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A WKB ANALYSIS OF THE ALFVEN SPECTRUM OF THE LINEARIZED MAGNETOHYDRODYNAMICS EQUATIONS

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Summary. Small perturbations of an equilibrium plasma satisfy the linearized magnetohydrodynamics equations. These form a mixed elliptic-hyperbolic system that in a straight-field geometry and for a fixed time frequency may be reduced to a single scalar equation $\operatorname{div}(A_1 \nabla u) + A_2 u = 0$, where A_1 may have singularities in the domain U of definition. We study the case when U is a half-plane and u possesses high Fourier components, analyzing the changes brought about by the singularity $A_1 = \infty$. We show that absorption of energy takes place precisely at this singularity, that the solutions have a near harmonic character, and the integrability characteristics of the boundary data are kept throughout U.

Keywords: Magnetohydrodynamics, Alfvén waves, Fourier analysis, singularity

AMS classification: 76W05, 34E05

1. INTRODUCTION

Let a stationary plasma fill the half space U: $x > 0$, and assume that its density ϱ and the magnetic field $\mathbf{B} = (0, 0, B)$ through it depend only on the coordinate x. Also assume that the plasma satisfies the state equation of polytropic gases, $p = S\,\varrho^\gamma$, where p is the pressure, S and γ constants. Consider now a small perturbation induced by a surface current at the plane $x = -h$, and let, after taking Fourier components, $p_\star = p_\star(x, y)e^{i\omega t + ilz}$ be the perturbed magnetic pressure. Then $p_\star$ satisfies the linearized Magnetohydrodynamics equation

$$\operatorname{div}\left(\frac{1}{\omega^2\varrho - l^2B^2}\nabla p_\star\right) - \frac{l^2}{(1+\frac{B^2}{\gamma p})}\frac{1}{\omega^2\varrho(1+\frac{B^2}{\gamma p}) - l^2B^2}p_\star = 0, \tag{1}$$

which is singular at every point such that

$$f_1(x) = \omega^2\varrho(x) - l^2B^2(x) = 0, \tag{2}$$

and also where

$$f_2(x) = \omega^2 \varrho(x) \left(1 + \frac{B^2(x)}{\gamma\, p(x)}\right) - l^2 B^2(x) = 0\,. \tag{3}$$

In the incompressible case ($\gamma \to \infty$) both conditions coincide. Usually, however, γ is taken as $5/3$ and the rate p/B^2 (the beta of the plasma) is low, so that for fixed l the ω's for which (3) holds at some point of U are much smaller than those for which (2) is satisfied. We will consider an ω such that there are zeroes of f_1 within U, but not of f_2. This assumption is not an ad hoc hypothesis; in fact it lies at the very heart of many important physical phenomena. The ω's such that $f_1(x) = 0$ somewhere in U are called the Alfvén frequencies; an external perturbation of the frequency ω resonates with the points of the plasma where f_1 vanishes, and absorption of energy takes place there, as will be shown later. Plasma heating by Alfvén waves, with its possible application to nuclear fusion, was proposed by Grad [1], and its physical mechanism has been analyzed in depth ([2, 3]). Also long-period geomagnetic pulsations are attributed to Alfvén resonance ([4, 5]). The analysis of equation (1) for these particular frequencies may help to elucidate the behaviour of the perturbation as affected by the singularity. We will restrict our attention to solutions with relatively high Fourier components in y, which appear with sheet currents narrowly localized in y, i.e. following magnetic field lines. This case is interesting because it leaves the plasma structure relatively unharmed ([2]) and because several characteristics modes are known to appear ([6]).

A plasma state excited by a surface current has been also studied in [7]. Hence we must explain to which extent our work differs from the deep analysis of J.A. Tataronis in the classical paper mentioned above. Although some hypotheses do not coincide (in [7] an incompressible plasma and a finite symmetric slab are considered) the main difference lies in the piecewise linear density and magnetic field profiles assumed in [7] (and in fact in other related papers, such as [2, 8, 9]). This linear character in a neighbourhood of the singularity enables these authors to solve exactly the relevant equations with the help of Bessel functions, thus avoiding the asymptotic analysis in the Fourier mode k of the variable y which we have to develop, as well as the delicate bounds used in Section 6 to estimate the difference between the WKB and the real solutions. As one could expect, what one gains in generality by considering a generic profile is lost in another respect by having to assume a large $|k|$. Another aspect analyzed in [7, 8, 9] is the contribution of surface waves. They correspond to frequencies which are roots of the dispersion relation, i.e. poles of the analytic continuation of the solution into the lower half plane. Although the contribution of these waves decays exponentially in time, they may be relevant in a finite time interval. We are not dealing with these poles in this paper, mainly because such a deep study as the one in [8] is only possible with an exact solution in terms of Bessel functions, which enables the author to locate precisely those poles. In a generic case such as ours we would be limited to generalities, depending on the unknown location of the roots of the dispersion equation.

