag{3.5}$$

Let us denote

$$\mu = \lambda_1 \alpha + \sqrt{\lambda_1}\,\beta - q \quad (> 0),$$
$$\nu = -p - \frac{g(0)}{2} \quad (> 0).$$

Taking into account (3.4) and (3.5) we find $T > 0$ such that for all $t \geqslant T$

$$\|\widetilde{G}(t,\cdot) - g(0)\|_{L_\infty} < 2\nu,$$
$$\sqrt{\lambda_1}\,|\tilde{b}(t)| < \mu.$$

Choose $\eta > 0$ such that $-p < \sqrt{q - \eta}$ and

$$|J| > \vartheta_p^{q-\eta} + \vartheta_0^{q-\eta} > \vartheta_p^q + \vartheta_0^q.$$

This choice is possible since the function ϑ is continuous and decreasing with respect to q, see (A.4). Define

$$\psi(t) = C(t - t_1, q - \eta, p, 0),$$

where C is the universal comparison function defined in (A.8). Let us mention some properties of the function ψ we shall need.

$$\psi \in C^2[t_1, t_1 + \vartheta_p^{q-\eta} + \vartheta_0^{q-\eta}], \tag{3.6}$$

$$\ddot{\psi} + 2p\,\dot{\psi}^+ + (q-\eta)\,\psi = 0, \qquad t \in [t_1, t_1 + \vartheta_p^{q-\eta} + \vartheta_0^{q-\eta}], \tag{3.7}$$

$$\psi(0) = \psi(\vartheta_p^{q-\eta} + \vartheta_0^{q-\eta}) = 0, \qquad \psi(t) > 0, \quad t \in (t_1, t_1 + \vartheta_p^{q-\eta} + \vartheta_0^{q-\eta}), \tag{3.8}$$

$$\begin{gathered} \dot{\psi}(t) > 0, \qquad t \in [t_1, t_1 + \vartheta_p^{q-\eta}), \\ \dot{\psi}(t_1 + \vartheta_p^{q-\eta}) = 0, \\ \dot{\psi}(t) < 0, \qquad t \in (t_1 + \vartheta_p^{q-\eta}, t_1 + \vartheta_p^{q-\eta} + \vartheta_0^{q-\eta}]. \end{gathered} \tag{3.9}$$

Multiplying Eq. (1.1) by $\psi(t)\,v_1(x)$ where v_1 is given in (1.8) and integrating over $t \in (t_1, t_1 + \vartheta_p^{q-\eta} + \vartheta_0^{q-\eta})$ and $x \in (0, \ell)$ we obtain

$$\begin{aligned} 0 = -\int_0^\ell u\,v_1\,\dot{\psi}\,\mathrm{d}x\bigg|_{t=t_1}^{t=t_1+\vartheta_p^{q-\eta}+\vartheta_0^{q-\eta}} &+ \int_0^\ell \int_{t_1}^{t_1+\vartheta_p^{q-\eta}+\vartheta_0^{q-\eta}} u\,v_1\Big[\ddot{\psi} - \widetilde{G}(t,x)\,\dot{\psi} \\ &+ \Big(\lambda_1\alpha + \sqrt{\lambda_1}\,\beta + \sqrt{\lambda_1}\,\tilde{b}(t) + f(u(t,x))/u(t,x)\Big)\Big]\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

(where $f(u(t,x))/u(t,x)$ is to be replaced by 0 if $u(t,x) = 0$). Employing the above mentioned properties of the function ψ, the positivity of the first eigenfunction v_1 on $(0, \ell)$, the sign condition

$$u\,f(u) \geqslant 0, \qquad u \in \mathbf{R}$$

which follows from (1.6), and the following two inequalities which are easily seen to hold for all $t \geqslant T$ and $x \in (0, \ell)$:

$$\begin{gathered} \widetilde{G}\,\dot{\psi} \leqslant -2p\,\dot{\psi}^+, \\ \lambda_1\alpha + \sqrt{\lambda_1}\,\beta + \sqrt{\lambda_1}\,\tilde{b}(t) \geqslant q, \end{gathered}$$