2. Boundary conditions

Let us see how the sheet current $\mathbf{j}$ (located outside of the plasma, at $x = -h$) determines a boundary condition on $p_\star$ at $x = 0$. Assuming that there are no free charges at the sheeet $x = -h$, we have $\operatorname{div} \mathbf{j} = 0$. Then

Proposition 1. *$\mathbf{j}$ determines the values $\partial p_\star/\partial x(0, y)$ for all $y \in \mathbb{R}$.*

Proof. We shall make a Fourier transform $y \mapsto k$ on all variables, and denote by $\mathbf{J}$ and P the Fourier transform of $\mathbf{j}$ and $p_\star$. Within the vacuum $-h < x < 0$ the magnetic field is irrotational and solenoidal, so it can be written as $\mathbf{b}_v = \nabla\varphi$, φ being a harmonic function. The Fourier transform Φ of φ must hence satisfy

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} - (k^2 + l^2)\Phi = 0$$

so $\mathbf{b}_v$, for fixed k, is a linear combination

$$\mathbf{b}_v = C_1 \mathrm{e}^{qx+\mathrm{i}ky+\mathrm{i}lz}(q, \mathrm{i}k, \mathrm{i}l) + C_2 \mathrm{e}^{-qx+\mathrm{i}ky+\mathrm{i}lz}(-q, \mathrm{i}k, \mathrm{i}l)$$

where $q = (k^2 + l^2)^{1/2}$. The same may be said for the vacuum field $\mathbf{b}_w$ to the left of $-h$, but since this one must also vanish at $x = -\infty$, necessarily

$$\mathbf{b}_w = D\mathrm{e}^{qx+\mathrm{i}ky+\mathrm{i}lz}(q, \mathrm{i}k, \mathrm{i}l).$$

The Maxwell boundary conditions at the sheet are first, that the normal component of the magnetic field is continuous through it, so $D = C_1 - C_2\mathrm{e}^{2qh}$; and second, that the jump $\mathbf{b}_v - \mathbf{b}_w$ satisfies $\mathbf{n} \times (\mathbf{b}_v - \mathbf{b}_w) = \mathbf{J}$ ($\mathbf{n}$ being the normal vector to the sheet), which means

$$\mathbf{J} = (0, -\mathrm{i}lC_1\mathrm{e}^{qh} - \mathrm{i}lC_2\mathrm{e}^{-qh} + \mathrm{i}lD\mathrm{e}^{-qh}, \mathrm{i}kC_1\mathrm{e}^{qh} + \mathrm{i}kC_2\mathrm{e}^{-qh} - \mathrm{i}kD\mathrm{e}^{-qh}),$$

which confirms the fact that $kJ_y + lJ_z = 0$, i.e. $\operatorname{div} \mathbf{j} = 0$.

Writing $\mathbf{J}$ as $I(k, l)(0, -\mathrm{i}l, \mathrm{i}k)$, we have that $C_1\mathrm{e}^{qh} + C_2\mathrm{e}^{-qh} - D\mathrm{e}^{-qh} = I(k, l)$, i.e.

$$C_1 - C_2 = \frac{I(k, l)}{2\cosh(qh)},$$

which determines $\mathbf{b}_v$ up to a constant:

$$(4) \qquad \mathbf{b}_v = \left(C_1[\mathrm{e}^{qx}(q, \mathrm{i}k, \mathrm{i}l) + \mathrm{e}^{-qx}(-q, \mathrm{i}k, \mathrm{i}l)] - \frac{I(k,l)}{2\cosh(qh)}(-q, \mathrm{i}k, \mathrm{i}l) \right) \mathrm{e}^{\mathrm{i}ky+\mathrm{i}lz}$$

Now we must consider the interface $x = 0$ between the vacuum and the plasma. Two jump conditions hold there: the continuity of both the normal component of the field and the perturbed magnetic pressure. The first condition yields

$$b_x(0) = \frac{q\, I(k,l)}{2\cosh(qh)}.$$

The linearized MHD equations relate b_x and P by means of the formula

$$b_x = \frac{\mathrm{i}lB}{\omega^2\varrho - l^2B^2} \frac{\mathrm{d}P}{\mathrm{d}x}$$

which, together with our last equation, determines the value of $\mathrm{d}P/\mathrm{d}x(0)$ (i.e. $\partial p_\star/\partial x(0,y)$) as

$$\frac{\mathrm{d}P(0)}{\mathrm{d}x} = \frac{\mathrm{i}lB(0)qI(k,l)}{2\cosh(qh)}.$$

□

Let us consider the remaining condition, i.e. the continuity of the perturbed magnetic pressure. The magnetic pressure is defined as $p + \frac{1}{2}B^2$; thus its perturbation $p_\star$ is $p_1 + \mathbf{B} \cdot \mathbf{b}$, p_1 being the increment in hydrostatic pressure and $\mathbf{b}$ the added magnetic field. In our case, to the left of $x = 0$ there is no fluid, so $p_\star = \mathbf{B}_v \cdot \mathbf{b}_v$, and to the right its value is what we have denoted as $p_\star(0)$. Another boundary condition must hold: $p_\star$ vanishes at $x = \infty$. This apparently overdetermined problem is not really so, as we have yet an arbitrary constant at the expression of $\mathbf{b}_v$ in (4). In practice, the condition $p_\star(\infty) = 0$ will select a one-dimensional space of solutions to (1); the known value $\partial p_\star/\partial x$ will determine the real solution, and its value $p_\star(0)$ the constant C_1 at (4).

3. The WKB approximation

We will denote the expressions

$$\omega^2\varrho - l^2B^2 \qquad \text{and} \qquad \frac{l^2}{1+\frac{B^2}{\gamma p}} \frac{\omega^2\varrho - l^2B^2}{\omega^2\varrho(1+\frac{B^2}{\gamma p}) - l^2B^2}$$

by f and g, respectively. Equation (1) then becomes

$$\frac{\partial^2 p_\star}{\partial x^2} + \frac{\partial^2 p_\star}{\partial y^2} - \frac{f'}{f}\frac{\partial p_\star}{\partial x} - g\, p_\star = 0$$

or, in terms of P,

$$P'' - \frac{f'}{f}P' - (k^2 + g)P = 0 \tag{5}$$

Let us recall (see e.g. [10]) that in order to find the WKB approximation one must consider a formal solution of the form

$$P \sim \exp\left(k \sum_{n=0}^{\infty} \frac{1}{k^n} S_n(x)\right)$$

and equal all the powers of k to zero in equation (5). It is easy to see that $S_0' = \sigma = \pm 1$ and $S_1 = \log \sqrt{|f(x)|}$, so that the first order (physical optics) approximation is

$$P_W(x, k) = e^{\sigma k x} |f(x)|^{1/2}. \tag{6}$$

On the other hand, S_2 satisfies

$$2\sigma S_2' = \frac{3}{4}\left(\frac{f'}{f}\right)^2 + g - \frac{1}{2}\frac{f''}{f}.$$

The solution is singular at the zeros of f; in fact all the recurrence relations break there, and must be understood excluding a band around each zero. In order to simplify the argument we will assume that there is a unique zero x_0 of f, that f and g are regular, and even that f is analytic in a neigbourhood of x_0. Later we will see how the solutions in (6) connect through the turning point x_0. With a reasonable behaviour of f and g (for example, if ϱ, p and B tend to a constant quickly enough) S_1 remains bounded from $x_0 + \varepsilon$ (any $\varepsilon > 0$) to ∞, and certainly from 0 to $x_0 - \varepsilon$, and $S_1' \to 0$ as $x \to \infty$. The remaining recurrence relations

$$2\sigma S_{p+1}' = \frac{f'}{f} S_p' - S_p'' - \sum_{\substack{n+m=p+1 \\ n,m\neq 0}} S_n' S_m'$$

behave in the same way, so we may assume that S_2 is bounded outside the band $(x_0 - \varepsilon, x_0 + \varepsilon)$ (say by M) and $(1/k)S_{p+1} \ll S_p$ for k big enough, uniformly outside this band. These are the hypotheses needed for the relative error of P_W with respect to P to be of the order $1/|k|$:

$$(7) \qquad |P(x,k) - P_W(x,k)| \leqslant \frac{N}{|k|}|P_W(x,k)|$$

uniformly outside $(x_0 - \varepsilon, x_0 + \varepsilon)$. As a matter of fact we should write $|e^{M/|k|} - 1|$ instead of $N/|k|$, but for $|k|$ large enough they both have the same order. The bound in (7) will be used later in our analysis. Let us point out that in most real cases the WKB approximation is fairly good, much better than (7) even for low values of $|k|$.

4. Analysis of the boundary layer

From now on we will assume that f posssesses a single zero x_0 in $(0,\infty)$ and x_0 has order one, which implies in particular that $f(x) < 0$ for $x < x_0$, that $f(x) > 0$ for $x > x_0$ and $f'(x_0) > 0$. This restriction is not important, as higher order zeros are unstable and under a small modification of ϱ, l or B they either vanish or become first order, so they are unlikely to occur in a real case.

Proposition 2. *In a small enough neighbourhood of x_0 and for k large enough, two independent first order approximations to the solutions of* (5) *are*

$$F\colon x \mapsto (x - x_0)I_1(|k|(x - x_0)), \quad G\colon x \mapsto (x - x_0)K_1(|k|(x - x_0))$$

where I_1 and K_1 are the modified Bessel functions of order one.