we get

$$\begin{aligned} 0 \geqslant \operatorname{sgn} u \Big\{ -\int_0^\ell u\,v_1\,\dot{\psi}\,\mathrm{d}x\bigg|_{t=t_1}^{t=t_1+\vartheta_p^{q-\eta}+\vartheta_0^{q-\eta}} \\ + \int_0^\ell \int_{t_1}^{t_1+\vartheta_p^{q-\eta}+\vartheta_0^{q-\eta}} u\,v_1\Big[\ddot{\psi} + 2p\,\dot{\psi}^+ + (q-\eta)\,\psi\Big]\,\mathrm{d}x\,\mathrm{d}t\Big\} \\ \geqslant \eta \operatorname{sgn} u \int_0^\ell \int_{t_1}^{t_1+\vartheta_p^{q-\eta}+\vartheta_0^{q-\eta}} u\,v_1\,\psi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Consequently, $u \equiv 0$ on $(t_1, t_1+\vartheta_p^{q-\eta}+\vartheta_0^{q-\eta}) \times (0,\ell)$ and hence on $\mathbf{R}^+ \times (0,\ell)$ because of the uniqueness of solutions ensured by Proposition 2.2.

The proof of Proposition 3.1 and, in fact, of Theorem 1.1 is complete. □

APPENDIX

The main tool for studying oscillatory phenomena in dissipative systems is the universal comparison function c. This function is a special solution of the equation

$$(A.1) \qquad \ddot{c} + 2(p\,\dot{c}^+ + n\,\dot{c}^-) + q\,c = 0$$

which is non-negative and has positive local maximum. So, it is defined for (q,p), $(q,n) \in \mathcal{O}$ on the interval $t \in [0, \vartheta_p^q + \vartheta_n^q]$. Here

$$(A.2) \qquad \mathcal{O} = \{(q,p) \in \mathbf{R}^2 \mid q > 0,\ p > -\sqrt{q}\}$$

and ϑ is the so-called summit function defined by

$$(A.3) \quad \vartheta(q,p) = \vartheta_p^q = \begin{cases} \dfrac{\pi}{\sqrt{q-p^2}} + \dfrac{1}{\sqrt{q-p^2}} \arctan \dfrac{\sqrt{q-p^2}}{p}, & -\sqrt{q} < p < 0, \\ \dfrac{\pi}{2\sqrt{q}}, & p = 0, \\ \dfrac{1}{\sqrt{q-p^2}} \arctan \dfrac{\sqrt{q-p^2}}{p}, & p > 0,\ p \neq \sqrt{q}, \\ \dfrac{1}{\sqrt{q}}, & p = \sqrt{q}. \end{cases}$$

The function ϑ is real positive continuous on $\mathcal{O}$ and

(A.4) decreasing in each variable while the other is fixed.

The function c possesses the following properties

$$(A.5) \qquad c \in C^2[0, \vartheta_p^q + \vartheta_n^q],$$

$$(A.6) \qquad c(0) = c(\vartheta_p^q + \vartheta_n^q) = 0, \qquad c(t) > 0, \quad t \in (0, \vartheta_p^q + \vartheta_n^q),$$

$$(A.7) \qquad \dot{c}(t) > 0, \quad t \in [0, \vartheta_p^q), \qquad \dot{c}(\vartheta_p^q) = 0, \qquad \dot{c}(t) < 0, \quad t \in (\vartheta_p^q, \vartheta_p^q + \vartheta_n^q].$$

Explicitly, denoting $c(t) = C(t,q,p,n)$, we have

$$(A.8)\quad C(t,q,p,n) = \begin{cases} A(t,q,p), & t \in [0, \vartheta_p^q], \\ \exp(-p\,\vartheta_p^q + n\,\vartheta_n^q)\,A(\vartheta_p^q + \vartheta_n^q - t, q, n), & t \in (\vartheta_p^q, \vartheta_p^q + \vartheta_n^q], \end{cases}$$

where

$$(A.9)\quad A(t,q,p) = \begin{cases} \dfrac{1}{\sqrt{q-p^2}} \exp(-p\,t)\,\sin(\sqrt{q-p^2}\,t), & p > -\sqrt{q},\ p \neq \sqrt{q}, \\ t\exp(-\sqrt{q}\,t), & p = \sqrt{q}. \end{cases}$$

For details see Herrmann (1991).

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