Proof. In a neighbourhood of x_0, f'/f will have the form $1/(x - x_0)$, as x_0 has order one. Since g is small by comparison to k^2 or $1/(x - x_0)$, we take instead of (5)

$$(8) \qquad P'' - \frac{1}{x - x_0}P' - k^2 P = 0,$$

which after defining

$$u(z) = \frac{1}{z}P\left(x_0 + \frac{z}{|k|}\right)$$

becomes

$$u'' + \frac{1}{z}u' - \left(\frac{1}{z^2} + 1\right)u = 0$$

whose two fundamental solutions are I_1 and K_1. The result is proved. □

The power series for F and G are ([11])

$$I_1(z) = \frac{z}{2}\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!(n+1)!},$$

$$K_1(z) = \frac{1}{z} + \frac{z}{2}\left(\log\frac{z}{2}\right)\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!(n+1)!} - \frac{z}{4}\sum_{n=0}^{\infty}\frac{\psi(n+1)+\psi(n+2)}{n!(n+1)!}\left(\frac{z}{2}\right)^{2n}$$

where $\psi(n+1) = -C + 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ and C is the Euler-Mascheroni constant. As we see, F is even around x_0, and K_1 multivalued by the presence of the logarithm. If we take it to be real for $z > 0$, for small $r > 0$ we have

$$G(x_0 - r) = G(x_0 + r) + \mathrm{i}\delta\pi F(x_0 + r)$$

where $\delta = \pm 1$,the sign depending on the branch of the logarithm.

Proposition 3. *δ is the sign of ω.*

Proof. Let us remember that we are dealing with the Fourier-Laplace transform $\int_0^\infty e^{\mathrm{i}\omega t} p_\star(t,x)\,dt$ of the perturbed magnetic pressure as a function of time; hence the solution to (5) for real ω is the limit of the solutions for $\omega + \mathrm{i}\varepsilon$, $\varepsilon \downarrow 0$; i.e., any oscillatory solution is the limit of damped solutions. Denoting by $\omega(x)$ the point which makes $f(\omega(x), x) = 0$, we have

$$0 = \frac{\partial f}{\partial \omega}\omega' + f' = 2\omega\omega'\varrho + f', \text{ so that } \omega'(x) = \frac{-f'(\omega(x),x)}{2\omega(x)\varrho(x)}.$$

Hence, near x_0,

$$\omega(x) - \omega(x_0) = -\frac{f'(\omega(x_0),x_0)}{2\omega(x_0)\varrho(x_0)}(x - x_0) + o(x - x_0)$$

Let $x(z)$ be the complex point making $f(z, x(z)) = 0$; then

$$x(z) - x_0 = x(z) - x(\omega) = -(z-\omega)\frac{2\omega\varrho(x_0)}{f'(\omega,x_0)} + o(z-\omega).$$

which implies that for $z = \omega + \mathrm{i}\varepsilon$, $x(z)$ has an imaginary part whose sign is the sign of $-\omega/f'(x_0)$. Since $f'(x_0) > 0$, this sign is $-\operatorname{sgn}(\omega)$. For $\omega > 0$, $x(z)$ lies in the lower half plane. Since the function $x \mapsto \log(x - x(z))$ must be continuous, the logarithm must also be continuous in the upper half plane, so we need to make a counterclockwise detour around x_0, which changes $\log(x - x_0)$ to $\log(x - x_0) + \mathrm{i}\pi$. If $\omega < 0$, $x(z)$ has a positive imaginary part and the contour around x_0 should be clockwise, so that $\log(x - x_0)$ becomes $\log(x - x_0) - \mathrm{i}\pi$. □

Theorem 4. *The WKB solution to (5) which vanishes at $x = \infty$ follows the pattern*

$$\begin{array}{rr}
e^{-|k|(x_0-x)}|f(x)|^{1/2} + \mathrm{i}\,\mathrm{sgn}(\omega)e^{|k|(x_0-x)}|f(x)|^{1/2} & x < x_0 - \varepsilon \\
\to (2|k|f'(x_0)/\pi)^{1/2}(x_0 - x)K_1(|k|(x_0 - x)) & \\
+ \mathrm{i}\,\mathrm{sgn}(\omega)(2\pi|k|f'(x_0))^{1/2}(x_0 - x)I_1(|k|(x_0 - x)) & x_0 - \varepsilon < x < x_0 \\
\to (2|k|f'(x_0)/\pi)^{1/2}(x - x_0)K_1(|k|(x - x_0)) & x_0 < x < x_0 + \varepsilon \\
\to e^{-|k|(x-x_0)}|f(x)|^{1/2} & x > x_0 + \varepsilon
\end{array}$$

as we go from 0 *to* ∞ *crossing* x_0.

Proof. The leading terms in the asymptotic expansions of I_1 and K_1 as $r \to \infty$ are ([11])

$$I_1(r) \sim \frac{e^r}{(2\pi r)^{1/2}},$$

$$K_1(r) \sim (\pi/2r)^{1/2}e^{-r}.$$

For $x > x_0$, the approximation obtained in (6) which vanishes at $x = \infty$ is

$$e^{-|k|(x-x_0)}|f(x)|^{1/2},$$

where the multiplicative constant $e^{|k|x_0}$ has been added to clarify the argument. In the region $|x - x_0| < \varepsilon$, $|k(x - x_0)| \to \infty$, this approximation and the one obtained above must be valid. Near x_0, $f(x)$ behaves like $f'(x_0)(x - x_0)$, so that the solution continues as

$$(2|k|f'(x_0)/\pi)^{1/2}(x - x_0)K_1(|k|(x - x_0))$$

within the band to the right of x_0. This function crosses x_0 as

$$\begin{aligned}&(2|k|f'(x_0)/\pi)^{1/2}(x_0 - x)K_1(|k|(x_0 - x))\\ &\qquad + \mathrm{i}\,\mathrm{sgn}(\omega)(2\pi|k|f'(x_0))^{1/2}(x_0 - x)I_1(|k|(x_0 - x)).\end{aligned}$$

To the left of the band, the solution becomes

$$e^{-|k|(x_0-x)}|f(x)|^{1/2} + \mathrm{i}\,\mathrm{sgn}(\omega)e^{|k|(x_0-x)}|f(x)|^{1/2}.$$

Obviously the only linear combination, except for constants, which tends to zero as $x \to \infty$ of these solutions is the one stated in the theorem. □

Now we must find the constant $A(k,l)$ which multiplies this function in order to find the WKB solution:

Proposition 5. *$A(k,l)$ behaves asymptotically as*

$$A(k,l) \sim \operatorname{sgn}(\omega)\operatorname{sgn}(lB(0))I(k,l)\mathrm{e}^{-|k|(x_0+h)}. \tag{9}$$

The relative error is at most of the order $1/|k|$.

P r o o f. The value of the derivative P'_W at $x = 0$ is $A(k,l)$ times

$$|k|(\mathrm{e}^{-|k|x_0} - \mathrm{i}\operatorname{sgn}(\omega)\mathrm{e}^{|k|x_0})|f(0)|^{1/2} + (\mathrm{e}^{-|k|x_0} + \mathrm{i}\operatorname{sgn}(\omega)\mathrm{e}^{|k|x_0})(|f|^{1/2})'(0), \tag{10}$$

which, together with the value of $P'(0)$ found in Section 2, yields $A(k,l)$ as the quotient between

$$\mathrm{i}lB(0)qI(k,l)/2\cosh(qh) \tag{11}$$

and the formula in (10). For large $|k|$, $q \sim |k|$, $2\cosh(qh) \sim \mathrm{e}^{|k|h}$, and the leading term in (10) is $-\mathrm{i}\operatorname{sgn}(\omega)|k|\mathrm{e}^{|k|x_0}|f(0)|^{1/2}$, so we get the desired result.

Moreover, the relative errors of both the numerator and the denominator are at most $1/|k|$, so the same occurs for $A(k,l)$. □

At this point we must make some comments concerning the solutions thus obtained. Equation (1) is a linearized one and as such only reliable for small solutions, i.e. small perturbations of the equilibrium state. This seems to contradict its singular character at the resonance points, but the fact is that both solutions of the equation are bounded at the singularity (although K_1 has a singularity, when multiplied by $x - x_0$ it becomes regular) and therefore they remain small for small boundary conditions. Also the imaginary part of the solution starts from zero at x_0, so that it can be also considered a small perturbation.

From the formulae in Theorem and (9) we may deduce that for any reasonable $I(k,l)$ (such as bounded), the solution decreases exponentially in $|k|$ for $x > x_0$. Therefore it is smaller than any negative power of $|k|$, and must be considered zero when dealing with asymptotic power series. However, our aim is to recover $p_\star(x,y)$ from P_W by means of an inverse Fourier transform, and a better approximation is needed. The bound

$$|P(x,k) - P_W(x,k)| \leqslant \frac{N}{|k|}|P_W(x,k)|$$

will give us the necessary accuracy.

5. Energy absorption

The rate of absorption of energy by the plasma contained in X is given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\int_{\partial X} p_\star \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\sigma \tag{12}$$

where $\mathbf{v}$ is the velocity of the fluid.

The particular configuration of our plasma makes this formula easier when X is a slab bounded by $x = a$, $x = b$. Hence we will study the integral in (12) for a fixed $x = a$, i.e.

$$-\int_{\mathbb{R}\times\mathbb{R}} p_\star(t,a,y,z)v_x(t,a,y,z)\,\mathrm{d}y\,\mathrm{d}z. \tag{13}$$

We will take an alternating current at the sheet $x = -h$ of the form

$$\mathbf{j}(t,y,z) = \mathbf{j}(y,z)\sin(\omega t + \theta)$$

with $\omega > 0$, i.e.

$$\mathbf{j}(t,y,z) = \mathbf{j}(y,z)\frac{1}{2\mathrm{i}}\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\omega t} - \mathbf{j}(y,z)\frac{1}{2\mathrm{i}}\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\omega t}.$$

Therefore its transform $\mathbf{J}$ has the form

$$\mathbf{J}(\omega,k,l)\mathrm{e}^{\mathrm{i}\omega t} + \mathbf{J}(-\omega,k,l)\mathrm{e}^{-\mathrm{i}\omega t}.$$

Since

$$\mathbf{J}(\omega,k,l) = \hat{\mathbf{j}}(y,z)\frac{1}{2\mathrm{i}}\mathrm{e}^{\mathrm{i}\theta} \qquad \text{and} \qquad \mathbf{J}(-\omega,k,l) = -\hat{\mathbf{j}}(y,z)\frac{1}{2\mathrm{i}}\mathrm{e}^{-\mathrm{i}\theta}$$

we have that $|\mathbf{J}(\omega,k,l)| = |\mathbf{J}(-\omega,k,l)|$, and since $|\mathbf{J}| = \sqrt{k^2+l^2}\,|I|$ we conclude

$$|I(\omega,k,l)| = |I(-\omega,k,l)|.$$

On the other hand, since $\mathbf{j}(t,y,z)$ is real, its Fourier transform satisfies

$$\mathbf{J}(-\omega,-k,-l) = \overline{\mathbf{J}(\omega,k,l)};$$

given that $\mathbf{J} = (0,-\mathrm{i}l,\mathrm{i}k)I$, the formula

$$I(-\omega,-k,-l) = \overline{I(\omega,k,l)}$$

also holds.

We have shown that A is the quotient between the expressions (11) and (10). With the above relations in mind, we find

$$|A(-\omega, k, l)| = |A(\omega, k, l)|,$$

$$A(-\omega, -k, -l) = \overline{A(\omega, k, l)} \qquad \text{and} \qquad A(\omega, -k, -l) = \overline{A(-\omega, k, l)}.$$

Since we are dealing with a fixed resonant surface, and this is determined by the zeros of $f = \omega^2 \varrho - l^2 B^2$, we have considered from the beginning functions which only depend on the two Fourier modes l and $-l$ in z, so that x_0 is the only root of f. As we know, with some restrictions upon the integrability of $p_\star$ and v_x,

$$\begin{aligned}
&- \int_{\mathbb{R}\times\mathbb{R}} p_\star(t, a, y, z)\, v_x(t, a, y, z)\, \mathrm{d}y\, \mathrm{d}z \\
&= - \int_{\mathbb{R}\times\mathbb{R}} \hat{p}_\star(t, a, k, l)\, \check{v}_x(t, a, k, l)\, \mathrm{d}k\, \mathrm{d}l \\
&= - \int_{\mathbb{R}} P(t, a, k, l)\, V(t, a, -k, -l)\, \mathrm{d}k - \int_{\mathbb{R}} P(t, a, k, -l)\, V(t, a, -k, l)\, \mathrm{d}k
\end{aligned} \tag{14}$$

where V stands for the Fourier transform of v_x. The MHD equations yield the following expression for V:

$$V = \frac{\mathrm{i}\omega}{f} \frac{\mathrm{d}P}{\mathrm{d}x}.$$

We will take the WKB approximation for P and V. Let α, β be the functions

$$\alpha = \begin{cases} \mathrm{e}^{-|k|(x_0 - x)} |f(x)|^{1/2} & \text{for } x < x_0 \\ \mathrm{e}^{-|k|(x - x_0)} |f(x)|^{1/2} & \text{for } x > x_0, \end{cases}$$

$$\beta = \begin{cases} \mathrm{e}^{|k|(x_0 - x)} |f(x)|^{1/2} & \text{for } x < x_0 \\ 0 & \text{for } x > x_0 \end{cases}$$

obtained in Theorem 4. Let us note that f, α and β depend only on the parameters ω^2, k^2 and l^2. Therefore

$$P(t, a, k, l) = A(\omega, k, l)(\alpha(a) + \mathrm{i}\beta(a))\mathrm{e}^{\mathrm{i}\omega t} + A(-\omega, k, l)(\alpha(a) - \mathrm{i}\beta(a))\mathrm{e}^{-\mathrm{i}\omega t}$$

and

$$\begin{aligned}
V(t, a, -k, -l) = {} & \frac{\mathrm{i}\omega}{f(a)} A(\omega, -k, -l)(\alpha'(a) + \mathrm{i}\beta'(a))\mathrm{e}^{\mathrm{i}\omega t} \\
& - \frac{\mathrm{i}\omega}{f(a)} A(-\omega, -k, -l)(\alpha'(a) - \mathrm{i}\beta'(a))\mathrm{e}^{-\mathrm{i}\omega t}.
\end{aligned}$$

In the product $-P(t,a,k,l)V(t,a,-k,-l)$ the terms in $e^{2i\omega t}$ and $e^{-2i\omega t}$ are oscillating functions in time, so the mean $\langle dW/dt\rangle$ associated to them is zero. For the crossed product $e^{i\omega t}e^{-i\omega t}=1$ we have the coefficient

$$\begin{aligned}
&\frac{i\omega}{f(a)}A(\omega,k,l)A(-\omega,-k,-l)(\alpha(a)+i\beta(a))(\alpha'(a)-i\beta'(a))\\
&\quad-\frac{i\omega}{f(a)}A(-\omega,k,l)A(\omega,-k,-l)(\alpha(a)-i\beta(a))(\alpha'(a)+i\beta'(a))\\
&\quad=\frac{2\omega}{f(a)}|A(\omega,k,l)|^2(\alpha\beta'-\beta\alpha')(a).
\end{aligned}$$

It is easy to see that the value of the wronskian determinant $(\alpha\beta'-\beta\alpha')(a)$ is $2|k|f(a)$ if $a<x_0$, 0 if $a>x_0$. The same is valid for the product $-P(t,a,k,-l)\ V(t,a,-k,l)$. Thus the integrand in (14) becomes, in time mean,

$$\begin{array}{ll}
8\omega|A(\omega,k,l)|^2|k| & \text{for } a<x_0\\
0 & \text{for } a>x_0.
\end{array}$$

Consider now the different possibilities for the boundaries $x=a$, $x=b$. If both are greater than x_0, the integral is zero in mean, as both boundary integrals are zero. If both are less than x_0, the boundary integrals are the same and their difference, due to the opposed normal vectors, is zero. Only if $a<x_0$, $b>x_0$ we have

$$\left\langle\frac{dW}{dt}\right\rangle=\int_{\mathbb{R}}8\omega|A(\omega,k,l)|^2|k|\,dk.$$

This means that there is absorption of energy, and it occurs only at the resonant sheet $x=x_0$. For high $|k|$ we have

$$\left\langle\frac{dW}{dt}\right\rangle=\int_{|k|>k_0}8\omega|I(\omega,k,l)|^2e^{-2|k|(x_0+h)}\,dk,$$

thus the contribution of the higher modes decreases exponentially, with an exponent proportional to the distance x_0+h between the resonant sheet and the driving current.

We proceed to invert the Fourier transform $P_W(x,k)$ in order to obtain the WKB solution $p_{\star W}(x,y)$; later we will compare it with the real perturbed magnetic pressure $p_\star(x,y)$. We will denote by L the region $0 < x < x_0$ and by R the remaining half space $x_0 < x < \infty$, excluding from them a narrow band around x_0.

Proposition 6. *Let j_y be the y-component of the sheet current $\mathbf{j}$. In L, the function $p_{\star W}/|f|^{1/2}$ is the sum of two harmonic functions: the first is defined in the whole half plane $-h < x < \infty$, is bounded there and its value at $x = -h$ is*

$$-\frac{\operatorname{sgn}(B(0))}{|l|} j_y .$$

The second is defined in $-\infty < x < 2x_0 + h$, is bounded and its value at $x = 2x_0 + h$ is

$$\mathrm{i}\frac{\operatorname{sgn}(\omega)\operatorname{sgn}(B(0))}{|l|} j_y .$$

P r o o f. Taking the asymptotic value of A found in (9) and the WKB normalized solution of Theorem 4, we find

$$\begin{aligned} P_W(x,k) = {} & \mathrm{i}\mathrm{e}^{-|k|(x+h)} I(k)|f(x)|^{1/2}\operatorname{sgn}(lB(0)) \\ & + \mathrm{e}^{-|k|(2x_0+h-x)} I(k)|f(x)|^{1/2}\operatorname{sgn}(\omega)\operatorname{sgn}(lB(0)) \end{aligned} \tag{15}$$

in L.

The inverse Fourier transform of $k \mapsto \mathrm{e}^{-|k|r}$ is

$$y \mapsto \frac{1}{\pi}\frac{r}{y^2 + r^2}$$

and $I(k)$ is the Fourier transform of $\mathrm{i}j_y/l$. This means

$$p_{\star W}(x,y) = -|f(x)|^{1/2}\frac{\operatorname{sgn}(B(0))}{|l|}\int_{\mathbb{R}} \frac{1}{\pi}\frac{x+h}{(x+h)^2 + (y-s)^2} j_y(s)\, ds +$$

$$+\mathrm{i}|f(x)|^{1/2}\frac{\operatorname{sgn}(\omega)\operatorname{sgn}(B(0))}{|l|}\int_{\mathbb{R}} \frac{1}{\pi}\frac{2x_0+h-x}{(2x_0+h-x)^2 + (y-s)^2} j_y(s)\, ds.$$

These are the Poisson integrals which yield the harmonic functions detailed in the proposition. □

Proposition 7. *In R, the function $p_{\star W}/|f|^{1/2}$ is the harmonic function defined in $-h < x < \infty$, bounded, and whose value at $x = -h$ is*

$$\mathrm{i}\,\frac{\operatorname{sgn}(\omega)\operatorname{sgn}(B(0))}{|l|}\,j_y.$$

The proof is identical to the last one. □

Notice that j_y extends within the plasma as a newtonian potential from the sheet $x = -h$. For $x < x_0$ there is a second ideal sheet mirroring the first at the other side of the singularity, i.e. at $x = 2x_0 + h$. The presence of the factors $\mathrm{i}\operatorname{sgn}(\omega)$ means that the potential due to the real sheet crosses the singularity unaltered except for a phase-shift, whereas the ideal one does not extend beyond x_0.

Corollary 8. *For every $r \in [1, \infty]$, the L^r-norm of $p_{\star W}(x, \cdot)$ is bounded for $0 < x < x_0$ by*

$$|f(x)|^{1/2}\frac{2}{|l|}||j_y||_r$$

whereas for $x > x_0$, it is bounded by

$$|f(x)|^{1/2}\frac{1}{|l|}||j_y||_r.$$

This follows from the classical theorems about the Poisson integral ([12]).

Proposition 9. *Assume that* **J** *vanishes for $|k| < k_0$, k_0 large enough. Let $r \geqslant 2$, α, $\beta \geqslant 1$ be such that $\frac{1}{r} + \frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then there exist a constant M such that*

$$||p_\star(x,\cdot) - p_{\star W}(x,\cdot)||_r \leqslant M|f(x)|^{1/2}||I||_\alpha \frac{1}{k_0}\left(\mathrm{e}^{-k_0(x+h)}\frac{1}{(\beta(x+h))^{1/\beta}} + \mathrm{e}^{-k_0(2x_0+h-x)}\frac{1}{(\beta(2x_0+h-x))^{1/\beta}}\right)$$

for $0 < x < x_0$, and

$$||p_\star(x,\cdot) - p_{\star W}(x,\cdot)||_r \leqslant M|f(x)|^{1/2}||I||_\alpha \frac{1}{k_0}\mathrm{e}^{-k_0(x+h)}\frac{1}{(\beta(x+h))^{1/\beta}}$$

for $x > x_0$.

Proof. The relative errors $|P - P_W|/|P_W|$ and that of A with the approximation of (9), which has been used to find P_W as in (15), are at most of the order $1/|k|$. Let k_0 be such that for all x, $|k| > k_0$,

$$|P(x,k) - P_W(x,k)| \leqslant \frac{N}{|k|}|P_W(x,k)|.$$

Let r' be the conjugate of r. Then

$$\|P(x,\cdot)-P_W(x,\cdot)\|_{r'} \leqslant 2N\Big(\int_{k_0}^{\infty}\Big|\frac{P_W(x,k)}{k}\Big|^{r'}\mathrm{d}k\Big)^{1/r'}$$
$$\leqslant 2N|f(x)|^{1/2}\|I\|_\alpha\Big[\Big(\int_{k_0}^{\infty}\Big(\frac{\mathrm{e}^{-k(x+h)}}{k}\Big)^{\beta}\mathrm{d}k\Big)^{1/\beta}+\Big(\int_{k_0}^{\infty}\Big(\frac{\mathrm{e}^{-k(2x_0+h-x)}}{k}\Big)^{\beta}\mathrm{d}k\Big)^{1/\beta}\Big]$$

by Hölder's inequality, given that $\frac{1}{r'}=\frac{1}{\alpha}+\frac{1}{\beta}$. The integral $\int_{k_0}^{\infty}(\mathrm{e}^{-kc}/k)^{\beta}\mathrm{d}k$ may be bounded by

$$\frac{1}{k_0^{\beta}}\frac{\mathrm{e}^{-\beta k_0 c}}{\beta c},$$

so that

$$\|P(x,\cdot)-P_W(x,\cdot)\|_{r'} \leqslant 2N|f(x)|^{1/2}\|I\|_\alpha\,\frac{1}{k_0}\Big(\mathrm{e}^{-k_0(x+h)}\frac{1}{(\beta(x+h))^{1/\beta}}$$
$$+\,\mathrm{e}^{-k_0(2x_0+h-x)}\,\frac{1}{(\beta(2x_0+h-x))^{1/\beta}}\Big).$$

By Hausdorff-Young's inequality,

$$\|p_\star(x,\cdot)-p_{\star W}(x,\cdot)\|_r \leqslant \|P(x,\cdot)-P_W(x,\cdot)\|_{r'}$$

so that the bound in L follows. The argument for R is identical. □

Corollary 10. *Let $r\geqslant 2$, $1\leqslant a\leqslant 2$, $b\geqslant 1$, $\frac{1}{r}+\frac{1}{b}=\frac{1}{a}$. Then the same bounds of Proposition 9 hold if $\|I\|_\alpha$ is replaced by $(1/|l|)\|j_y\|_a$ and β for b.*

Proof. Let a be the conjugate of α in Proposition 9. When $1\leqslant a\leqslant 2$, by the inequality of Hausdorff-Young $\|I\|_\alpha\leqslant(1/|l|)\|j_y\|_a$. The relation $\frac{1}{r}+\frac{1}{\alpha}+\frac{1}{\beta}=1$ is converted into $\frac{1}{r}+\frac{1}{b}=\frac{1}{a}$. □

We may see that the approximation improves exponentially with k_0 and x, so that even for moderate values of k_0 the perturbed magnetic pressure behaves much like the harmonic function whose values at the current sheet are given by j_y.

No attempt is made to obtain the solutions as functions of time because we would need to take into account the values of the perturbed field and velocity at $t=0$ in order to perform an inversion of the Fourier-Laplace transform. This would add an independent term to our equation and many difficulties to our analysis. By taking them zero as we have done we are still able to study the behaviour of the spectrum and the effects of resonance.

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